

# Renormalization-Group Improved Resummation of Super-Leading Logarithms using SCET

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based on

2107.01212

2307.06359

2307.11089 (Glauber series for IS-quarks)

2311.18811 (Glauber series for IS-gluons)

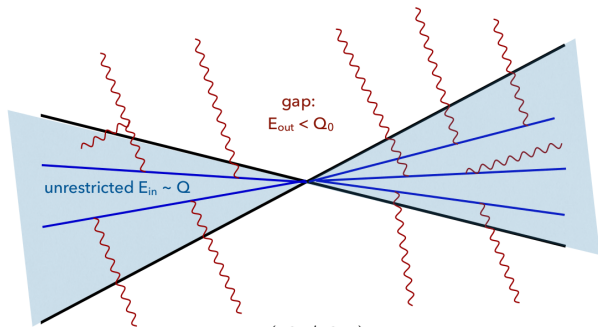
→ 2405.05305 (RG-improved resummation)

2407.01691 (Resummation at large- $N_c$ )

in collaboration with

Patrick Hager, Matthias Neubert, Michel Stillger, Xiaofeng Xu

# Super-leading logarithms in non-global observables

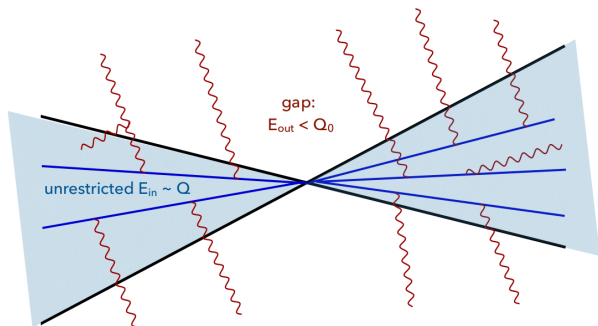


$$L = \ln(Q/Q_0) \gg 1$$

- Perturbative expansion includes “super-leading” logarithms starting at 4 loops

$$\sigma \sim \sigma_{\text{Born}} \times \{1 + \alpha_s L + \alpha_s^2 L^2 + \alpha_s^3 L^3 + \alpha_s^4 L^5 + \alpha_s^5 L^7 + \dots\}$$

# Super-leading logarithms in non-global observables



$$L = \ln(Q/Q_0) \gg 1$$

- A double-logarithmic series starting at **3-loop order**:

$$\sigma \sim \sigma_{\text{Born}} \times \{1 + \alpha_s L + \alpha_s^2 L^2 + (\alpha_s \pi^2) (\alpha_s^2 L^3 + \alpha_s^3 L^5 + \dots)\}$$

- $(\alpha_s \pi^2)$  from **imaginary part** of the large logarithm  $\ln(-Q^2/Q_0^2) = 2L - i\pi$ .
- Breakdown of color-coherence due to **soft (Glauber) gluon exchange** between **partonic initial-states**.

[Forshaw, Kyrieleis, Seymour 2006]

# The Glauber series

- SLLs require **two** Glauber phases.
- Not a large logarithm, but associated with large numerical factor of  $\pi^2$ .
- For realistic values, e.g  $Q \sim 1\text{TeV}$  and  $Q_0 \sim 40\text{GeV}$ , find  $L \sim \pi$ .
- Resum **both** the SLLs and associated factors of  $\pi^2$  in the **Glauber series**.
- Cross section takes the form

$$\sigma^{\text{SLL+G}} \sim \frac{\alpha_s L}{\pi N_c} \sum_{\ell=1}^{\infty} \sum_{n=0}^{\infty} c_{\ell,n} w_{\pi}^{\ell} w^{n+\ell}, \quad c_{\ell,n} \sim (-1)^{n+\ell}$$

