High Precision for Hard Processes (HP2), Turin, Italy 10 September 2024

Feynman Integrals in Parameter Space: Hidden Regions and Contour Deformation

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SJ, Olsson, Stone [WIP]
Gardi, Herzog, SJ, Ma [2407.13738]
Gardi, Herzog, SJ, Ma, Schlenk [2211.14845]
Heinrich, Jahn, SJ, Kerner, Langer, Magerya, Olsson, Põldaru, Schlenk, Villa [2108.10807, 2305.19768]

Outline

Introduction

Parameter space & Newton polytopes

Method of Regions (MoR)

Integrals with Pinch Singularities

Finding and evaluating integrals with pinch singularities for *generic* kinematics

Hidden Regions due to Cancellation

Evaluating Integrals in the Minkowski Regime w/o Contour Deformation

Concept & Examples

Introduction

Parameter Space

Can exchange integrals over loop momenta for integrals over parameters

Feynman Parametrisation

$[\mathrm{d}\boldsymbol{\alpha}] = \prod_{e \in G} \frac{\mathrm{d}\alpha_e}{\alpha_e}$ $\quad \boldsymbol{\alpha}^{\nu} = \prod_{e \in G} \alpha_e^{\nu_e}$

$$I(\mathbf{s}) = \frac{\Gamma(\nu - LD/2)}{\prod_{e \in G} \Gamma(\nu_e)} \int_0^\infty [d\boldsymbol{\alpha}]\, \boldsymbol{\alpha}^{\nu} \delta\left(1 - H(\boldsymbol{\alpha})\right) \frac{\left[\mathcal{U}(\boldsymbol{\alpha})\right]^{\nu - (L+1)D/2}}{\left[\mathcal{F}(\boldsymbol{\alpha}; \mathbf{s})\right]^{\nu - LD/2}}$$

$\mathcal{U}, \mathcal{F}$ homogeneous polynomials of degree L and $L+1$

Lee-Pomeransky Parametrisation

$$I(\mathbf{s}) = \frac{\Gamma(D/2)}{\Gamma\left((L+1)\,D/2 - \nu\right) \prod_{e \in G} \Gamma(\nu_e)} \int_0^\infty [d\mathbf{x}]\, \mathbf{x}^{\nu} \left(\mathcal{G}(\mathbf{x}, \mathbf{s})\right)^{-D/2}$$

$$\mathcal{G}(\mathbf{x}; \mathbf{s}) = \mathcal{U}(\mathbf{x}) + \mathcal{F}(\mathbf{x}; \mathbf{s})$$

Lee, Pomeransky 13

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^{N+1}_{\geq 0}} [\mathrm{d}\boldsymbol{\alpha}]\, \boldsymbol{\alpha}^{\nu} \frac{[\mathscr{U}(\boldsymbol{\alpha})]^{N-(L+1)D/2}}{[\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) - i\delta]^{N-LD/2}} \delta(1 - H(\boldsymbol{\alpha}))$$

Singularities

1. UV/IR singularities when some $\alpha \to 0$ simultaneously $\Longrightarrow$ Sector Decomposition
2. Thresholds when $\mathscr{F}$ vanishes inside integration region $\Longrightarrow$ Contour Deformation

Sector decomposition

Find a local change of coordinates for each singularity that factorises it (blow-up)

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t$$

$$\mathcal{N}(I) = \mathrm{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

Normal vectors incident to each extremal vertex define a local change of variables*

Kaneko, Ueda 10

$$x_i = \prod_{f \in S_j} y_f^{\langle \mathbf{n}_f, \mathbf{e}_i \rangle}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_0^1 \left[\mathrm{d}\mathbf{y}_f\right] \underbrace{\prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle - t a_f}}_{\text{Singularities}} \Bigg(\underbrace{c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f}}_{\text{Finite}} \Bigg)^t$$

Singularities **Finite**

*If $|S_j| > N$, need triangulation to define variables (simplicial normal cones $\sigma \in \Delta^T_{\mathcal{N}}$)

→Talk of Leonardo on Thursday

Method of Regions

Consider expanding an integral about some limit:

$p_i^2 \sim \lambda Q^2$, $p_i \cdot p_j \to \lambda Q^2$ or $m^2 \sim \lambda Q^2$ for $\lambda \to 0$

Issue: integration and series expansion do not necessarily commute

Method of Regions

$$I(\mathbf{s}) = \sum_R I^{(R)}(\mathbf{s}) = \sum_R T_{\mathbf{t}}^{(R)} I(\mathbf{s})$$

1. Split integrand up into regions (R)
2. Series expand each region in λ
3. Integrate each expansion over the whole integration domain
4. Discard scaleless integrals (= 0 in dimensional regularisation)
5. Sum over all regions

Smirnov 91; Beneke, Smirnov 97; Smirnov, Rakhmetov 99; Pak, Smirnov 11; Jantzen 2011; …

Finding Regions

Assuming all c_i have the same sign we rescale $\boldsymbol{s} \to \lambda^{\boldsymbol{\omega}}\boldsymbol{s}$ ← $s_i \to \lambda^{\omega_i} s_i$

Newton Polytope

$$I \sim \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t \to \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i} \lambda^{r_{i,N+1}}\right)^t \to \mathcal{N}^{N+1}$$

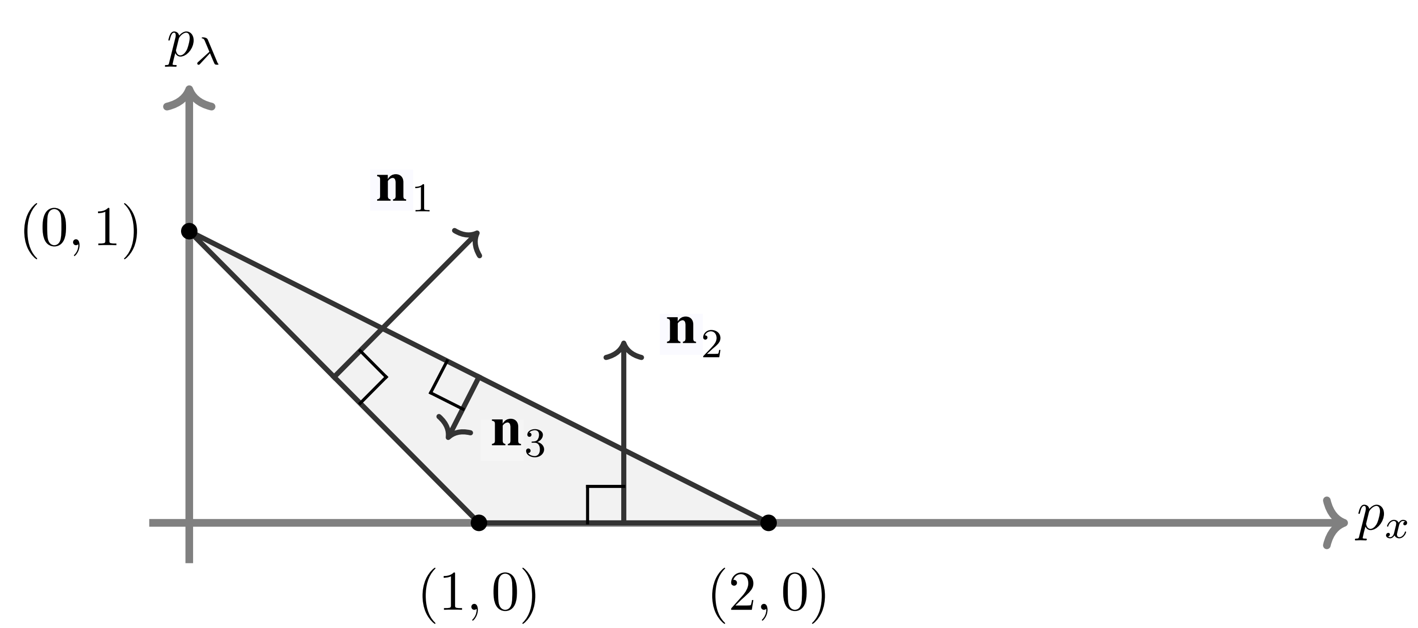
Normal vectors w/ positive λ component define change of variables $\mathbf{n}_f = (v_1, \ldots, v_N, 1)$

$$\boldsymbol{x} = \lambda^{\mathbf{n}_f}\, \mathbf{y}\,, \qquad \lambda \to \lambda$$

Pak, Smirnov 10; Semenova, A. Smirnov, V. Smirnov 18

Example

$p(x, \lambda) = \lambda + x + x^2$



$1{,}2 \in F^+$

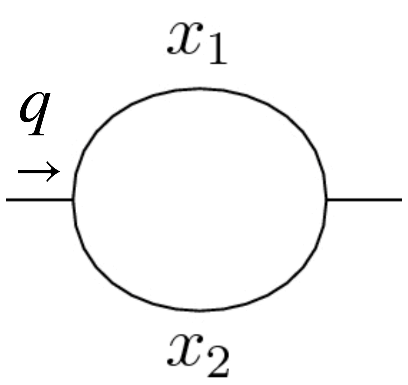
$3 \notin F^+$

Original integral I may then be approximated as $I = \sum_{f \in F^+} I^{(f)} + \ldots$

Regions due to Cancellation

What happens if c_i have different signs?

Example: 1-loop massive bubble at *threshold* $y = m^2 - q^2/4 \to 0$



$$I = \Gamma(\epsilon) \int d\alpha_1 d\alpha_2 \frac{\delta(1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2)^{-2+2\epsilon}}{\left(\mathcal{F}_{\text{bub}}(\alpha_1,\alpha_2;q^2,y)\right)^{\epsilon}}$$

$$\mathcal{F}_{\text{bub}} = \frac{q^2}{4}(\alpha_1 - \alpha_2)^2 + y(\alpha_1+\alpha_2)^2$$

Can split integral into two subdomains $\alpha_1 \le \alpha_2$ and $\alpha_2 \le \alpha_1$ then remap

$$\begin{aligned} \alpha_1 &= \alpha_1'/2 \\ \alpha_2 &= \alpha_2' + \alpha_1'/2 \end{aligned} : \quad \mathcal{F}_{\text{bub},1} \to \frac{q^2}{4}\alpha_2'^2 + y(\alpha'_1 + \alpha'_2)^2 \quad \text{(for first domain)}$$

Before split: only **hard** region found ($\alpha_1 \sim y^0, \alpha_2 \sim y^0$)

After split: also **potential** region found ($\alpha_1 \sim y^0, \alpha_2 \sim y^{1/2}$)

Existing tools attempt to find such re-mappings using **linear** changes of variables

ASY: Jantzen, Smirnov, Smirnov 12; **ASPIRE:** Ananthanarayan, (Pal, Ramanan), Sarkar 18 + Das 20;

This is not generally enough to expose all regions in parameter space

Integrals with Pinch Singularities

Based on:

Gardi, Herzog, SJ, Ma [2407.13738]

Gardi, Herzog, SJ, Ma, Schlenk [2211.14845]

Landau Equations

Polynomials $\mathscr{U}, \mathscr{F}$ can vanish (gives singularities) for some $\alpha_i \to 0$ (end-point)

Additionally, due to signs in $\mathscr{F}$ it can vanish due to cancellation of terms

Avoid poles on real axis by deforming contour (roughly speaking…):

$$\alpha_k \to \alpha_k - i\varepsilon_k(\boldsymbol{\alpha})$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) \to \mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) - i\sum_k \varepsilon_k \frac{\partial \mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_k} + \mathcal{O}(\varepsilon^2)$$

If $\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = 0$ and $\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})/\partial\alpha_j = 0 \ \forall j$ simultaneously, contour will vanish exactly where the deformation is required, above conditions are just the Landau equations

Landau Equations (parameter space):

$$(L+1)\mathscr{F} = \sum_{k=1}^{N} \alpha_k \frac{\partial \mathscr{F}}{\partial \alpha_k}.$$

$$1) \quad \mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = 0$$

$$2) \quad \alpha_j \frac{\partial \mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_j} = 0 \quad \forall j$$

Leading: $\alpha_j \neq 0 \forall j$

Solutions are *pinched surfaces* of the integral where IR divergences may arise

Looking for Trouble: Algorithm

Generally, solutions of the Landau equations depend on $\mathbf{s}$.

Let us restrict our search to solutions with *generic* kinematics

$$\mathcal{F} = -\sum_i s_i \left[f_i(\boldsymbol{\alpha}) - g_i(\boldsymbol{\alpha})\right] = \sum_i \mathcal{F}_{i,-} + \mathcal{F}_{i,+}$$

$$\mathcal{F}_{i,-} = -s_i f_i(\boldsymbol{\alpha}), \quad \mathcal{F}_{i,+} = s_i g_i(\boldsymbol{\alpha}), \quad f_i(\boldsymbol{\alpha}), g_i(\boldsymbol{\alpha}) \geq 0$$

Algorithm (finds integrals which *potentially* have a pinch in the massless case)

For each s_i:

1) Compute $\mathcal{F}_{i,-}$, $\mathcal{F}_{i,+}$

2) If $\mathcal{F}_{i,-} = 0$ or $\mathcal{F}_{i,+} = 0 \rightarrow$ **Exit (no cancellation)**

3) If $\partial \mathcal{F}_{i,-}/\partial \alpha_j = 0$ or $\partial \mathcal{F}_{i,+}/\partial \alpha_j = 0$ set $\alpha_j = 0 \rightarrow$ Goto 1

Else $\rightarrow$ **Exit (potential cancellation)**

Much more sophisticated algorithms for solving Landau equations exist

(E.g.) Mizera, Simon Telen 21; Fevola, Mizera, Telen 23

(See also) Gambuti, Kosower, Novichkov, Tancredi 23

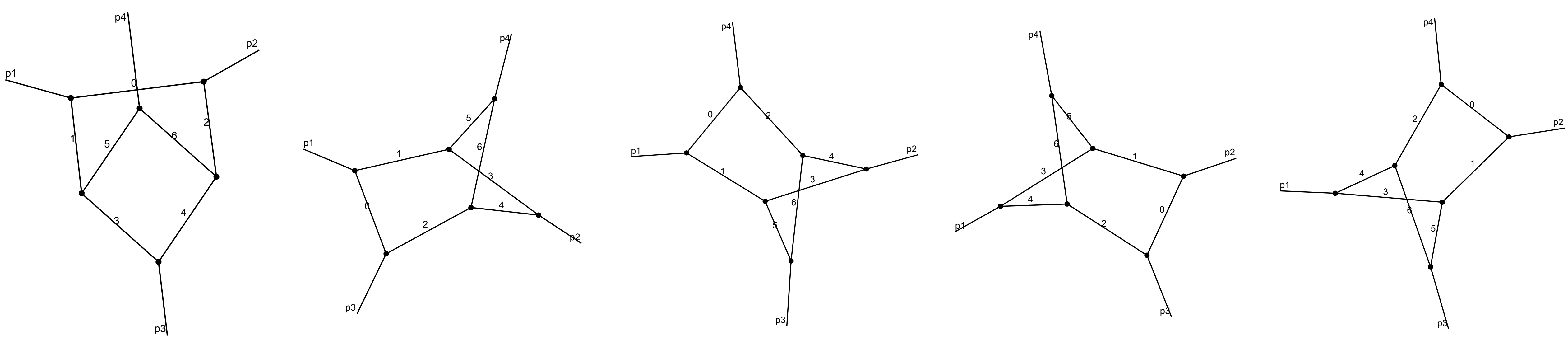
→**Talk of Pavel on Thursday**

Looking for Trouble: 1- & 2-loops

We considered massless 4-point scattering amplitudes ($s_{23} = -s_{12} - s_{13}$)

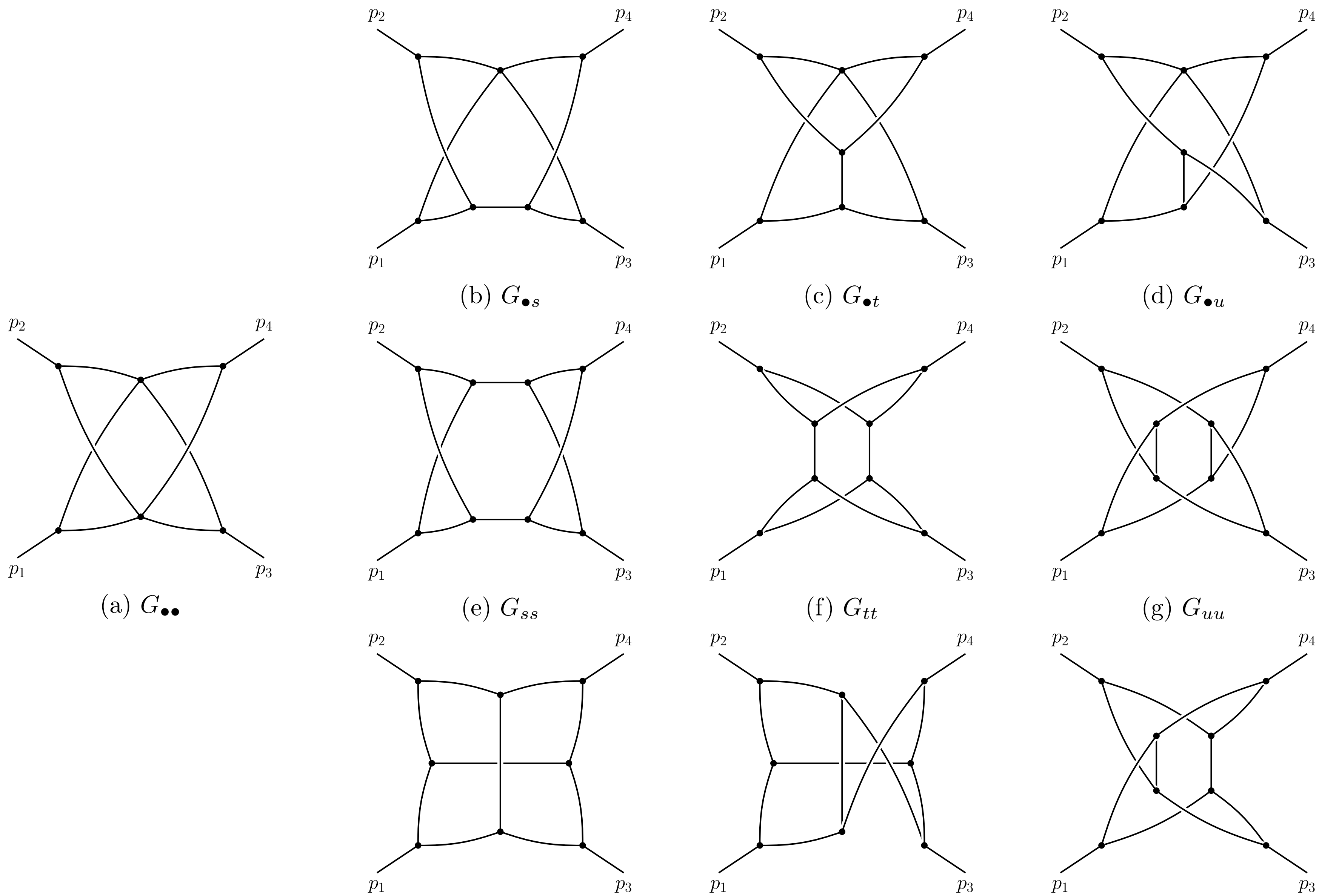
@1-loop: found no candidates (trivially)

@2-loop: no candidates (!)