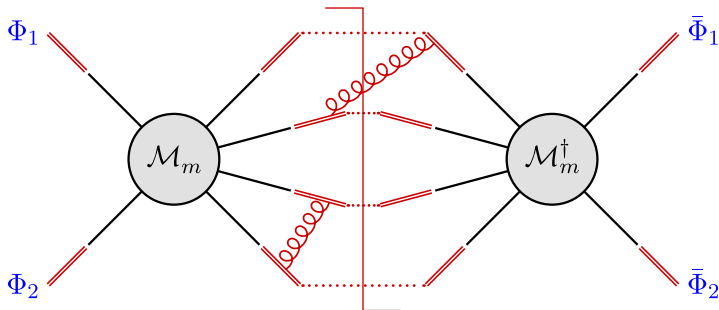
with  $\mathcal{O}(1)$  expansion parameters  $w_{\pi} = \frac{N_c \alpha_s}{\pi} \pi^2$  and  $w = \frac{N_c \alpha_s}{\pi} L^2$

# Factorisation Theorem

For gap-between-jets cross sections

$$\sigma_{2 \rightarrow M}(Q, Q_0) = \int dx_1 dx_2 \sum_{m=2+M}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, x_1, x_2, \mu) \otimes \mathcal{W}_m(\{\underline{n}\}, Q_0, x_1, x_2, \mu) \rangle$$

[Becher, Neubert, Shao (2021); Becher, Neubert, Rothen, Shao (2015,2016)]



# Factorisation Theorem

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- Hard function in color space

$$\mathcal{H}_m(\{\underline{n}\}) = \int dE_m |\mathcal{M}_m(\{\underline{p}\})\rangle \langle \mathcal{M}_m(\{\underline{p}\})|$$

where energy integrals contain the phase-space constraints

- Renormalisation-group equation:

$$\frac{d}{d \ln \mu} \mathcal{H}_l(\{\underline{n}\}, Q, \mu) = - \sum_{m \leq l} \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \star \Gamma_{ml}^H(\{\underline{n}\}, Q, \mu)$$

# Resummation of super-leading logarithms

- Evaluate factorisation theorem at low scale  $\mu_s \sim Q_0$ :
- Low-energy matrix element reduces to PDFs

$$\mathcal{W}_m(\{\underline{n}\}, Q_0, x_1, x_2, \mu_s) = f_1(x_1)f_2(x_2) \mathbf{1} + \mathcal{O}(\alpha_s)$$

- Evolve hard function from  $\mu_h \sim Q$  to  $\mu_s \sim Q_0$

$$\mathcal{H}_m(\mu_s) = \sum_{l \leq m} \mathcal{H}_l(\mu_h) \star \mathbf{P} \exp \left[ \int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu} \Gamma^H(\mu) \right]_{lm}$$

- Task: need to manage the **P**-ordered evolution operator.



# Anomalous Dimension

- One-loop anomalous dimension contains **virtual** and **real** contributions

$$\Gamma^H(\{\underline{n}\}, s, \mu) = \frac{\alpha_s}{4\pi} \begin{pmatrix} \mathbf{V}_{2+M} & \mathbf{R}_{2+M} & 0 & 0 & \dots \\ 0 & \mathbf{V}_{2+M+1} & \mathbf{R}_{2+M+1} & 0 & \dots \\ 0 & 0 & \mathbf{V}_{2+M+2} & \mathbf{R}_{2+M+2} & \dots \\ 0 & 0 & 0 & \mathbf{V}_{2+M+3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Action on hard functions:

$$\mathcal{H}_m \bar{\mathbf{V}}_m = \sum_{(ij)} \text{diagram 1} + \text{diagram 2}$$

$$\mathcal{H}_m \bar{\mathbf{R}}_m = \sum_{(ij)} \text{diagram 3}$$

- Real-emission** contributions **extend** the color space!