Looking for Trouble: 3-loops

@3-loop: finally some interesting candidates

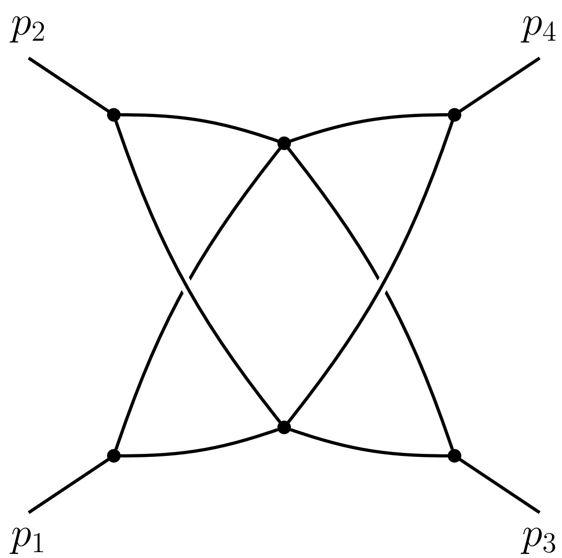


(a) $G_{\bullet\bullet}$ (b) $G_{\bullet s}$ (c) $G_{\bullet t}$ (d) $G_{\bullet u}$ (e) G_{ss} (f) G_{tt} (g) G_{uu} (h) G_{st} (i) G_{su} (j) G_{tu}

The complete set of corresponding master integrals for generic s_{12}, s_{13} are known

Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21

Interesting Example



$$= \int_0^\infty \mathrm{d}x_0 \ldots \mathrm{d}x_7 \frac{\mathcal{U}(\mathbf{x})^{4\epsilon}}{\mathcal{F}(\mathbf{x};\mathbf{s})^{2+3\epsilon}} \delta(1 - x_7)$$

$$\mathcal{U}(\alpha) = \alpha_0\alpha_2\alpha_4 + \alpha_0\alpha_2\alpha_5 + \alpha_0\alpha_2\alpha_6 + (29 \text{ terms})$$

$$\mathcal{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4 - \alpha_0\alpha_5)\,(\alpha_3\alpha_6 - \alpha_2\alpha_7) - s_{13}\,(\alpha_1\alpha_2 - \alpha_0\alpha_3)\,(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\frac{\partial \mathcal{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_0} = s_{12}\,\alpha_5(\alpha_3\alpha_6 - \alpha_2\alpha_7) + s_{13}\,\alpha_3(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\vdots$$

$$\frac{\partial \mathcal{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_7} = s_{12}\,\alpha_2(\alpha_1\alpha_4 - \alpha_0\alpha_5) + s_{13}\,\alpha_4(\alpha_1\alpha_2 - \alpha_0\alpha_3)$$

Can have a leading Landau singularity with *generic kinematics* (arbitrary s_{12}, s_{13}) when each factor of $\mathcal{F}$ vanishes!

Contour Deformation

For these candidates $\mathcal{F}(\boldsymbol{\alpha})$ and all $\partial\mathcal{F}(\boldsymbol{\alpha})/\partial\alpha_i$ vanish at the same point inside the integration domain → *pinch* singularity

Example

$$\begin{aligned}
\mathcal{F}(\boldsymbol{\alpha};\mathbf{s}) &= -s_{12}\ (\alpha_1\alpha_4-\alpha_0\alpha_5)\,(\alpha_3\alpha_6-\alpha_2\alpha_7)-s_{13}\ (\alpha_1\alpha_2-\alpha_0\alpha_3)\,(\alpha_5\alpha_6-\alpha_4\alpha_7),\\
\frac{\partial\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_0} &= s_{12}\ \alpha_5(\alpha_3\alpha_6-\alpha_2\alpha_7)+s_{13}\ \alpha_3(\alpha_5\alpha_6-\alpha_4\alpha_7),\\
&\vdots\\
\frac{\partial\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_7} &= s_{12}\ \alpha_2(\alpha_1\alpha_4-\alpha_0\alpha_5)+s_{13}\ \alpha_4(\alpha_1\alpha_2-\alpha_0\alpha_3)
\end{aligned}$$

vanish for

$$\alpha_2=\frac{\alpha_0\alpha_3}{\alpha_1},\qquad \alpha_4=\frac{\alpha_0\alpha_5}{\alpha_1},\qquad \alpha_6=\frac{\alpha_0\alpha_7}{\alpha_1}\,.$$

Computing this integral with contour deformation in parameter space fails!

Resolution

The problem is that we have monomials with different signs…

Asy2.1 PreResolve->True

```
MACTHXJONES:fiesta sj$ cat diagram2636.m
Get["asy2.1.m"];

Print["Diagram2636"];
result = AlphaRepExpand[{k1,k2,k3}, {k1^2, k2^2,k3^2, (k1+p1)^2, (k2+p2)^2, (k3+p3)^2,
(k1+k2+k3)^2, (k1+k2+k3+p1+p2+p3)^2}, {p1^2->0, p2^2->0, p3^2->0, p1*p2->s12/2, p1*p3->
s13/2, p2*p3->-s12/2-s13/2}, {s12 -> 1, s13 -> -1/5}, PreResolve->True] (* 3-loop box *
)
Print[result];
Print["=========="];
Exit[];

MACTHXJONES:fiesta sj$ wolframscript -file diagram2636.m
Diagram2636
Asy2.1
Variables for UF: {k1, k2, k3, p1, p2, p3}
WARNING: preresolution failed
{}
==========
MACTHXJONES:fiesta sj$
```

Correctly identifies that iterated linear changes of variables are not sufficient to resolve the singularity and reports that pre-resolution has failed

Resolution

1) Rescale parameters to *linearise* singular surfaces

$$\mathcal{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4-\alpha_0\alpha_5)\,(\alpha_3\alpha_6-\alpha_2\alpha_7) - s_{13}\,(\alpha_1\alpha_2-\alpha_0\alpha_3)\,(\alpha_5\alpha_6-\alpha_4\alpha_7)$$

$$\alpha_0 \to \alpha_0\alpha_1,\ \alpha_2 \to \alpha_2\alpha_3,\ \alpha_4 \to \alpha_4\alpha_5,\ \alpha_6 \to \alpha_6\alpha_7$$

$$\mathcal{F}(\boldsymbol{\alpha};\mathbf{s}) = \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_4-\alpha_0)(\alpha_6-\alpha_2) - s_{13}(\alpha_2-\alpha_0)(\alpha_6-\alpha_4)\right]$$

2) Split the integral by imposing $\alpha_i \geq \alpha_j \geq \alpha_k \geq \alpha_l$

$$\begin{aligned}\alpha_0 &\to \alpha_0+\alpha_2+\alpha_4+\alpha_6,\\ \alpha_2 &\to \alpha_2+\alpha_4+\alpha_6,\\ \alpha_4 &\to \alpha_4+\alpha_6,\\ \alpha_6 &\to \alpha_6\end{aligned} \qquad +\text{perms}$$

$$\begin{aligned}\mathcal{F}_1(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_0+\alpha_2)(\alpha_2+\alpha_4) - s_{13}(\alpha_0)(\alpha_4)\right]\\ \mathcal{F}_2(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2)(\alpha_0+\alpha_2+\alpha_6) + s_{13}(\alpha_0)(\alpha_6)\right]\\ &\vdots\\ \mathcal{F}_{24}(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2+\alpha_4)(\alpha_4+\alpha_6) - s_{13}(\alpha_2)(\alpha_6)\right]\end{aligned}$$

All coefficients of s_{12}, s_{13} now have definite sign

Result

Can now obtain results numerically ($s_{12} = 1,\ s_{13} = -1/5$) with $C_\epsilon = \Gamma(2+3\epsilon)$

$$\mathcal{I}_1/C_\epsilon = \epsilon^{-5}\left[0.55555538\mathbf{27}\right] + \epsilon^{-4}\left[-3.884290\mathbf{14} + 5.235983\mathbf{13}\, i\right] + \mathcal{O}(\epsilon^{-3}),$$
$$\mathcal{I}_2/C_\epsilon = \epsilon^{-5}\left[2.222232\mathbf{11}\right] + \epsilon^{-4}\left[-7.92923\mathbf{11} + 20.94388\mathbf{18}\, i\right] + \mathcal{O}(\epsilon^{-3}),$$
$$\mathcal{I}_3/C_\epsilon = \epsilon^{-5}\left[-2.77778888\mathbf{83}\right] + \epsilon^{-4}\left[18.519682\mathbf{69} - 15.708041\mathbf{67}\, i\right] + \mathcal{O}(\epsilon^{-3}),$$
$$\mathcal{I}_4/C_\epsilon = \epsilon^{-5}\left[2.2222211\mathbf{19}\right] + \epsilon^{-4}\left[-13.294002\mathbf{23}\right] + \mathcal{O}(\epsilon^{-3}),$$
$$\mathcal{I}_5/C_\epsilon = \epsilon^{-5}\left[-2.7777713\mathbf{46}\right] + \epsilon^{-4}\left[12.74345\mathbf{17} - 23.56186\mathbf{15}\, i\right] + \mathcal{O}(\epsilon^{-3}),$$
$$\mathcal{I}_6/C_\epsilon = \epsilon^{-5}\left[0.55555546\mathbf{19}\right] + \epsilon^{-4}\left[-4.0702347\mathbf{61}\right] + \mathcal{O}(\epsilon^{-3}),$$

Agrees with analytic result

$$\begin{aligned} I &= 4\ (I_1 + I_2 + I_3 + I_4 + I_5 + I_6) \\ &= \epsilon^{-4}\left[8.340\mathbf{55} - 52.36\mathbf{08}j\right] + \mathcal{O}(\epsilon^{-3}) \\ I_{\text{analytic}} &= \epsilon^{-4}\left[8.3400403922 - 52.3598775598j\right] + \mathcal{O}(\epsilon^{-3}) \end{aligned}$$

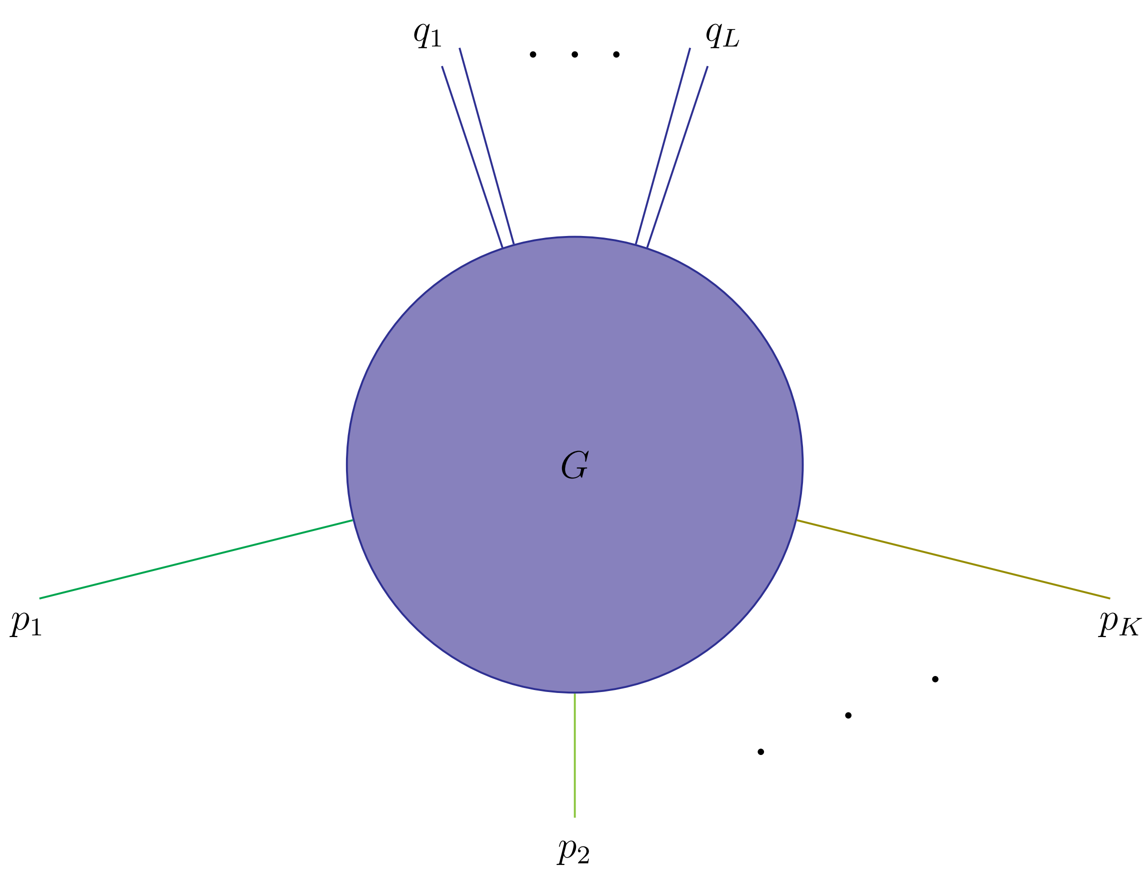
But: still slow to compute numerically, possible to vastly improve performance by avoiding contour deformation entirely (we will return to this point shortly)

MoR and Hidden Regions

On-Shell Expansion

On-shell expansion provides a way to explore emergence of IR singularities starting from an object free of IR singularities (off-shell Green's function)

Consider an arbitrary loop, $(K+L)$-leg wide-angle scattering graph



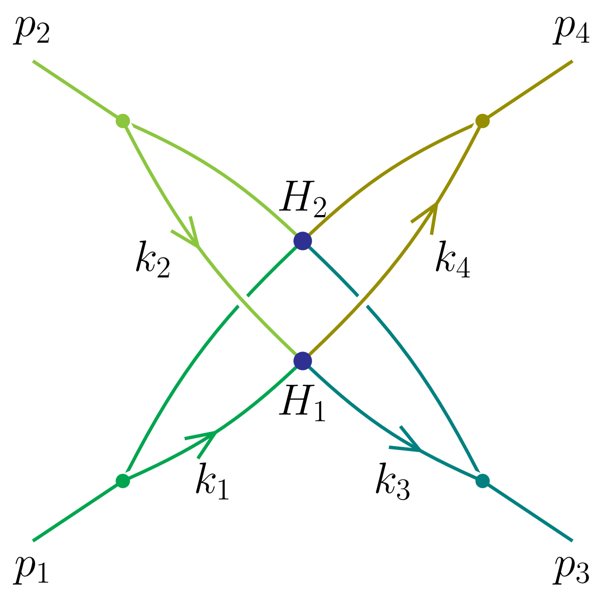
on-shell: $p_i^2 \sim \lambda Q^2 \quad (i = 1, \ldots, K),$

off-shell: $q_j^2 \sim Q^2 \quad (j = 1, \ldots, L),$

wide-angle: $p_k \cdot p_l \sim Q^2 \quad (k \neq l)\,.$

Cancellations of the type just observed lead to new regions that are *hidden* in the Newton polytope approach as they do not originate from an end-point singularity

On-Shell Expansion



Consider a collinear/jet configuration

$$p_i^2 = \lambda Q^2, \quad p_i \cdot v_i \sim \lambda Q, \quad p_i \cdot \bar{v}_i \sim Q, \quad p_i \cdot v_{i\perp} \sim \sqrt{\lambda} Q$$

Let us introduce a fourth (extra) loop momentum and consider the mode with all k_i collinear to p_i

$$k_i^\mu = Q\left(\xi_i v_i^\mu + \lambda \kappa_i \bar{v}_i^\mu + \sqrt{\lambda}\tau_i u_i^\mu + \sqrt{\lambda}\nu_i n^\mu\right)$$

Botts, Sterman 89

Momentum conservation at H_1 vertex ($k_1 + k_2 = k_3 + k_4$) implies not all ξ_i are independent:

$$\xi_2 = \xi_1 - \frac{1}{2}\sqrt{\lambda}\cos^2(\theta)\left(\tan\left(\frac{\theta}{2}\right)\Delta\tau - \cot\left(\frac{\theta}{2}\right)\Sigma\tau\right) + \lambda(\kappa_2 - \kappa_1),$$

$$\xi_3 = \xi_1 + \frac{1}{2}\sqrt{\lambda}\tan\left(\frac{\theta}{2}\right)\Delta\tau + \lambda(\kappa_2 - \kappa_4),$$

$$\xi_4 = \xi_1 - \frac{1}{2}\sqrt{\lambda}\cot\left(\frac{\theta}{2}\right)\Sigma\tau + \lambda(\kappa_2 - \kappa_3)\,.$$

$$\Delta\tau \equiv \tau_1 + \tau_2 - \tau_3 - \tau_4$$
$$\Sigma\tau = \tau_1 + \tau_2 + \tau_3 + \tau_4$$