# Anomalous Dimension

- Decompose into **soft emission**, **Glauber phase** and **collinear emission**:

$$\Gamma^H = \Gamma^C + \gamma_{\text{cusp}}(\alpha_s) \left( \Gamma^c \ln \frac{\mu^2}{\mu_h^2} + \mathbf{V}^G \right) + \frac{\alpha_s}{4\pi} \bar{\Gamma} + \mathcal{O}(\alpha_s^2)$$

- where

$$\Gamma^c = \sum_{i=1,2} [C_i \mathbf{1} - \mathbf{T}_{i,L} \circ \mathbf{T}_{i,R} \delta(n_k - n_i)],$$

$$\mathbf{V}^G = -2i\pi (\mathbf{T}_{1,L} \cdot \mathbf{T}_{2,L} - \mathbf{T}_{1,R} \cdot \mathbf{T}_{2,R}),$$

$$\begin{aligned} \bar{\Gamma} &= 2 \sum_{(ij)} (\mathbf{T}_{i,L} \cdot \mathbf{T}_{j,L} + \mathbf{T}_{i,R} \cdot \mathbf{T}_{j,R}) \int \frac{d\Omega(n_k)}{4\pi} \bar{W}_{ij}^k \\ &\quad - 4 \sum_{(ij)} \mathbf{T}_{i,L} \circ \mathbf{T}_{j,R} \bar{W}_{ij}^k \Theta_{\text{hard}}(n_k) \end{aligned}$$

# Anomalous Dimension

- Decompose into **soft emission**, **Glauber phase** and **collinear emission**:

$$\mathbf{\Gamma}^H = \mathbf{\Gamma}^C + \gamma_{\text{cusp}}(\alpha_s) \left( \mathbf{\Gamma}^c \ln \frac{\mu^2}{\mu_h^2} + \mathbf{V}^G \right) + \frac{\alpha_s}{4\pi} \mathbf{\bar{\Gamma}} + \mathcal{O}(\alpha_s^2)$$

- Important properties:

$$\begin{array}{lll} \left[ \mathbf{\Gamma}^c, \mathbf{\bar{\Gamma}} \right] = 0, & \langle \mathcal{H}_m \mathbf{\Gamma}^c \otimes \mathbf{1} \rangle = 0, & \langle \mathcal{H}_m \mathbf{V}^G \otimes \mathbf{1} \rangle = 0 \\ \text{(color-coherence)} & \text{(collinear safety)} & \text{(cyclicity of trace)} \end{array}$$

- rightmost structure in color-trace must be  $\mathbf{V}^G \mathbf{\bar{\Gamma}}$ .

# SLLs in RG-Improved PT

- keep log-enhanced soft-collinear piece in exponential

$$U_{\text{SLL}}(\{\underline{n}\}, \mu_h, \mu_s) = \int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1} \int_{\mu_s}^{\mu_1} \frac{d\mu_2}{\mu_2} \int_{\mu_s}^{\mu_2} \frac{d\mu_3}{\mu_3} \\ \times U_c(\mu_h, \mu_1) \gamma_{\text{cusp}}(\alpha_s(\mu_1)) \mathbf{V}^G U_c(\mu_1, \mu_2) \gamma_{\text{cusp}}(\alpha_s(\mu_2)) \mathbf{V}^G \frac{\alpha_s(\mu_3)}{4\pi} \bar{\Gamma}$$

- Sudakov-like evolution operator resums double-log's

$$U_c(\mu_i, \mu_j) = \exp \left[ \mathbf{\Gamma}^c \int_{\mu_j}^{\mu_i} \frac{d\mu}{\mu} \gamma_{\text{cusp}}(\alpha_s(\mu)) \ln \frac{\mu^2}{\mu_h^2} \right].$$

- One scale integral for each insertion of  $\mathbf{V}^G$  and  $\bar{\Gamma}$ .
- Running coupling implemented.

# Color basis for Quark-initiated Processes

- 5 color structures:

(20 for gg, 14 for qg)

$$\mathbf{X}_1 = \sum_{j>2} J_j i f^{abc} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_j^c,$$

$$\mathbf{X}_4 = \frac{1}{N_c} J_{12} \mathbf{T}_1 \cdot \mathbf{T}_2,$$

$$\mathbf{X}_2 = \sum_{j>2} J_j (\sigma_1 - \sigma_2) d^{abc} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_j^c,$$

$$\mathbf{X}_5 = J_{12} \mathbf{1}.$$

$$\mathbf{X}_3 = \frac{1}{N_c} \sum_{j>2} J_j (\mathbf{T}_1 - \mathbf{T}_2) \cdot \mathbf{T}_j,$$

- Matrix representation of elements of anomalous dimension

$$\mathbf{\Gamma}^c \rightarrow N_c \mathbf{\Gamma}^c, \quad \mathbf{V}^G \rightarrow i\pi N_c \mathbf{W}^G, \quad \mathbf{V}^G \bar{\mathbf{\Gamma}} \otimes \mathbf{1} \rightarrow 16i\pi \mathbf{X}_1 = 16i\pi \mathbf{X}^T \zeta$$

$$\mathbf{\Gamma}^c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -\frac{C_F}{N_c} & 0 & 0 \end{pmatrix}, \quad \mathbf{W}^G = \begin{pmatrix} 0 & -2\delta_{q\bar{q}} \frac{N_c^2 - 4}{N_c^2} & \frac{4}{N_c^2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

# Representation of the evolution operator

- $\mathbf{\Gamma}^c$  appears inside the **Sudakov-like** matrix exponential

$$\mathbb{U}_c(\mu_i, \mu_j) = \exp \left[ N_c \mathbf{\Gamma}^c \int_{\mu_j}^{\mu_i} \frac{d\mu}{\mu} \gamma_{\text{cusp}}(\alpha_s(\mu)) \ln \frac{\mu^2}{\mu_h^2} \right].$$

- The exponential can be performed to yield

$$\mathbb{U}_c(\mu_i, \mu_j) = \begin{pmatrix} U_c(1; \mu_i, \mu_j) & 0 & 0 & 0 & 0 \\ 0 & U_c(1; \mu_i, \mu_j) & 0 & 0 & 0 \\ 0 & 0 & U_c(\frac{1}{2}; \mu_i, \mu_j) & 0 & 0 \\ 0 & 0 & 2[U_c(\frac{1}{2}; \mu_i, \mu_j) - U_c(1; \mu_i, \mu_j)] & U_c(1; \mu_i, \mu_j) & 0 \\ 0 & 0 & \frac{2C_F}{N_c} [1 - U_c(\frac{1}{2}; \mu_i, \mu_j)] & 0 & 1 \end{pmatrix}$$

with the scalar Sudakov factor

$$U_c(v; \mu_i, \mu_j) = \exp \left[ v N_c \int_{\mu_j}^{\mu_i} \frac{d\mu}{\mu} \gamma_{\text{cusp}}(\alpha_s(\mu)) \ln \frac{\mu^2}{\mu_h^2} \right]$$

# LO in RG-improved perturbation theory

- The cross section can be expressed as a matrix product

$$\hat{\sigma}_{2 \rightarrow M}^{\text{SLL}+\text{G}}(Q_0) = \sum_{l=1}^{\infty} \langle \mathcal{H}_{2 \rightarrow M}(\mu_h) \mathbf{X}^T \rangle \mathbb{U}_{\text{SLL}}^{(l)}(\mu_h, \mu_s) \mathbb{S}$$

with the evolution operator

$$\begin{aligned} \mathbb{U}_{\text{SLL}}^{(l)}(\mu_h, \mu_s) &= 16 (i\pi)^l N_c^{l-1} \int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1} \dots \int_{\mu_s}^{\mu_l} \frac{d\mu_{l+1}}{\mu_{l+1}} \mathbb{U}_c(\mu_h, \mu_1) \\ &\times \left[ \prod_{i=1}^{l-1} \gamma_{\text{cusp}}(\alpha_s(\mu_i)) \mathbb{V}^{\text{G}} \mathbb{U}_c(\mu_i, \mu_{i+1}) \right] \gamma_{\text{cusp}}(\alpha_s(\mu_l)) \frac{\alpha_s(\mu_{l+1})}{4\pi} \end{aligned}$$