On-Shell Expansion

Now let us analyse the leading behaviour of this integrand for small λ,

1) Loop measure can be expressed as $\int d^D k_1 d^D k_2 d^D k_3 = Q^{3D} \int \prod_{i=1}^{3} d\xi_i d\kappa_i d\tau_i d\nu_i$
2) Trade large components of k_2, k_3 for small components of k_4, $\{\xi_2, \xi_3\} \to \{\kappa_4, \tau_4\}$

   Jacobian of transformation: $\det\left(\frac{\partial(\xi_2, \xi_3)}{\partial(\kappa_4, \tau_4)}\right) = \lambda^{3/2} \cos(\theta)\cot(\theta)$

Overall obtain the following scaling:

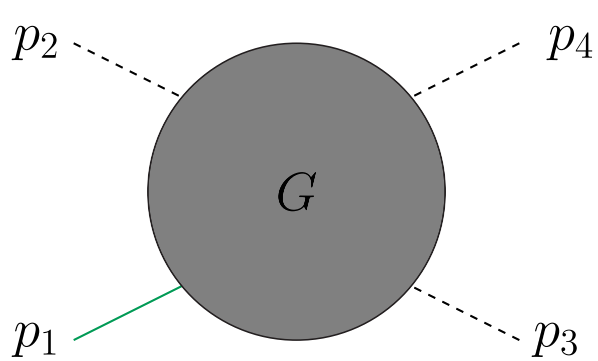
$$\int \prod_{i=1}^{3} d\xi_i \, d\kappa_i d\tau_i d\nu_i \sim \int_0^1 d\xi_1 \underbrace{\left(\int \prod_{i=1}^{3} (\lambda d\kappa_i)(\lambda^{\frac{1}{2}} d\tau_i)(\lambda^{\frac{1}{2}} d\nu_i)^{1-2\epsilon}\right)}_{\lambda^{6-3\epsilon}} \int d\kappa_4 d\tau_4 \underbrace{\det\left(\frac{\partial(\xi_2, \xi_3)}{\partial(\kappa_4, \tau_4)}\right)}_{\lambda^{3/2}}$$

Expect this region to scale as $\boxed{\mu = 6 - 3\epsilon + \frac{3}{2} - 8 = -\frac{1}{2} - 3\epsilon}$

Scaling of collinear propagators

On-Shell Expansion

Directly applying MoR in parameter space, we do not see this region…



$I \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	**order**
$(-2,-1,-2,-1,-2,-1,-2,-1;1)$	-6ϵ
$(-1,-2,-1,-2,-1,-2,-1,-2;1)$	-6ϵ
$(-1,-1,-1,0,-1,0,-1,0;1)$	$1-3\epsilon$
$(-1,-1,0,-1,0,-1,0,-1;1)$	$1-3\epsilon$
$(-1,-1,0,0,0,0,0,0;1)$	$-\epsilon$
$(0,0,0,0,0,0,0,0;1)$	0

Dissecting the polytope according to our resolution procedure eliminates monomials of different sign, we now see the region in each of the 24 new polytopes

$I_1 \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(y_0, x_1, y_2, x_3, y_4, x_5, y_6, x_7)$	$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	**order**
$(1/2,-1,1/2,-1,1/2,-1,0,-1;1)$	$(-2,-2,-2,-2,-2,-2,-2,-2;2)$	$-1/2-3\epsilon$
$(0,-1,1,-1,1,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	-3ϵ
$(1,-1,1,-1,0,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	-3ϵ
$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$(-2,-1,-2,-1,-2,-1,-2,-1;1)$	-6ϵ
$(1,-2,1,-2,1,-2,1,-2;1)$	$(-1,-2,-1,-2,-1,-2,-1,-2;1)$	-6ϵ
$(0,-1,0,0,0,0,0,0;1)$	$(-1,-1,0,0,0,0,0,0;1)$	$-\epsilon$
$(0,0,0,0,0,0,0,0;1)$	$(0,0,0,0,0,0,0,0;1)$	0

← $\mu = -\frac{1}{2} - 3\epsilon$

A similar analysis for forward scattering reveals hidden regions with Glauber modes

→Talk of Thomas

Avoiding Contour Deformation in the Minkowski Regime

Based on:
SJ, Olsson, Stone [LL24 Proceedings & WIP]

Minkowski Regime

Several conflicting definitions of the term *Minkowski regime* for Feynman Integrals

In the remainder of this talk I will use the following conventions:

(Pseudo-)Euclidean

$\mathscr{F}(\boldsymbol{\alpha}) \geq 0$ for $\boldsymbol{\alpha} \in \mathbb{R}^N_{\geq 0}$ and vanishes only on the boundary

Minkowski

Not Euclidean/Pseudo-Euclidean

We can have $\mathscr{F}(\boldsymbol{\alpha}) < 0$ for some values of $\boldsymbol{\alpha} \in \mathbb{R}^N_{\geq 0}$

Contour Deformation

Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07, 08; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14; Winterhalder, Magerya, Villa, SJ, Kerner, Butter, Heinrich, Plehn 22; …

Downsides of contour deformation:

1. Real valued integrand → complex valued integrand (slower numerics)
2. Large and complicated Jacobian from $\boldsymbol{\alpha} \to \mathbf{z}$ (can be optimised)
   Borinsky, Munch, Tellander 23
3. Increases variance of function (integrand can be both > 0 and < 0)
4. Sensitive to choice of contour
5. Sometimes fails analytically and/or numerically

Summary: it is **slow, arbitrary** and can **fail**

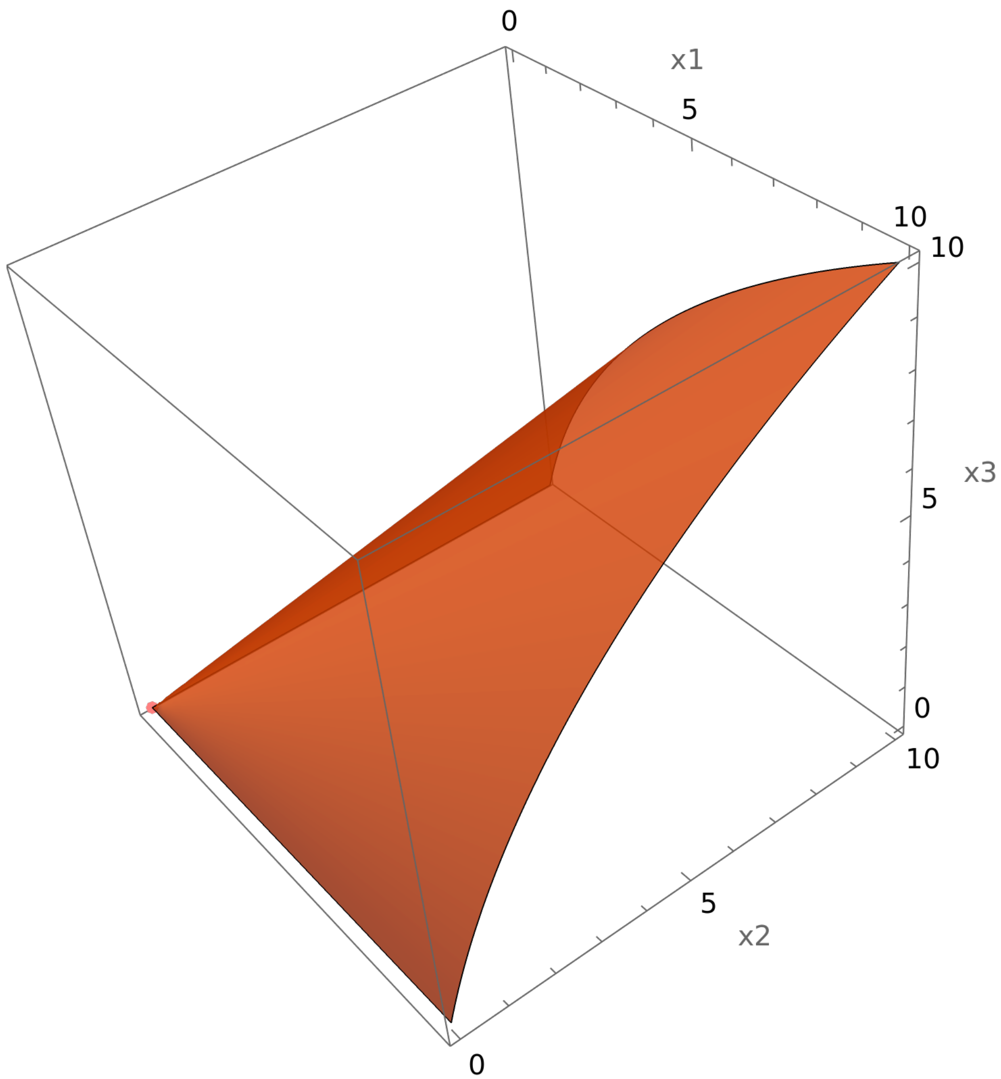
Can we find a way to avoid contour deformation? **Yes**

Always? **I don't know**

NoCD: Avoiding Contour Deformation

Idea:

1. Construct transformations of the Feynman parameters which map the zeroes of the $\mathscr{F}$-polynomial to the boundary of integration



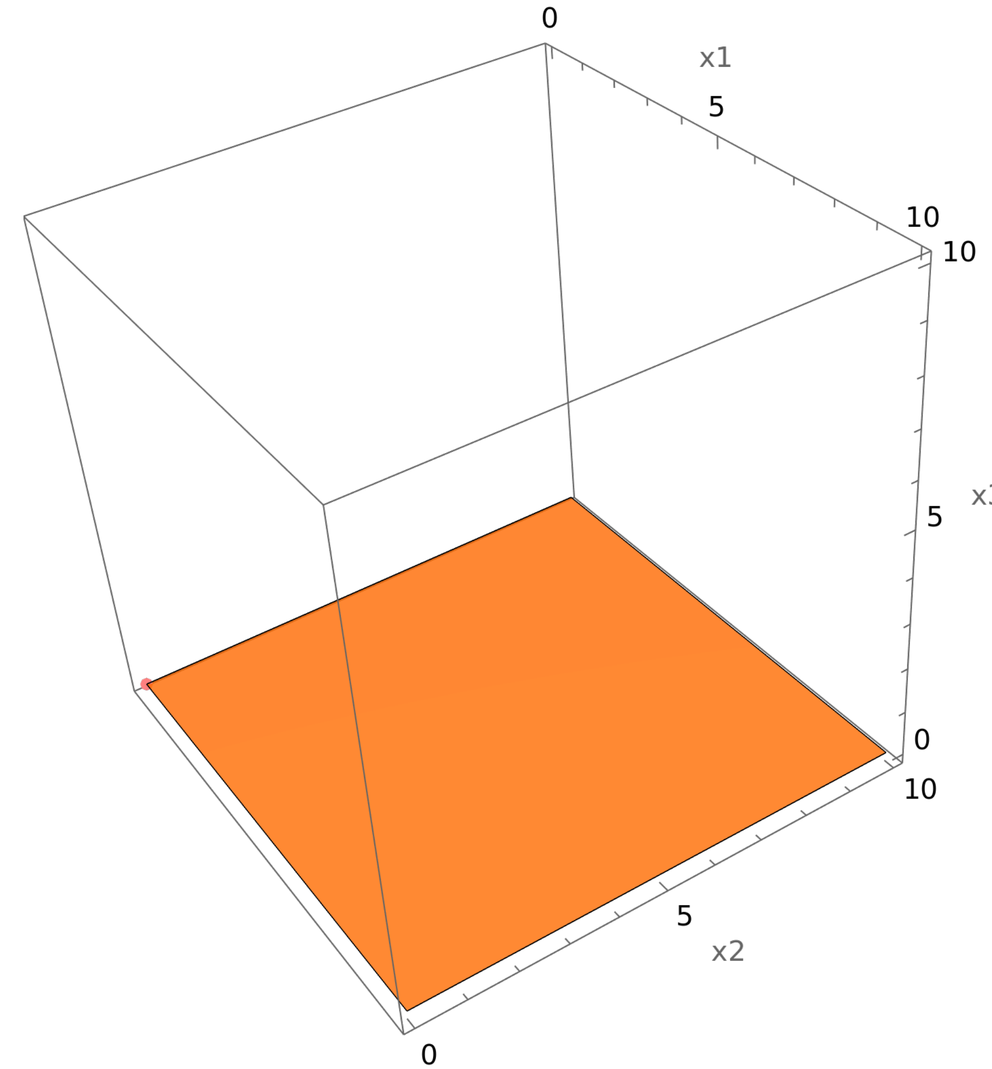


Figure: Thomas Stone

NoCD: Avoiding Contour Deformation

Idea:

2. For transformations which make $\mathscr{F}$ non-positive extract an overall minus sign (using the $i\delta$ prescription to generate the physically correct imaginary part)

3. Stitch together the resulting integrals

$$I = \sum_{n_+=1}^{N_+} I^+_{n_+} + (-1 - i\delta)^{-(\nu - LD/2)} \sum_{n_-=1}^{N_-} I^-_{n_-}$$

The individual integrals $\{I^+_{n_+}, I^-_{n_-}\}$ have *manifestly* non-negative integrands

$\implies$ no contour deformation, trivial analytic continuation, faster to integrate

NoCD: Avoiding Contour Deformation

Rules of the Game:

1. Transformations must not spoil the δ-func. constraint

   Cheng-Wu Theorem:

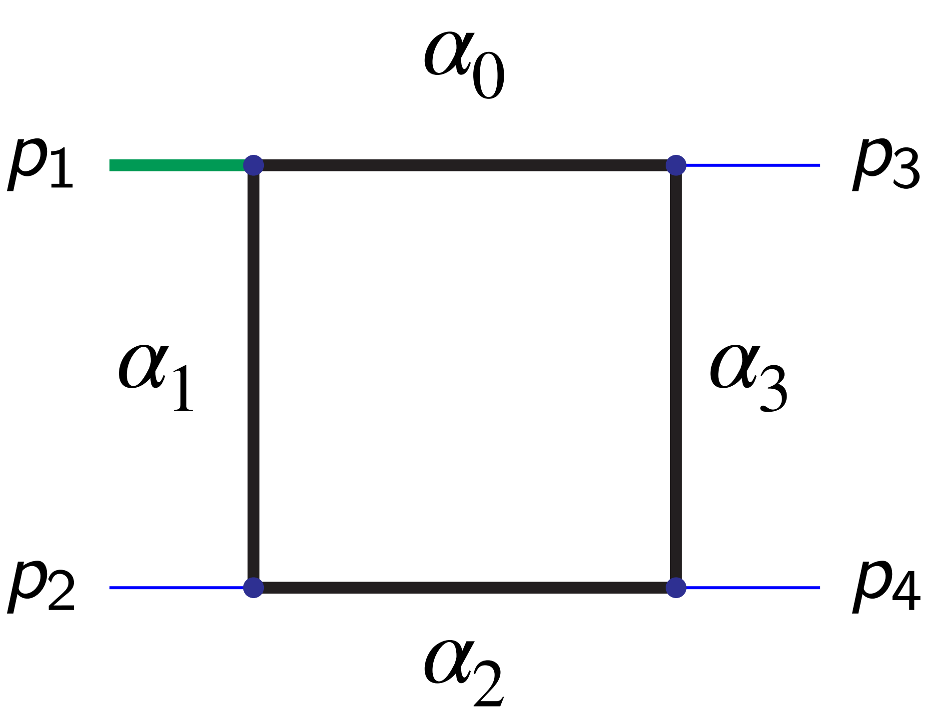
$$\forall S \subseteq \{1,\ldots,N\} \wedge S \neq \emptyset : \qquad \delta\left(1 - \sum_{j=1}^{N} \alpha_j\right) \to \delta\left(1 - \sum_{j\in S} \alpha_j\right)$$

2. Transformations must preserve the sign of $\mathscr{U} \geq 0$
3. Jacobian $\mathscr{J}$ of the transformation must have a definite sign

We found the following rational transformations useful:

1. Rescaling: $\alpha_j \to c\alpha_j$ with $c > 0$
2. Blow-up: $\alpha_j \to \alpha_i\alpha_j$
3. Decomposition: $\alpha_j \to \alpha_i + \alpha_j$ ← $1 = \theta(\alpha_a - \alpha_b) + \theta(\alpha_b - \alpha_a)$

Massless Example @ 1-loop



$$\mathcal{U} = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$$
$$\mathcal{F} = -s\alpha_0\alpha_2 - t\alpha_1\alpha_3 - p_1^2\alpha_0\alpha_1$$

Consider the regime: $s > 0,\ p_1^2 > 0\ \&\ t < 0$

Can have zeros of $\mathcal{F}$ within the integration volume for $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \in \mathbb{R}^4_{>0}$

Massless Example @ 1-loop

$$\mathcal{F} = -s\alpha_0\alpha_2 + |t|\alpha_1\alpha_3 - p_1^2\alpha_0\alpha_1$$

$\alpha_0 \to \frac{\alpha_0\alpha_1}{s},\ \alpha_3 \to \frac{\alpha_2\alpha_3}{|t|}$

$$\mathcal{F} \to \alpha_1\left(\alpha_2\left(\alpha_3 - \alpha_0\right) - \frac{p_1^2}{s}\alpha_0\alpha_1\right)$$

$\alpha_0 > \alpha_3 :\ \ \alpha_0 \to \alpha_0 + \alpha_3$

$$\boxed{\mathcal{F} \to -\frac{1}{s}\left(\alpha_1\left(s\alpha_0\alpha_2 + p_1^2\alpha_1\left(\alpha_0 + \alpha_3\right)\right)\right) =: -\mathcal{F}_1^-}$$

$\alpha_3 > \alpha_0 :\ \ \alpha_3 \to \alpha_3 + \alpha_0$

$$\mathcal{F} \to \alpha_1\left(-\frac{p_1^2}{s}\alpha_0\alpha_1 + \alpha_2\alpha_3\right)$$

$\alpha_2 \to \frac{p_1^2\alpha_0\alpha_2}{s},\ \alpha_1 \to \alpha_1\alpha_3$

$$\mathcal{F} \to \frac{p_1^2}{s}\alpha_0\alpha_1\alpha_3^2\left(\alpha_2 - \alpha_1\right)$$

$\alpha_2 > \alpha_1 :\ \ \alpha_2 \to \alpha_2 + \alpha_1$

$$\boxed{\mathcal{F} \to \frac{p_1^2}{s}\alpha_0\alpha_1\alpha_2\alpha_3^2 =: \mathcal{F}_1^+}$$

$\alpha_1 > \alpha_2 :\ \ \alpha_1 \to \alpha_1 + \alpha_2$

$$\boxed{\mathcal{F} \to -\frac{p_1^2}{s}\alpha_0\alpha_1\left(\alpha_1 + \alpha_2\right)\alpha_3^2 =: -\mathcal{F}_2^-}$$

Massless Example @ 1-loop

Generate $\mathscr{U}_1^+, \mathscr{U}_1^-, \mathscr{U}_2^-$ by applying the same transformations to $\mathscr{U}$
Compute the Jacobian determinants of the transformations $\mathscr{J}_1^+, \mathscr{J}_1^-, \mathscr{J}_2^-$

Each new integral is of the form:

$$I_{n_\pm}^\pm \sim \mathscr{J}_{n_\pm}^\pm \left(\mathscr{U}_{n_\pm}^\pm\right)^{2\varepsilon} \left(\mathscr{F}_{n_\pm}^\pm\right)^{-2-\varepsilon}$$

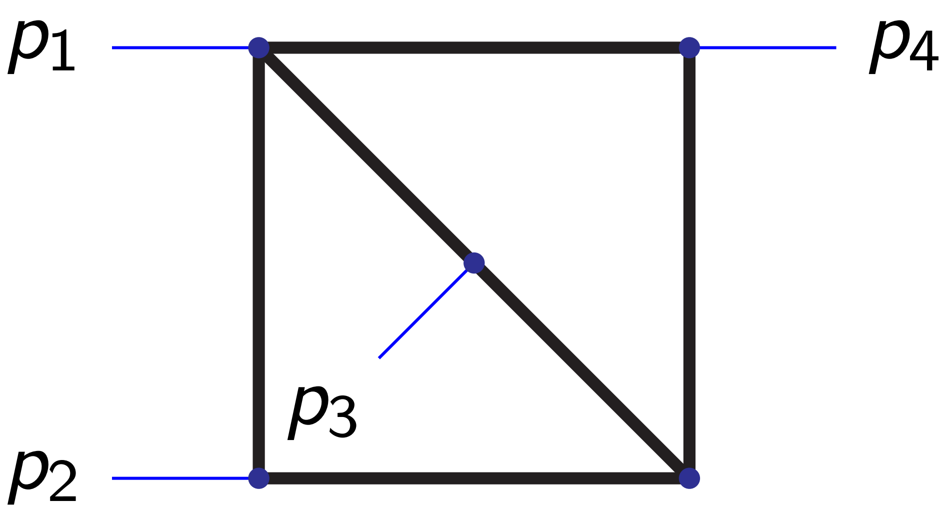
with *manifestly non-negative integrand*

We have converted the initial integral into sum of 3 integrals:

$$\boxed{I = I_1^+ + (-1 - i\delta)^{-2-\varepsilon}\left(I_1^- + I_2^-\right)}$$

Verified result numerically against known analytic result

Massless Example @ 2-loops



$$\mathscr{U} = \alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_0\alpha_3 + \alpha_0\alpha_4 + \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_5 + \alpha_2\alpha_4 + \alpha_2\alpha_5 + \alpha_3\alpha_4 + \alpha_3\alpha_5 + \alpha_4\alpha_5$$

$$\mathscr{F} = -s\alpha_1\alpha_2\alpha_5 - t\alpha_0\alpha_1\alpha_3 - u\alpha_0\alpha_2\alpha_4$$

Momentum conservation implies: $s + t + u = 0 \implies u = -(s+t)$

Hence $\mathscr{F}$ can be 0 within $\{\alpha_i\} \in \mathbb{R}^6_{>0}$ even with $s > 0,\ t > 0$

Not possible to define a Euclidean region at all!

Nevertheless, the method works

Massless Example @ 2-loops

We considered the cases:

1. $s > -t$
2. $s < -t$

We obtain *different* resolutions for each case

Nevertheless, in each case we find we need 6 integrals to cover the space:

$$I = \left(I_1^+ + I_2^+ + I_3^+\right) + (-1 - i\delta)^{-2-2\varepsilon}\left(I_1^- + I_2^- + I_3^-\right)$$

Verified result numerically against known analytic result

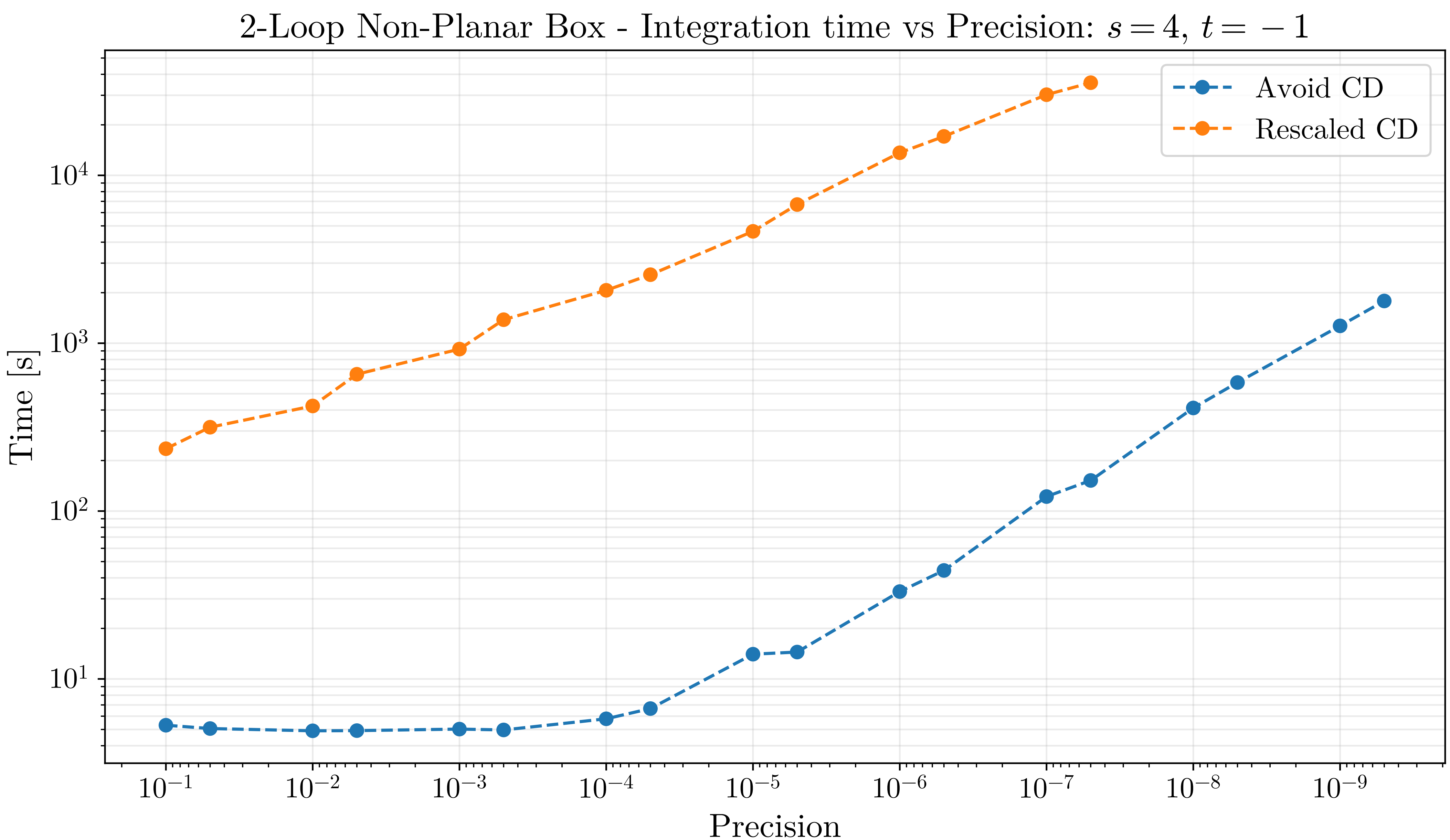
Tausk 99

Let's take a look at the time taken to numerically integrate this example…

Massless Example @ 2-loops

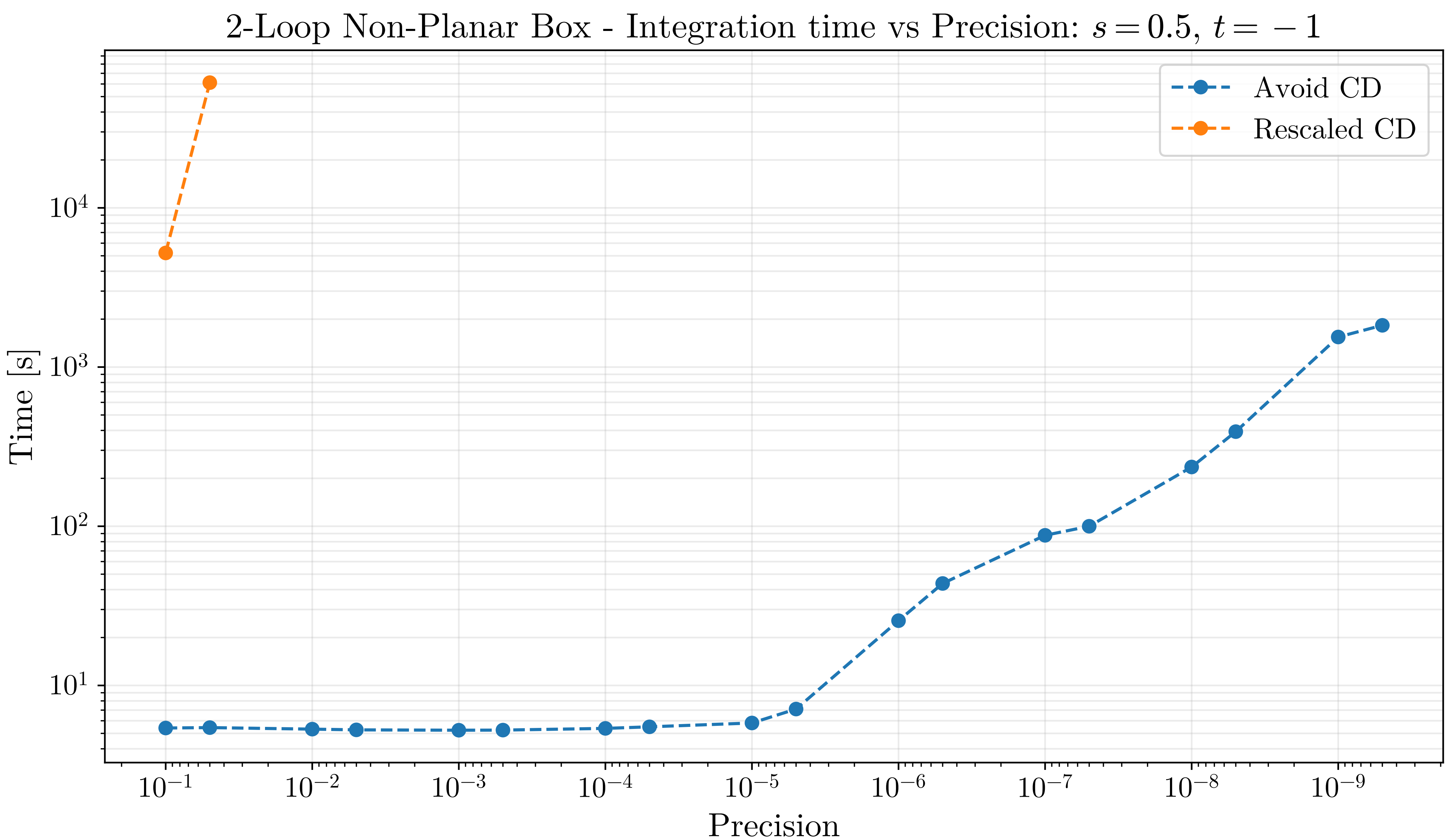
Evaluating up-to-and-including finite order with pySecDec

Heinrich, SPJ,
Kerner, Magerya,
Olsson, Schlenk 23



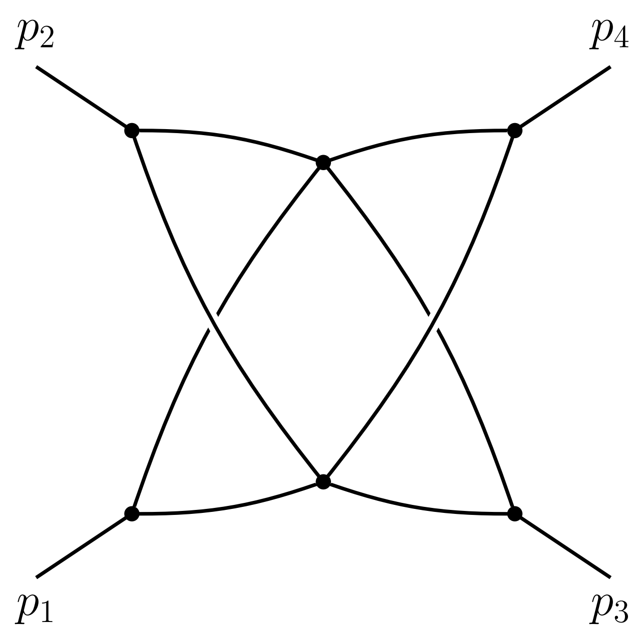
Massless Example @ 2-loops

Evaluating up-to-and-including finite order with pySecDec



Massless Example @ 3-loops

Returning to our 3-loop friend



$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s\ (\alpha_1\alpha_4 - \alpha_0\alpha_5)(\alpha_3\alpha_6 - \alpha_2\alpha_7) \\ -t\ (\alpha_1\alpha_2 - \alpha_0\alpha_3)(\alpha_5\alpha_6 - \alpha_4\alpha_7)$$

linearise

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = \alpha_1\alpha_3\alpha_5\alpha_7\big[-s(\alpha_4 - \alpha_0)(\alpha_6 - \alpha_2) \\ -t(\alpha_2 - \alpha_0)(\alpha_6 - \alpha_4)\big]$$

dissect

$\mathscr{F}_1, \ldots, \mathscr{F}_6$ + 18 integrals related by relabelling

Massless Example @ 3-loops

For $s > -t > 0$, two of the 6 independent integrals require contour deformation:

$$\mathcal{F}_3 = \alpha_1\alpha_3\alpha_5\alpha_7 \left[-s\alpha_0\alpha_2 + |t| \left(\alpha_0 + \alpha_4\right) \left(\alpha_2 + \alpha_4\right)\right]$$

$$\mathcal{F}_5 = \alpha_1\alpha_3\alpha_5\alpha_7 \left[s\alpha_6 \left(\alpha_0 + \alpha_2 + \alpha_6\right) - |t| \left(\alpha_0 + \alpha_6\right) \left(\alpha_2 + \alpha_6\right)\right]$$

Can express each of these in terms of 4 manifestly non-negative integrands

$$I = \sum_{n_+=1}^{8} I^+_{n_+} + (-1 - i\delta)^{-2-3\varepsilon} \sum_{n_-=1}^{4} I^-_{n_-}$$

pySecDec (~min integration) agrees with known analytic result

$$I(s_{12} = 1, s_{13} = -1/5) = \epsilon^{-4} \left[8.340\mathbf{55} - 52.36\mathbf{08}i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

$$I^{\mathrm{NoCD}}(s_{12} = 1, s_{13} = -1/5) = \epsilon^{-4} \left[8.3400403920\mathbf{28} - 52.3598775598\mathbf{3}\mathbf{47}i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

$$I^{\mathrm{analytic}}(s_{12} = 1, s_{13} = -1/5) = \epsilon^{-4} \left[8.34004039223768 - 52.35987755984493i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

Massive Integrals

Can this work also for massive integrals?

$$\mathscr{F}(\boldsymbol{\alpha};\boldsymbol{s}) = \mathscr{F}_0(\boldsymbol{\alpha};\boldsymbol{s}) + \mathscr{U}_0(\boldsymbol{\alpha}) \sum_{j=1}^{N} m_j^2 \alpha_j$$

Now α_j appears quadratically in $\mathscr{F}$
Transformations harder to find, even for trivial integrals

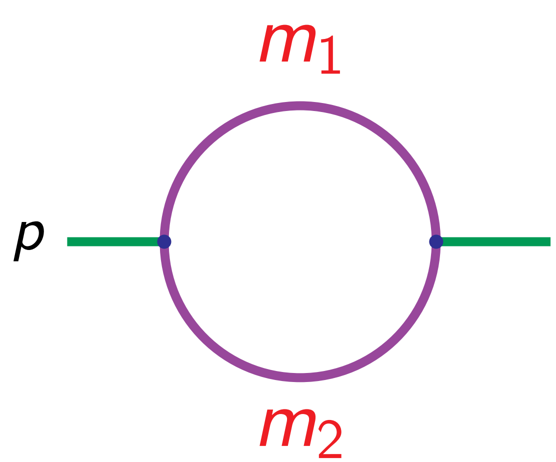
Ideas:

1. Can geometry guide us in the right direction?
2. Is this just singularity resolution? If so, how can we use existing technology?