- Two-loop** approx. in exponent ( $x_i = \alpha_s(\mu_i)/\alpha_s(\mu_h)$ ), one-loop elsewhere

$$\begin{aligned} U_c(v; \mu_i, \mu_j) &= \exp \left\{ \frac{\gamma_0 v N_c}{2\beta_0^2} \left[ \frac{4\pi}{\alpha_s(\mu_h)} \left( \frac{1}{x_i} - \frac{1}{x_j} - \ln \frac{x_j}{x_i} \right) \right. \right. \\ &\quad \left. \left. + \left( \frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right) \left( x_i - x_j + \ln \frac{x_j}{x_i} \right) + \frac{\beta_1}{2\beta_0} (\ln^2 x_j - \ln^2 x_i) \right] \right\}, \end{aligned}$$

# LO in RG-improved perturbation theory

- Process-independent **evolution functions** with two  $V^G$  insertions:

$$\mathbb{U}_{\text{SLL}}^{(2)}(\mu_h, \mu_s) \mathbb{S} = -\frac{32\pi^2}{\beta_0^3} N_c \int_1^{x_s} \frac{dx_2}{x_2} \ln \frac{x_s}{x_2} \int_1^{x_2} \frac{dx_1}{x_1} \times \begin{pmatrix} 0 \\ -\frac{1}{2} U_c(1; \mu_h, \mu_2) \\ U_c(\frac{1}{2}, 1; \mu_h, \mu_1, \mu_2) \\ 2 [U_c(\frac{1}{2}, 1; \mu_h, \mu_1, \mu_2) - U_c(1; \mu_h, \mu_2)] \\ \frac{2C_F}{N_c} [U_c(1; \mu_1, \mu_2) - U_c(\frac{1}{2}, 1; \mu_h, \mu_1, \mu_2)] \end{pmatrix}$$

- contains products of evolution factors with different eigenvalues
- Two remaining scale integrals required to resum SLLs



# Asymptotic Behaviour

- Asymptotic expansion for  $\alpha_s L \sim 1, \alpha_s L^2 \gg 1$  (restricting to a fixed coupling)

$$\mathbb{U}_{\text{SLL}}^{(2)}(\mu_h, \mu_s) \varsigma = -\frac{2\pi^2}{3} N_c \left(\frac{\alpha_s}{\pi} L_s\right)^3 \begin{pmatrix} 0 \\ -\frac{1}{2} \Sigma(1, 1; w) \\ \Sigma(\frac{1}{2}, 1; w) \\ 2 [\Sigma(\frac{1}{2}, 1; w) - \Sigma(1, 1; w)] \\ \frac{2C_F}{N_c} [\Sigma(0, 1; w) - \Sigma(\frac{1}{2}, 1; w)] \end{pmatrix}$$

- Expansions extracted from a method-of-regions analysis  $(w = \frac{N_c \alpha_s}{\pi} L^2)$

$$\Sigma(1, 1; w) = \frac{3}{w} - \frac{3\sqrt{\pi}}{2w^{3/2}} + \mathcal{O}(e^{-w}),$$

$$\Sigma(\frac{1}{2}, 1; w) = \frac{3\sqrt{2} \ln(1 + \sqrt{2})}{w} - \frac{3\sqrt{\pi}}{\sqrt{2}w^{3/2}} + \mathcal{O}(e^{-\frac{w}{2}}),$$

$$\Sigma(0, 1; w) = \frac{3}{2} \frac{\ln(4w) + \gamma_E - 2}{w} + \frac{3}{4w^2} + \mathcal{O}(w^{-3}).$$