Hironaka e.g. desing

Massive Example

Let's consider the simplest possible case

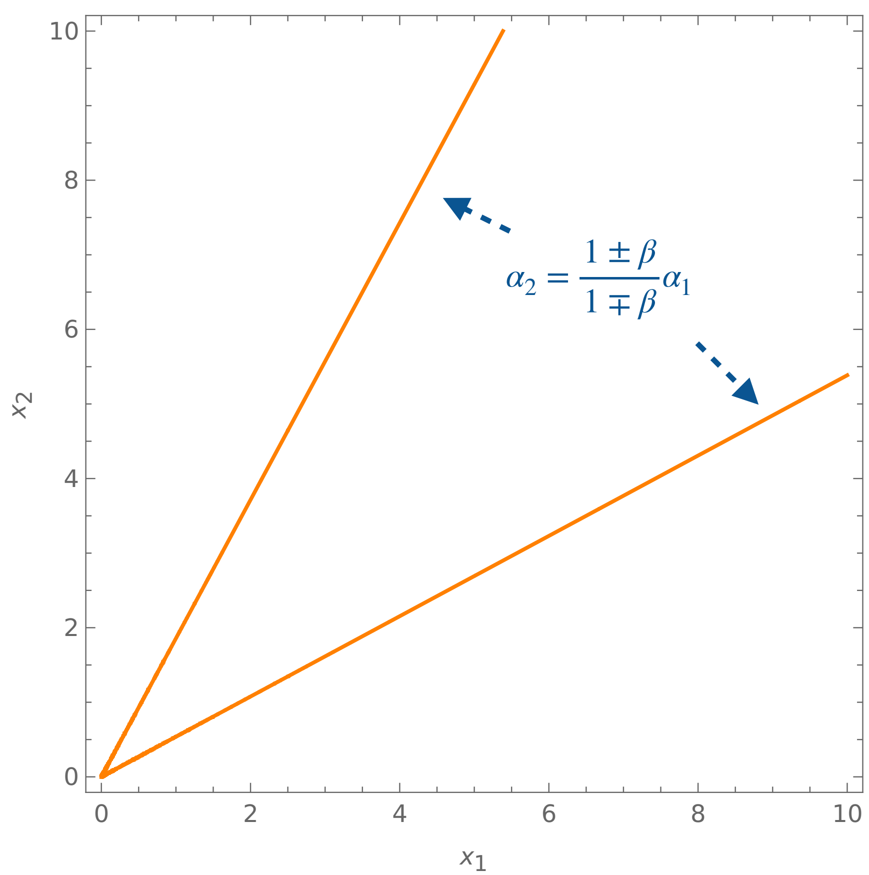


$$\mathcal{F} = -p^2\alpha_1\alpha_2 + (\alpha_1+\alpha_2)(m_1^2\alpha_1 + m_2^2\alpha_2)$$

Scale out d $\alpha_i \to \alpha_i/m_i$ and rewrite as

$$\tilde{\mathcal{F}} = \alpha_1^2 + \alpha_2^2 - 2\frac{1+\beta^2}{1-\beta^2}\alpha_1\alpha_2$$

$$\beta^2 \equiv \frac{p^2-(m_1+m_2)^2}{p^2-(m_1-m_2)^2} \in [0,1)$$

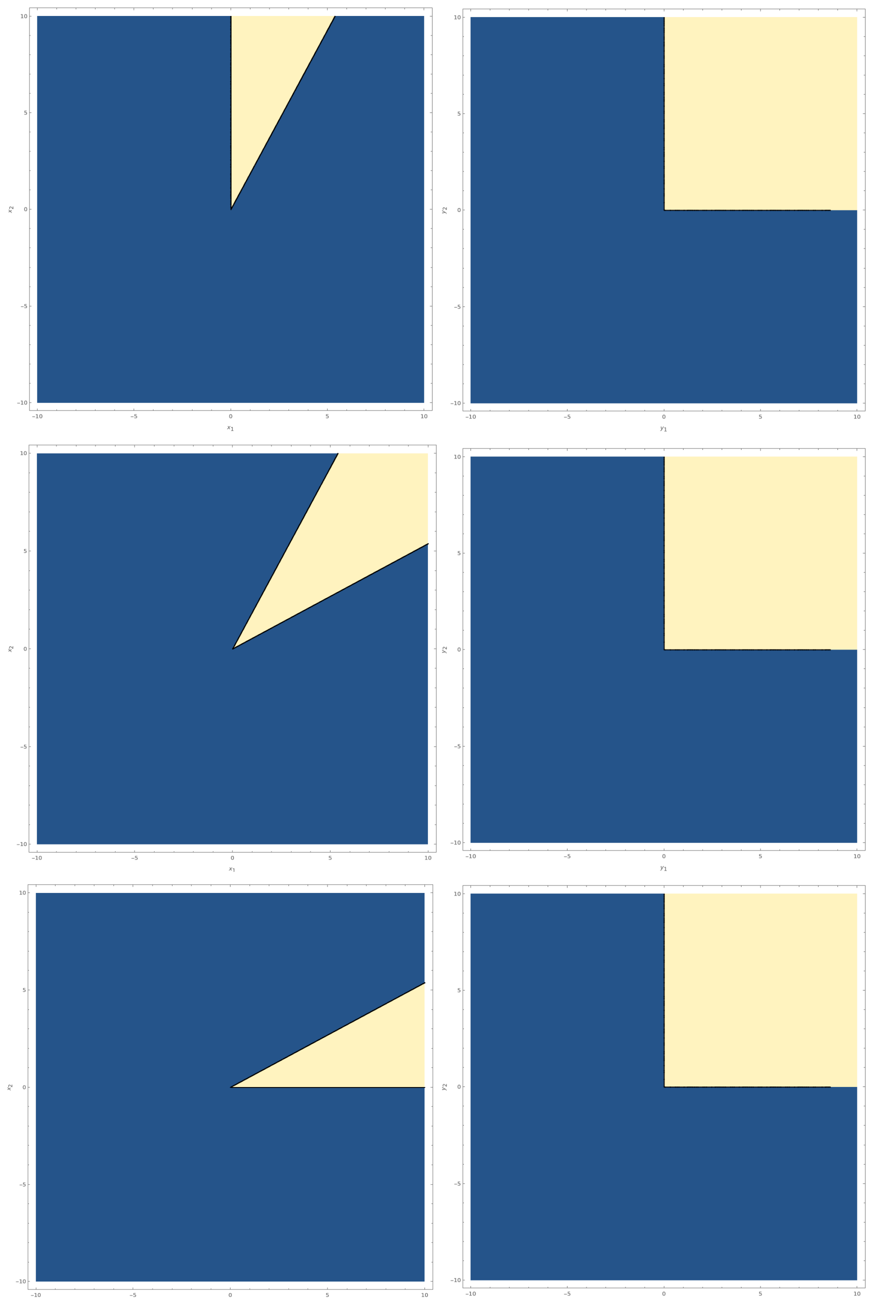


Studying the variety of $\tilde{\mathcal{F}}$ suggests that we will obtain 2 positive and 1 negative integrand

$$I = \sum_{n_+=1}^{2} I_{n_+}^{+} + (-1-i\delta)^{-\varepsilon} I_1^{-}$$

We can now construct transformations to send the variety of $\tilde{\mathcal{F}}$ to the integration boundary

Massive Example



$$\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + \frac{4\beta}{1-\beta^2} y_1 \right)$$

$$\widetilde{\mathcal{F}}_1^- = \frac{4\beta}{1-\beta^2} y_1 y_2$$

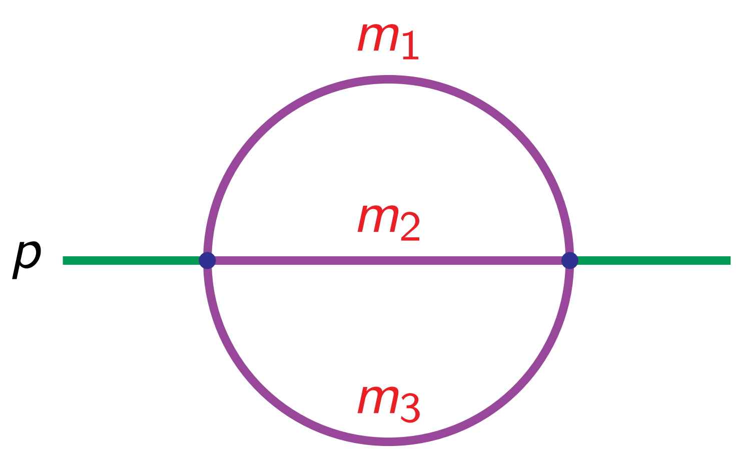
$$\widetilde{\mathcal{F}}_2^+ = \frac{y_1 \left(4\beta y_2 + (1+\beta)^2 y_1 \right)}{1-\beta^2}$$

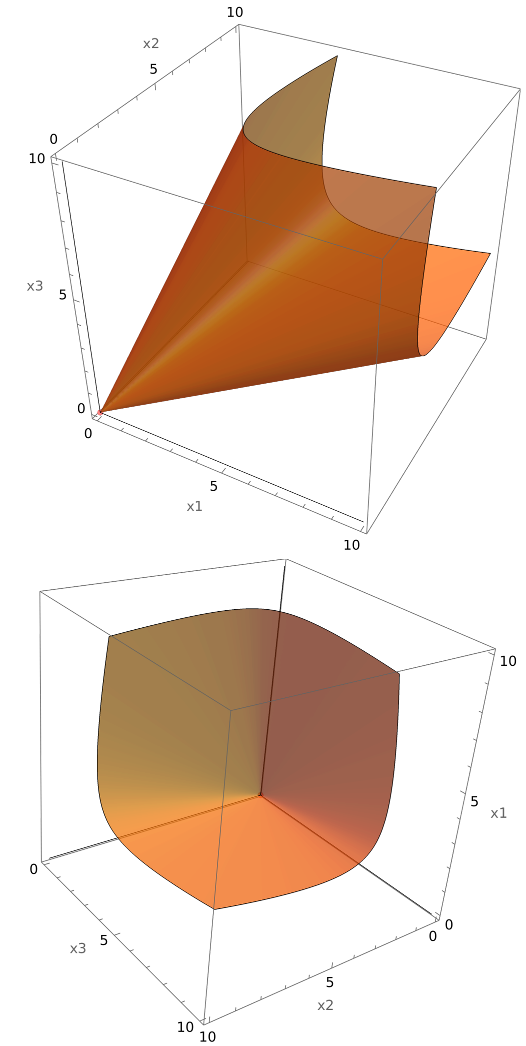
Verified result numerically **&** analytically ✓

Slide: Thomas Stone (Loops & Legs 2024)

Further Massive Examples

This works also for massive 1-loop triangles and boxes, but, it is less clear how to proceed in more involved cases





Rational transformations are not generally enough (Thanks E. Panzer)

However, algebraic transformations do not necessarily present a problem, we are currently investigating this direction

Conclusion

Pinched Feynman Integrals

Studied integrals with *pinched* contours independent of kinematics

Found a resolution procedure to remove the pinch, allowing us to obtain stable numerical results

MoR

Demonstrated that new regions can appear due to cancelling monomials either generically or at particular kinematic points

NoCD

Currently investigating a related method for evaluating integrals in the Minkowski regime without contour deformation

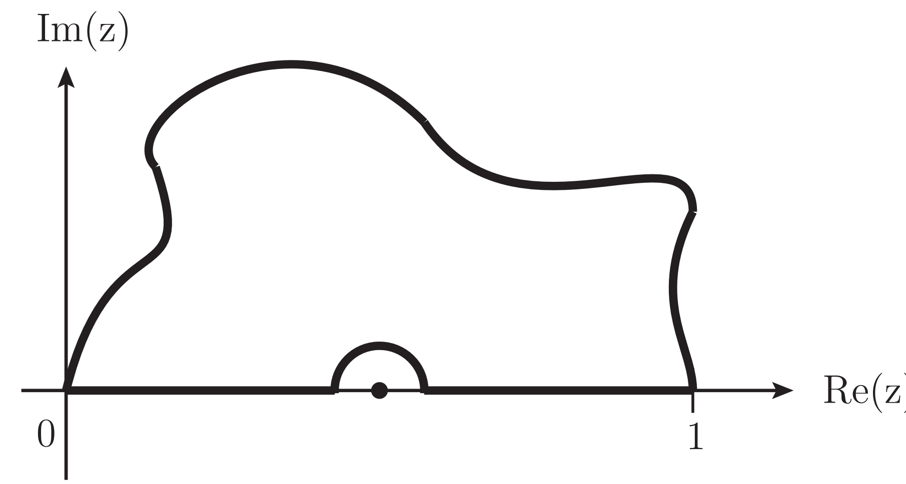
Much still to learn about the geometry of Feynman integrals and their singularity structure…

Backup

Contour Deformation

Feynman integral (after integrating δ-func.):

$$I \sim \int_0^1 [\mathrm{d}\boldsymbol{\alpha}]\, \boldsymbol{\alpha}^{\nu} \frac{[\mathscr{U}(\boldsymbol{\alpha})]^{N-(L+1)D/2}}{[\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})]^{N-LD/2}}$$



Deform our integration contour to avoid poles on real axis

Feynman prescription $\mathscr{F} \to \mathscr{F} - i\delta$ tells us how to do this

Expand $\mathscr{F}(z = \boldsymbol{\alpha} - i\boldsymbol{\tau})$ around $\boldsymbol{\alpha}$, $\quad \mathscr{F}(z) = \mathscr{F}(\boldsymbol{\alpha}) - i \sum_j \tau_j \frac{\partial \mathscr{F}(\boldsymbol{\alpha})}{\partial \alpha_j} + \mathscr{O}(\tau^2)$

Choose $\tau_j = \lambda_j \alpha_j (1 - \alpha_j) \frac{\partial \mathscr{F}(\boldsymbol{\alpha})}{\partial \alpha_j}$ with small constants $\lambda_j > 0$

Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07, 08; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14;…

Can also generalise $\lambda_j \to \lambda_j(\boldsymbol{\alpha})$ and train the deformation with a Neural Network

Winterhalder, Magerya, Villa, SJ, Kerner, Butter, Heinrich, Plehn 22

Additional Regulators/ Rapidity Divergences

MoR subdivides $\mathcal{N}(I) \to \{\mathcal{N}(I^R)\} \implies$ new (internal) facets $F^{\text{int.}}$

New facets can introduce spurious singularities not regulated by dim reg

Lee Pomeransky Representation:

$$\mathcal{N}(I^{(R)}) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_{\mathbb{R}^N_{\geq 0}} \left[\mathrm{d}\mathbf{y}_f \right] \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle + \frac{D}{2} a_f} \left(c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f} \right)^{-\frac{D}{2}}$$

If $f \in F^{\text{int}}$ have $a_f = 0$ need analytic regulators $\boldsymbol{\nu} \to \boldsymbol{\nu} + \boldsymbol{\delta\nu}$

Heinrich, Jahn, SJ, Kerner, Langer, Magerya, Põldaru, Schlenk, Villa 21; Schlenk 16

Regions due to Cancellation

Various tools attempt to find such re-mappings using **linear** changes of variables

ASY/FIESTA Jantzen, A. Smirnov, V. Smirnov 12

- Check all pairs of variables (α_1, α_2) which are part of monomials of opposite sign
- For each pair, try to build linear combination $\alpha_1 \to b\alpha_1', \alpha_2 \to \alpha_2' + b\alpha_1'$ s.t negative monomial vanishes
- Repeat until all negative monomials vanish **or** warn user

ASPIRE Ananthanarayan, Pal, Ramanan, Sarkar 18; B. Ananthanarayan, Das, Sarkar 20

- Consider Gröbner basis of $\left\{\mathcal{F}, \partial\mathcal{F}/\alpha_1, \partial\mathcal{F}/\alpha_2, \ldots\right\}$ (i.e. $\mathcal{F}$ and Landau equations)
- Eliminate negative monomials with linear transformations $\alpha_1 \to b\alpha_1', \alpha_2 \to \alpha_2' + b\alpha_1'$

This is not enough to straightforwardly expose all regions in parameter space

Interesting Example

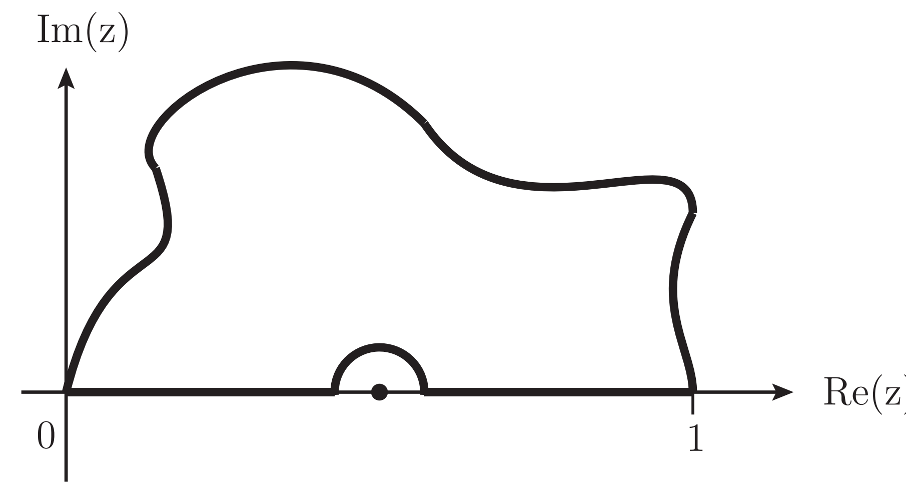
Let's try to compute this with sector decomposition (pySecDec)

Fails to find contour…

Contour Deformation

Feynman integral (after sector decomp):

$$I \sim \int_0^1 [\mathrm{d}\boldsymbol{\alpha}]\,\boldsymbol{\alpha}^{\nu} \frac{[\mathscr{U}(\boldsymbol{\alpha})]^{N-(L+1)D/2}}{[\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})]^{N-LD/2}}$$



Deform integration contour to avoid poles on real axis

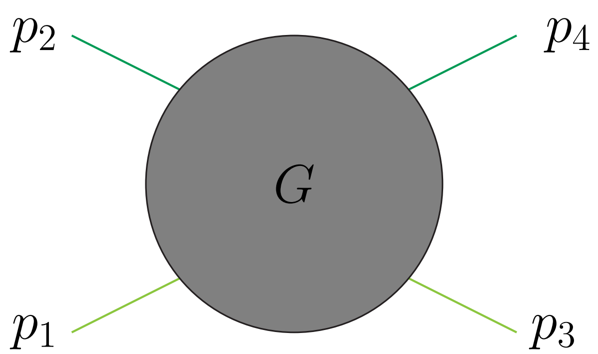
Feynman prescription $\mathscr{F} \to \mathscr{F} - i\delta$ tells us how to do this