- parametric suppression for higher Glauber phases

$$\Sigma(v^{(1)}, \dots, v^{(l)}; w) \sim w^{-l/2}$$

# Large $N_c$

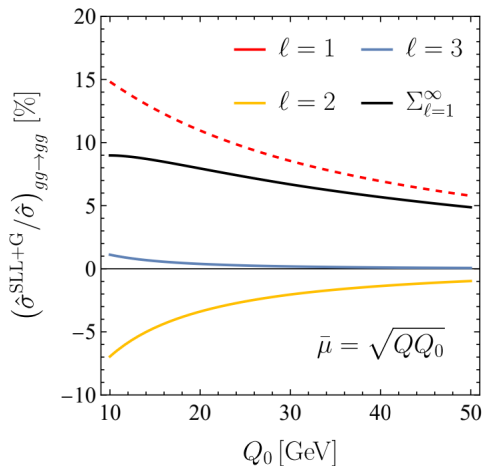
- Matrix structures simplify, such that all Glauber exchanges can be resummed in terms of a two-fold scale integral

$$\sum_{\ell=1}^{\infty} \mathbb{U}_{\text{SLL}}^{(2\ell-1)}(\mu_h, \mu_s) \varsigma = \frac{16i\pi}{\beta_0^2} \int_1^{x_s} \frac{dx_2}{x_2} \ln \frac{x_s}{x_2} U_c(1; \mu_h, \mu_2) \left[ 1 - 2\delta_{q\bar{q}} \sin^2 \left( \frac{\pi N_c}{\beta_0} \ln x_2 \right) \right] \varsigma,$$

$$\sum_{\ell=1}^{\infty} \mathbb{U}_{\text{SLL}}^{(2\ell)}(\mu_h, \mu_s) \varsigma = \frac{16\pi}{\beta_0^2} \frac{2\pi N_c}{\beta_0} \int_1^{x_s} \frac{dx_2}{x_2} \ln \frac{x_s}{x_2} \int_1^{x_2} \frac{dx_1}{x_1} \left[ 1 - 2\delta_{q\bar{q}} \sin^2 \left( \frac{\pi N_c}{\beta_0} \ln \frac{x_2}{x_1} \right) \right]$$

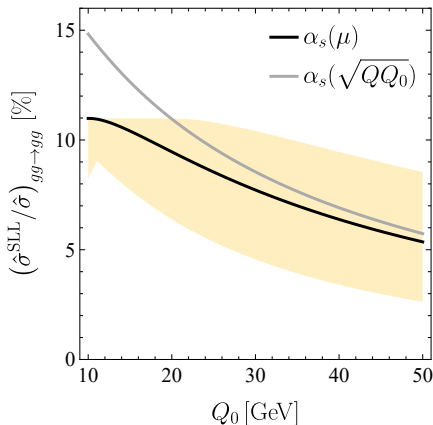
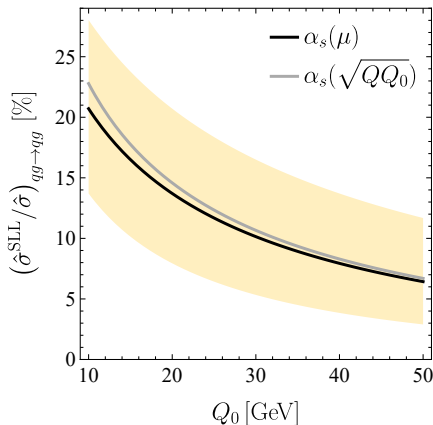
$$\times \begin{pmatrix} 0 \\ \frac{1}{2} U_c(1; \mu_h, \mu_2) \\ -U_c(\frac{1}{2}, 1; \mu_h, \mu_1, \mu_2) \\ 2[U_c(1; \mu_h, \mu_2) - U_c(\frac{1}{2}, 1; \mu_h, \mu_1, \mu_2)] \\ \frac{2C_F}{N_c} [U_c(\frac{1}{2}, 1; \mu_h, \mu_1, \mu_2) - U_c(1; \mu_1, \mu_2)] \end{pmatrix}$$

# Numerical Estimates



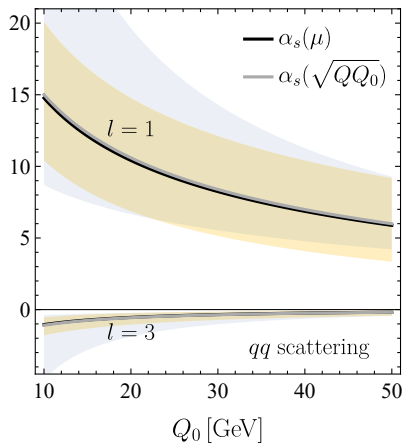
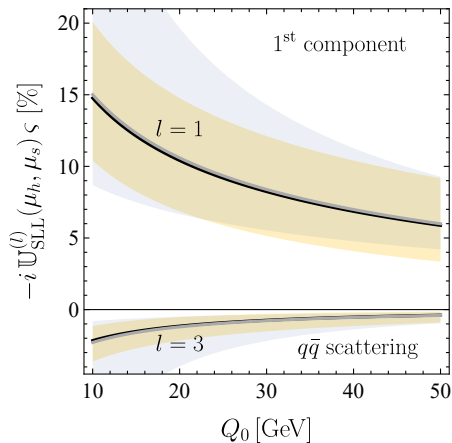
- $Q = 1\text{TeV}$  and rapidity gap  $\Delta Y = 2$ .

# Numerical Estimates



- $Q = 1\text{TeV}$  and rapidity gap  $\Delta Y = 2$ .
- **Error band** from scale variation  $\mu_s \in (\frac{Q_0}{2}, 2Q_0)$ .

# Numerical Estimates



- $Q = 1\text{TeV}$  and rapidity gap  $\Delta Y = 2$ .
- **Error band** from scale variation  $\mu_s \in (\frac{Q_0}{2}, 2Q_0)$ .

# Conclusion

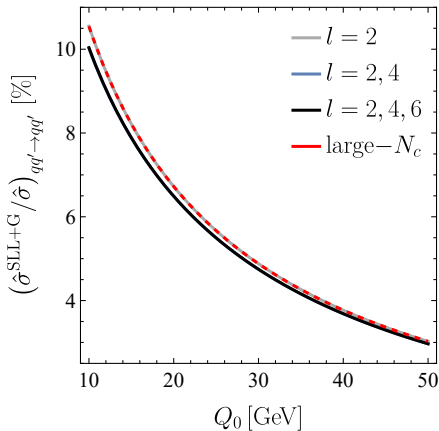
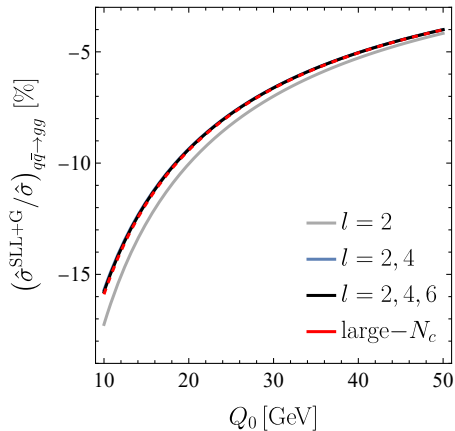
- Resummation of SLLs lifted to standard language of **RG-improved PT**
  - ▶ SLLs are integrals over Sudakov factors
  - ▶ Estimate of pert. uncertainties through scale variation
- Asymptotic scaling shows parametric suppression of higher Glauber exchanges
- Closed-form expression for entire Glauber series at large  $N_c$

## Outlook:

- How large is the impact **beyond the partonic level?**
- How to push the formalism to include double (multiple) soft emissions?
- Is double-logarithmic evolution leading to SLLs **compatible with PDF factorization?**
  - Plenary talk “Factorization restoration through Glauber gluons” by **Thomas Becher** today at **18:10**

## Backup-Slides

# Large $N_c$ comparison





# Large $N_c$ comparison