Expand $\mathscr{F}(z = \boldsymbol{\alpha} - i\boldsymbol{\tau})$ around $\boldsymbol{\alpha}$, $\mathscr{F}(z) = \mathscr{F}(\boldsymbol{\alpha}) - i \sum_j \tau_j \frac{\partial \mathscr{F}(\boldsymbol{\alpha})}{\partial \alpha_j} + \mathscr{O}(\tau^2)$

Choose $\tau_j = \lambda_j \alpha_j (1 - \alpha_j) \frac{\partial \mathscr{F}(\boldsymbol{\alpha})}{\partial \alpha_j}$ with small constants $\lambda_j > 0$

Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07, 08; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14;…

Forward Scattering



Inserting $\theta \sim \sqrt{\lambda}$ into the Botts-Sterman analysis leads to one of the loop momenta becoming Glauber:

$$k_4^\mu - k_2^\mu = k_1^\mu - k_3^\mu \sim Q(\lambda, \lambda; \sqrt{\lambda})$$

We obtain $\mu = -1 - 3\epsilon$

Alternatively, can expand known analytic result in the foward limit $x = -s_{13}/s_{12}$

Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21;

$$I(s_{12}, s_{13}; \epsilon) = s_{12}^{-2-3\epsilon} \mathcal{I}(x;\epsilon), \quad \mathcal{I}(x;\epsilon) \sum_{n=-4}^{\infty} \mathcal{I}^{(n)}(x)\,\epsilon^n = \sum_{n=-4}^{\infty} \sum_{k=-1}^{\infty} \mathcal{I}^{(n,k)}(L)\, x^k\, \epsilon^n$$

$L = \log(x)$

$$\mathcal{I}(x;\epsilon) = \mathrm{LP}\left\{I_{\mathrm{XX}}\right\}(L;\epsilon) + \mathcal{O}(x^0)$$

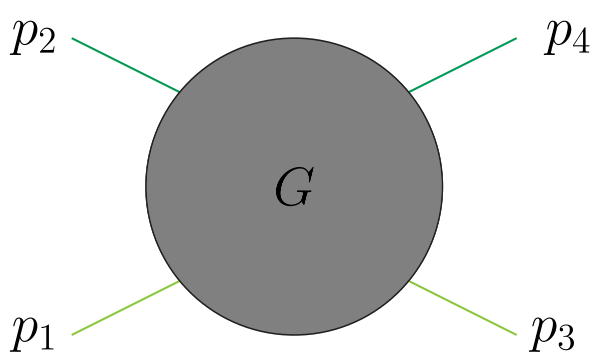
$$\mathrm{LP}\left\{\mathcal{I}\right\}(L;\epsilon) = i\pi x^{-1-3\epsilon}\Bigg(-\frac{8}{3\epsilon^4} + \frac{16}{\epsilon^3} + \frac{2\left(\pi^2 - 144\right)}{3\epsilon^2} - \frac{4\left(-58\zeta(3) + 3\pi^2 - 432\right)}{3\epsilon}$$

$$+\frac{1}{60}\left(-27840\zeta(3) + 71\pi^4 + 1440\pi^2 - 207360\right) + \cdots \Bigg),$$

gives $\mathcal{I}(x;\epsilon) \sim x^{-1-3\epsilon}$

Forward Scattering

Directly applying MoR in parameter space, no region with correct scaling…



$I \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	**order**
$(-1,-1,-1,0,-1,-1,-1,0;1))$	-3ϵ
$(-1,-1,0,-1,-1,-1,0,-1;1)$	-3ϵ
$(-1,0,-1,-1,-1,0,-1,-1;1)$	-3ϵ
$(0,-1,-1,-1,0,-1,-1,-1;1)$	-3ϵ
$(0,0,0,0,0,0,0,0;1)$	0

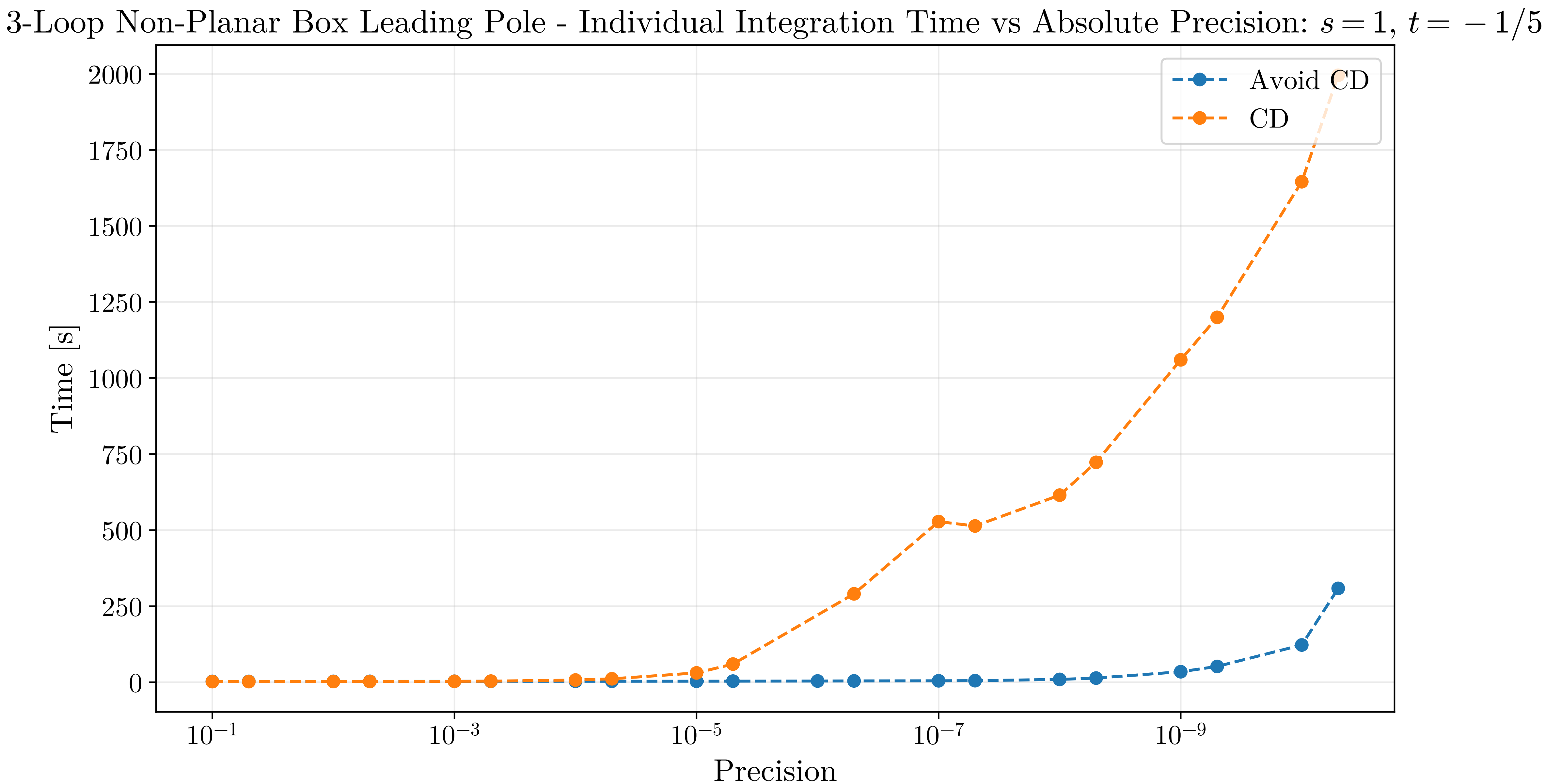
After resolution, in some polytopes we now directly see the leading region observed in the analytic result!

$I_1 \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(y_0, x_1, y_2, x_3, y_4, x_5, y_6, x_7)$	$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	**order**
$(0,-1,0,-1,0,-1,1,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$-1-3\epsilon$
$(1,-1,0,-1,0,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$-1-3\epsilon$
$(-1,0,0,-1,-1,0,0,-1;1)$	$(-1,0,-1,-1,-1,0,-1,-1;1)$	-3ϵ
$(0,0,0,0,0,0,0,0;1)$	$(0,0,0,0,0,0,0,0;1)$	0

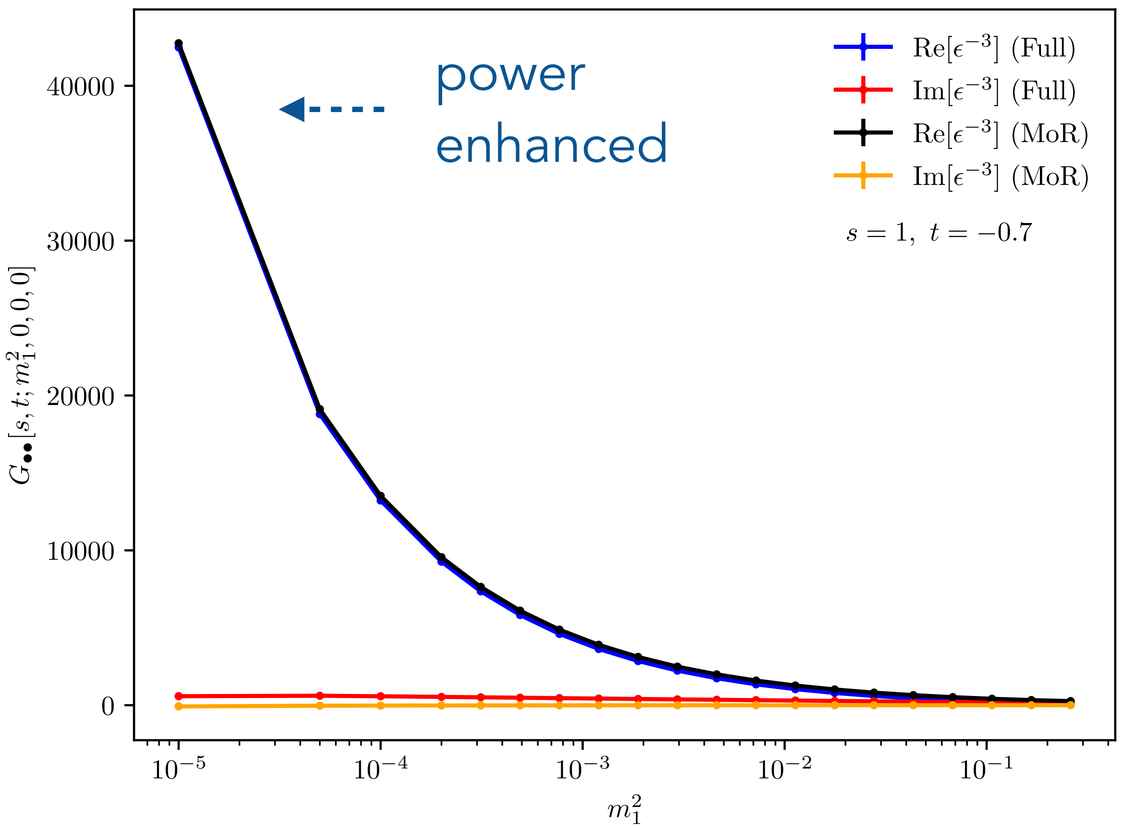
NoCD: Example 3

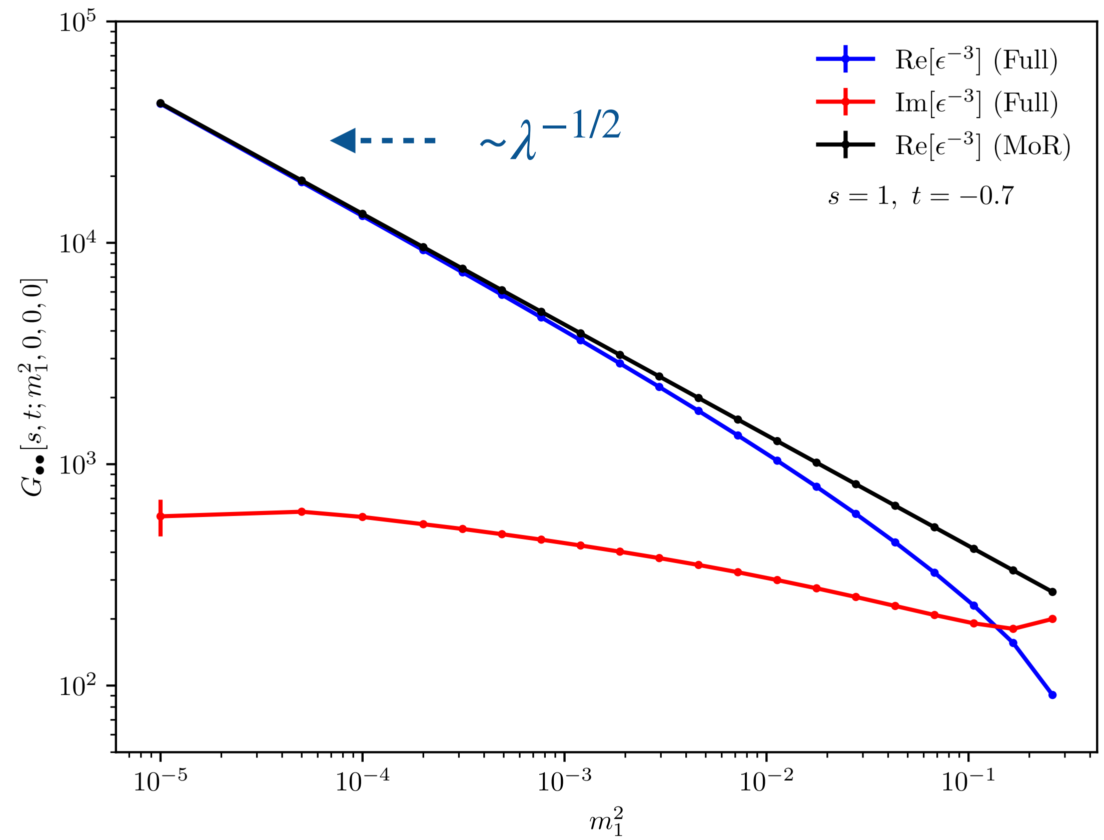
Evaluating leading pole with pySecDec



On-Shell Expansion

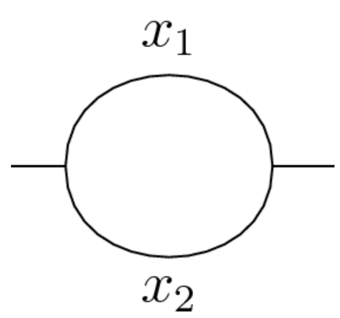
Use MoR on each of the split integrals $I_1, \ldots, I_{24}$ and summing only the leading region for each split (with $\mu = -1/2 - 3\epsilon$)





See strong numerical evidence that the split integrals (MoR) reproduce the leading behaviour of the full integral in the limit $p_1^2 \to 0$

Contour Deformation

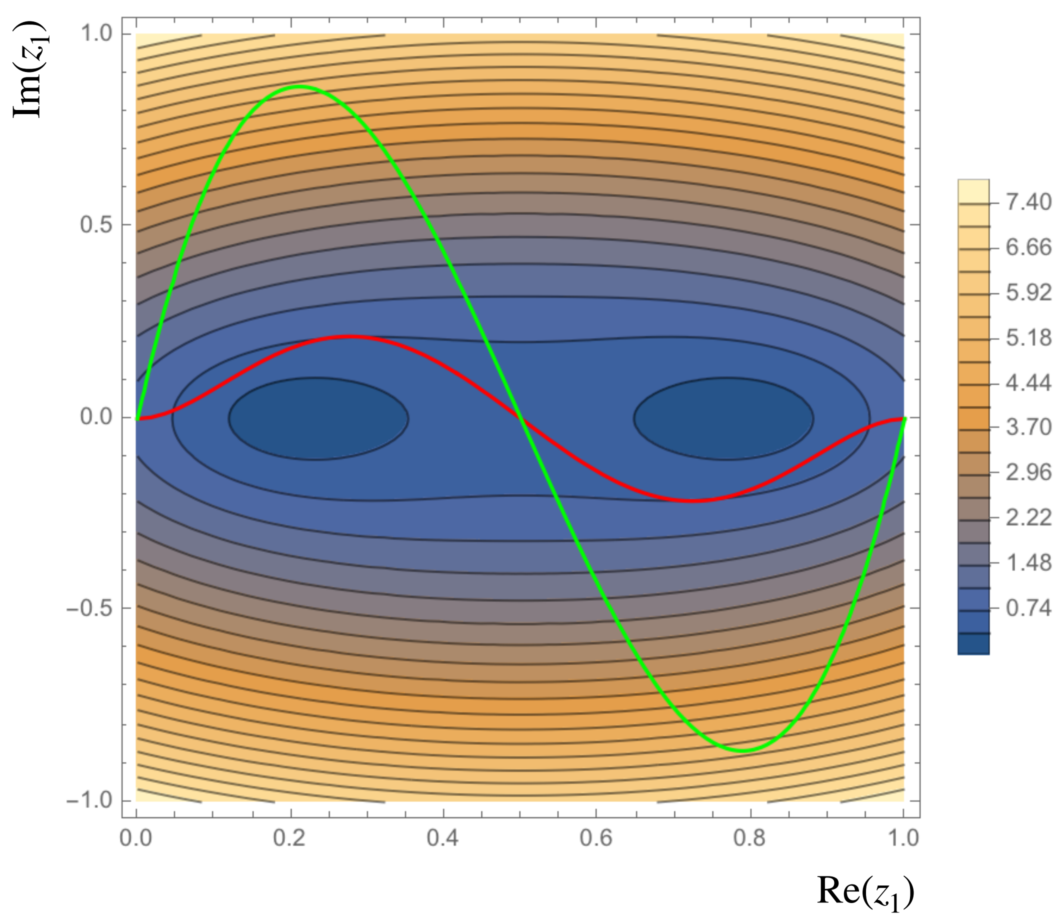


$$= \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2\,\frac{\mathscr{U}(\mathbf{x})^{-2+2\epsilon}}{\mathscr{F}(\mathbf{x},\mathbf{s})^{\epsilon}}\delta(1-x_1-x_2)$$

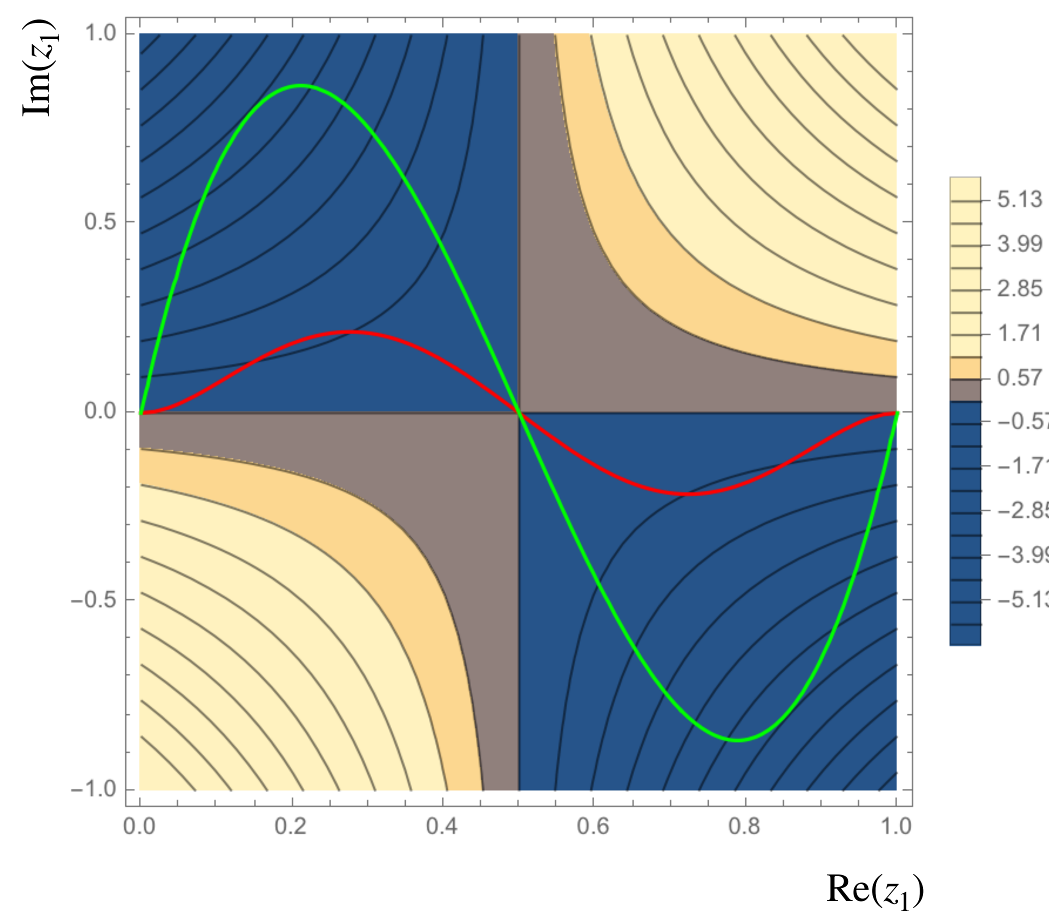
$$\mathscr{U}(\mathbf{x}) = x_1 + x_2$$

$$\mathscr{F}(\mathbf{x},\mathbf{s}) = -\,s x_1 x_2 + (m_1^2 x_1 + m_2^2 x_2)\,(x_1 + x_2)$$

$|\mathscr{F}|$

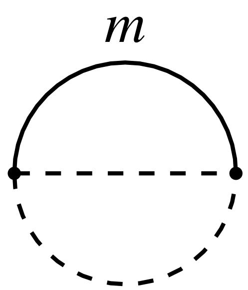


$\mathrm{Im}(\mathscr{F})$



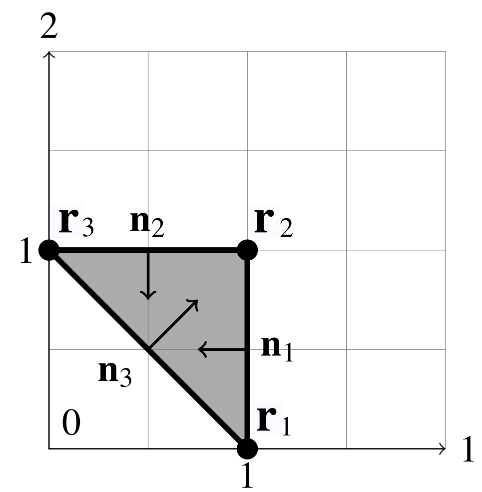
Sector Decomposition

Sector Decomposition in a Nutshell

$I =$ 

$$= -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon}\int_0^\infty \frac{\mathrm{d}x_1\mathrm{d}x_2}{\left(x_1^1x_2^0+x_1^1x_2^1+x_1^0x_2^1\right)^{2-\varepsilon}}.$$

$$\mathbf{r}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix}0\\1\end{pmatrix}$$

$\mathcal{N}(I) =$ 

$$= \quad \mathbf{n}_1 = \begin{pmatrix}-1\\0\end{pmatrix} \quad \mathbf{n}_2 = \begin{pmatrix}0\\-1\end{pmatrix} \quad \mathbf{n}_3 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$a_1 = 1 \quad a_2 = 1 \quad a_3 = -1$$

For each vertex make the local change of variables

e.g. $\mathbf{r}_1$: $x_1 = y_1^{-1}y_3^1,\ x_2 = y_1^0y_3^1$, $\mathbf{r}_2 : x_1 = y_1^{-1}y_2^0,\ x_2 = y_1^0y_2^{-1}$, $\mathbf{r}_3 : x_1 = y_2^0y_3^1, x_2 = y_2^{-1}y_3^1$

$$I = -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon}\int_0^1 \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\,\frac{y_1^{-\varepsilon}y_2^{-\varepsilon}y_3^{-1+\varepsilon}}{(y_1+y_2+y_3)^{2-\varepsilon}}\left[\delta(1-y_2)+\delta(1-y_3)+\delta(1-y_1)\right]$$

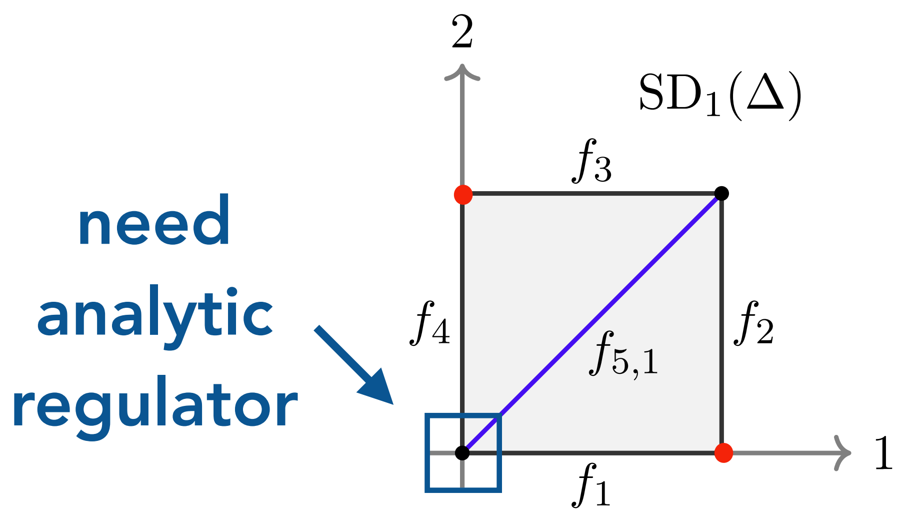
Schlenk 2016

Applications

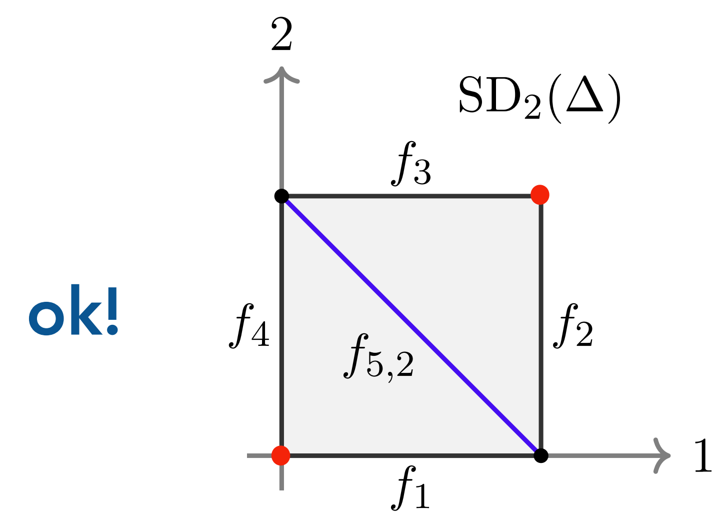
Additional Regulators (II)

Toy Example:

$$P_1(x,\lambda) = 1 + \lambda x_1 + x_1 x_2 + \lambda x_2$$



$$P_2(x,\lambda) = \lambda + x_1 + \lambda x_1 x_2 + x_2$$



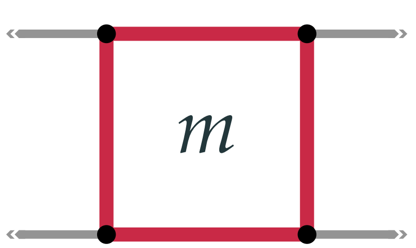
pySecDec can find the constraints on the analytic regulators for you

`extra_regulator_constraints()`:

$$\nu_2 - \nu_4 \neq 0, \quad \nu_1 - \nu_3 \neq 0$$

`suggested_extra_regulator_exponent()`:

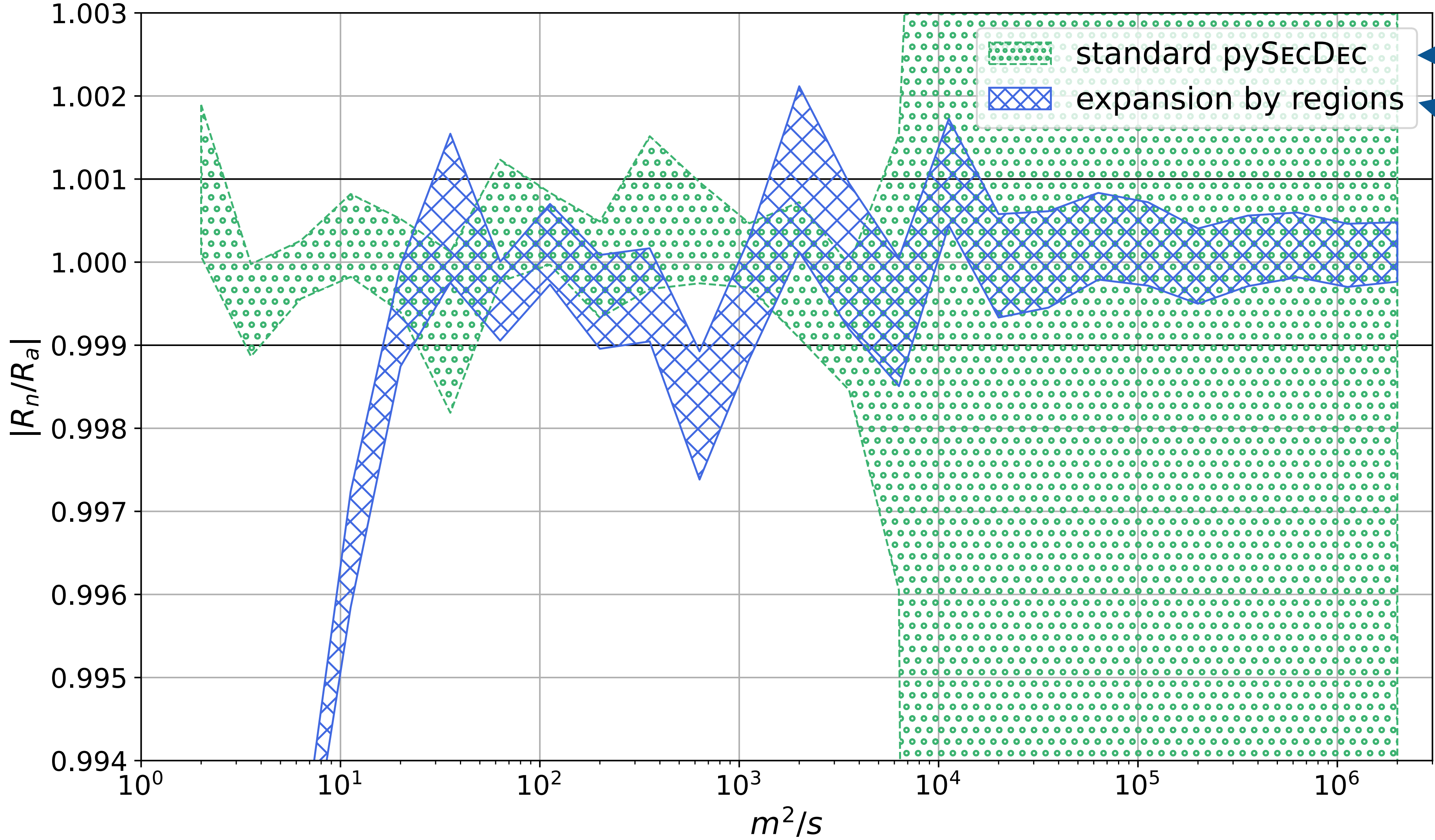
$$\{\delta\nu_1, \delta\nu_2, \delta\nu_3, \delta\nu_4\} = \{0, 0, \eta, -\eta\}$$



Small m expansion

Applying Expansion by Regions

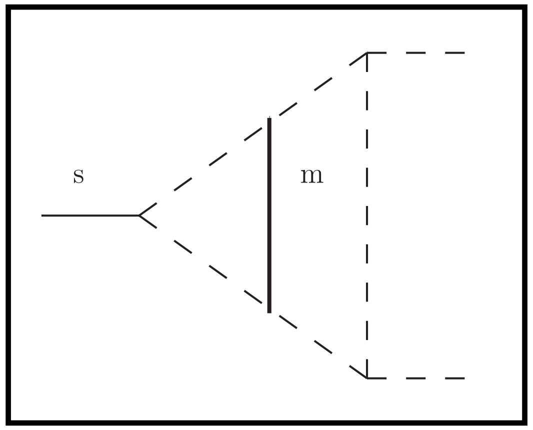
Ratio of the finite $\mathcal{O}(\epsilon^0)$ piece of numerical result R_n to the analytic result R_a



(at large m^2/s)

~1 day

~2 min



For large ratio of scales (m^2/s) the EBR result is **faster** & **easier** to integrate

Lee-Pomeransky and MoR

Building Bridges: LP $\leftrightarrow$ Propagator Scaling

Region vectors in momentum space and Lee-Pomeransky space are related, we can see this using Schwinger parameters $\tilde{x}_e$

$$\frac{1}{D_n^{\nu_e}} = \frac{1}{\Gamma(\nu_e)} \int_0^\infty \frac{\mathrm{d}\tilde{x}_e}{\tilde{x}_e}\, \tilde{x}_e^{\nu_e}\, e^{-\tilde{x}_e D_e}\ , \text{ with } x_e \propto \tilde{x}_e$$

$$(D_1^{-1}, \ldots, D_N^{-1}) \sim (\tilde{x}_1, \ldots, \tilde{x}_N) \sim (x_1, \ldots, x_N)$$

Example: 1-loop form factor

$$\text{Hard}: \quad (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^0, \lambda^0), \qquad (x_1, x_2, x_3) \sim (\lambda^0, \lambda^0, \lambda^0)$$

$$\text{Collinear to } \mathrm{p}_1: \quad (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1}), \qquad (x_1, x_2, x_3) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1})$$

$$\text{Collinear to } \mathrm{p}_2: \quad (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1}), \qquad (x_1, x_2, x_3) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1})$$

$$\text{Soft}: \quad (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2}), \qquad (x_1, x_2, x_3) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2})$$

Can connect the regions in mom. space with those we determine geometrically

Next step: automatically find (Sudakov decomposed) loop momentum scalings compatible with region vectors WIP w/ Yannick Ulrich

Building Bridges: Landau ↔ Regions

The **Landau equations** give the necessary conditions for an integral to diverge

$$1)\quad \alpha_e l_e^2(k,p,q) = 0 \qquad \forall e \in G$$

$$2)\quad \frac{\partial}{\partial k_a^\mu}\mathscr{D}(k,p,q;\alpha) = \frac{\partial}{\partial k_a^\mu}\sum_{e\in G}\alpha_e\left(-l_e^2(k,p,q) - i\varepsilon\right) = 0 \qquad \forall a \in \{1,\ldots,L\}$$

Solutions are *pinched surfaces* of the integral where IR divergences may arise

Idea is to explore the *neighbourhood of a pinched surface*, defined by

$$1)\quad \alpha_e l_e^2(k,p,q) \sim \lambda^p \qquad \forall e \in G, \quad \text{with} \quad p \in \{1,2\}$$

$$2)\quad \frac{\partial}{\partial k_a^\mu}\mathscr{D}(k,p,q;\alpha) \lesssim \lambda^{1/2} \qquad \forall a \in \{1,\ldots,L\}$$

with the goal of further understanding the connection between

Solutions of the Landau equations ↔ Regions

Gardi, Herzog, Ma, Schlenk 22

Method of Regions (Details/Examples)

Geometric Method

In Feynman parameter space, there is a **geometric method** for finding regions

Pak, Smirnov 10

Each region will be defined by a **region vector** $\mathbf{v} = (v_1, \ldots, v_N; 1)$, in each region we will perform a change of variables $x_i \to \lambda^{v_i} x_i$ and series expand about $\lambda = 0$

Let us start by considering some polynomial

$$P(\mathbf{x}, \lambda) = \sum_{i=1}^{m} c_i \, x_1^{r_{i,1}} \cdots x_N^{r_{i,N}} \, \lambda^{r_{i,N+1}}$$

c_i - non-negative coefficients

x_i - integration variables

λ - small parameter

$\mathbf{r}_i = (r_{i,1}, \ldots, r_{i,N+1}) \in \mathbb{N}^{N+1}$ - exponent vectors

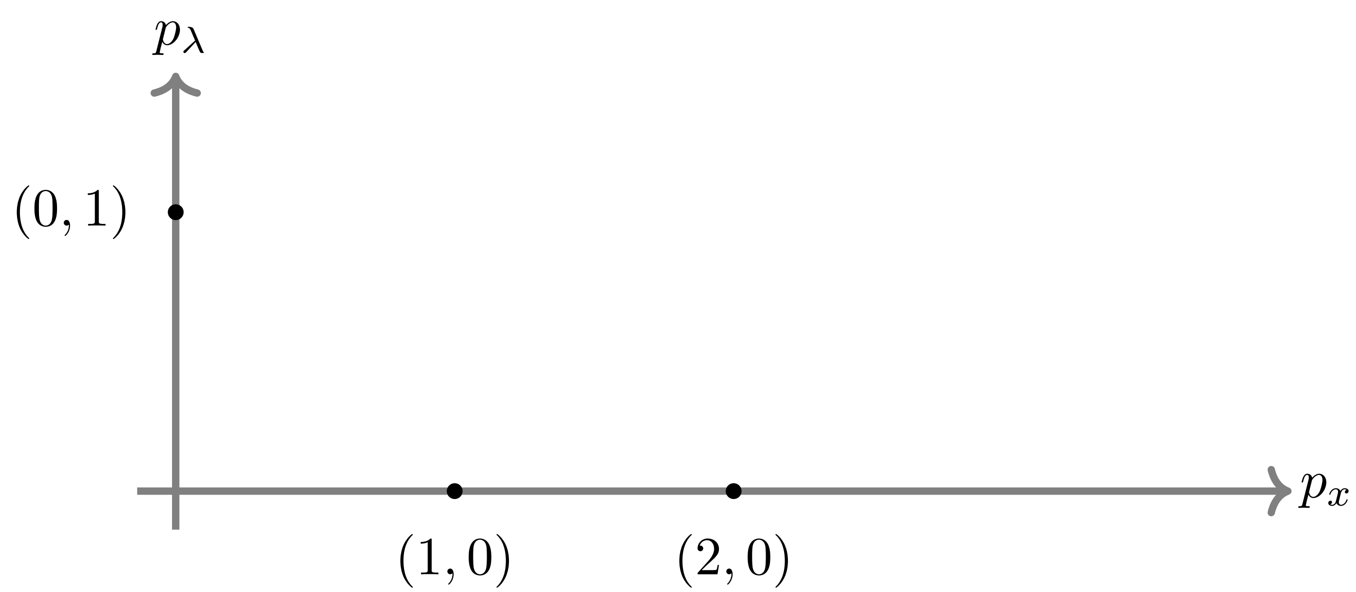
Geometric Method

Ignoring, for now, the coefficients c_i we can introduce a simple but useful picture for such polynomials:

- For each variable x_i or λ draw an orthogonal axis
- For each monomial, draw a dot at position $\mathbf{r}_i$

Example: $P(x,\lambda) = \lambda + x + x^2$ has exponent vectors

$\mathbf{r}_1 = (0{,}1), \mathbf{r}_2 = (1{,}0), \mathbf{r}_3 = (2{,}0)$



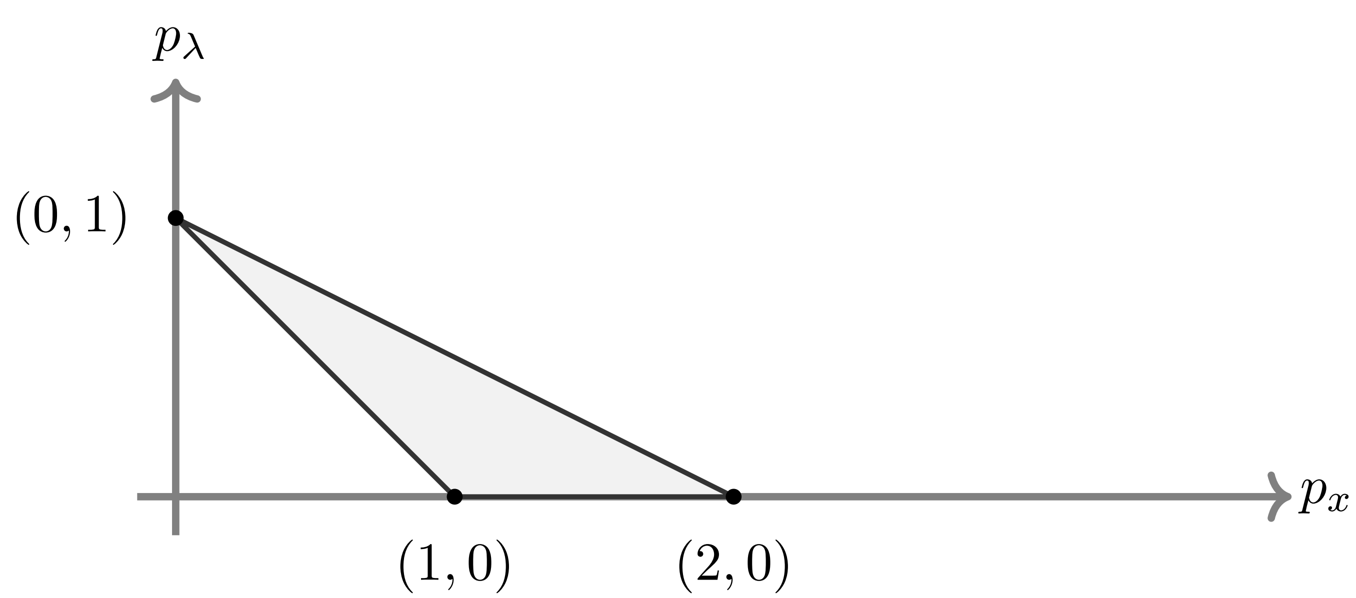
Geometric Method

We may define a **Newton polytope** of the polynomial, this is the convex hull of the exponent vectors:

$$\Delta = \text{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \left\{ \sum_j \alpha_j \mathbf{r}_j \,|\, \alpha_j \geq 0 \wedge \sum_j \alpha_j = 1 \right\}$$

Example: $P(x, \lambda) = \lambda + x + x^2$ has exponent vectors

$\mathbf{r}_1 = (0,1), \mathbf{r}_2 = (1,0), \mathbf{r}_3 = (2,0)$



Geometric Method

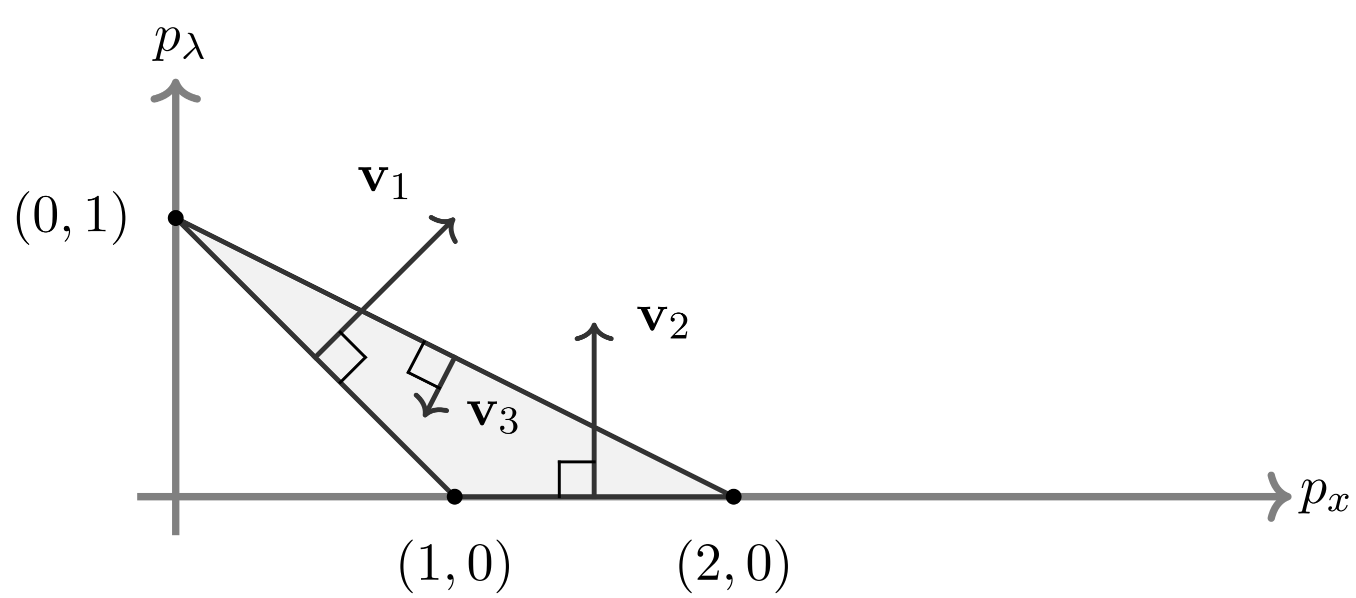
Alternatively, this polytope can also be described as the intersection of half spaces:

$$\Delta = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^{N+1} \mid \langle \mathbf{m}, \mathbf{v}_f \rangle + a_f \geq 0 \right\}$$

F - set of polytope facets, $a_f \in \mathbb{Z}$

$\mathbf{v}_f$ - inward-pointing normal vectors for each facet (co-dimension 1 face)

Several public tools exist for computing Newton polytopes/convex hulls and their representation in terms of facets exist, e.g. **Normaliz** and **Qhull**



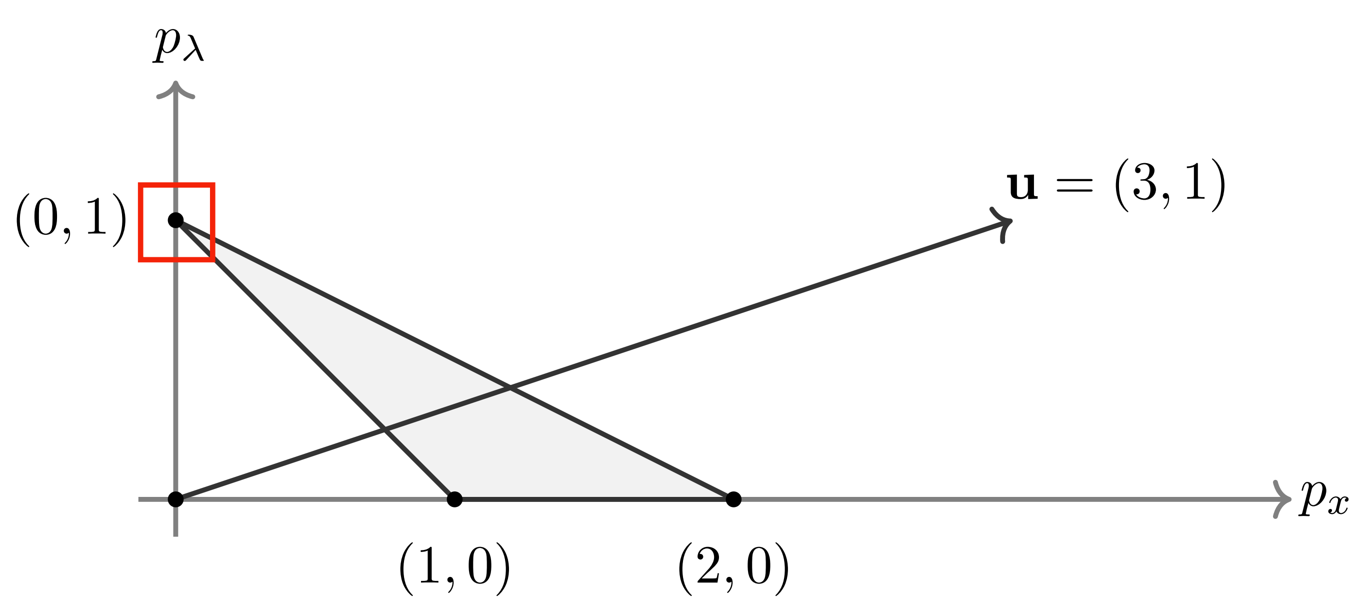
Geometric Method

Next, let us define a vector $\mathbf{u}$ such that $x_i = \lambda^{u_i}$ with $u_{N+1} = 1$ for each point $\mathbf{x}$ in the integration domain, we can write:

$$P(\mathbf{u}, \lambda) = \sum_{i=1}^{m} c_i \lambda^{\langle \mathbf{r}_i, \mathbf{u} \rangle}$$

Since $\lambda \ll 1$, the largest term in the polynomial has the smallest $\langle \mathbf{r}_i, \mathbf{u} \rangle$
Note that we can have several points with the same projection on $\mathbf{u}$, i.e. we can have several largest terms

Example: $P(x, \lambda) = \lambda + x + x^2$ with $\mathbf{u} = (3,1)$ gives $P(\mathbf{u}, \lambda) = \underline{\lambda} + \lambda^3 + \lambda^6$



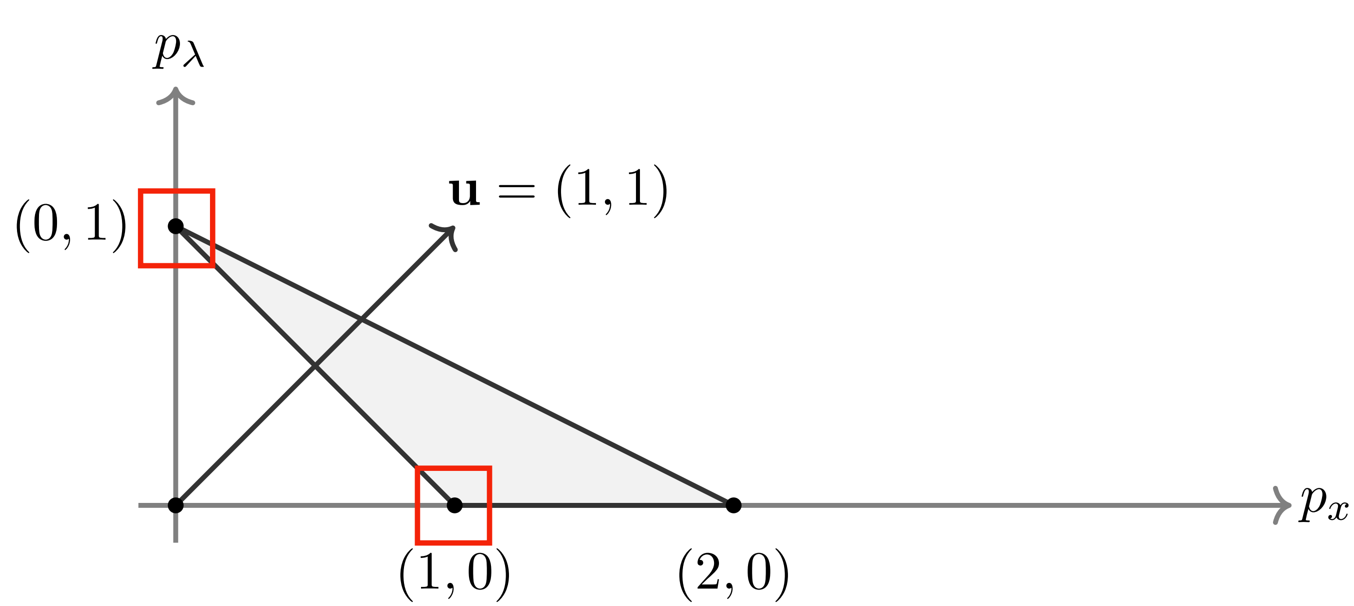
Geometric Method

Next, let us define a vector $\mathbf{u}$ such that $x_i = \lambda^{u_i}$ with $u_{N+1} = 1$ for each point $\mathbf{x}$ in the integration domain, we can write:

$$P(\mathbf{u}, \lambda) = \sum_{i=1}^{m} c_i \lambda^{\langle \mathbf{r}_i, \mathbf{u} \rangle}$$

Since $\lambda \ll 1$, the largest term in the polynomial has the smallest $\langle \mathbf{r}_i, \mathbf{u} \rangle$
Note that we can have several points with the same projection on $\mathbf{u}$, i.e. we can have several largest terms

Example: $P(x, \lambda) = \lambda + x + x^2$ with $\mathbf{u} = (1,1)$ gives $P(\mathbf{u}, \lambda) = \underline{\lambda + \lambda} + \lambda^2$



Expanding Regions

Rewrite our polynomial as: $P(\mathbf{x}) = Q(\mathbf{x}) + R(\mathbf{x})$

With $Q(\mathbf{x})$ defined such that it contains all of the lowest order terms in λ

The binomial expansion of

$P(\mathbf{x})^m = Q(\mathbf{x})^m \left(1 + \frac{R(\mathbf{x})}{Q(\mathbf{x})}\right)^m$ converges for $\mathbf{x} = \lambda^{\mathbf{u}}$ if $R(\mathbf{x})/Q(\mathbf{x}) < 1$

Some observations:

- An expansion with region vector $\mathbf{v}$ converges at a point $\mathbf{u}$ if the terms with minimum $< \mathbf{r}_i, \mathbf{u} >$ are contained in the terms with minimum $< \mathbf{r}_i, \mathbf{v} >$
- For any $\mathbf{u}$ the vertices with the smallest $< \mathbf{r}_i, \mathbf{u} >$ must be part of some facet F
- Since $u_{N+1} > 0$, the lowest order terms for any $\mathbf{u}$ must lie on a facet whose inwards pointing normal vector has a positive $(N+1)$-th component, let us call the set of such facets F^+ or lower facets

Claim: regions are defined by vectors normal to the facets in F^+, the integrand in each region consists of the monomials lying on the facet

Scaleless Integrals

Scaleless integrals seem to play quite an interesting role

Momentum space

In dimensional regularisation, **scaleless integrals are 0**

$$I(\{k_i\}_a, \{ck_i\}_b) = c^q\, I(\{k_i\}) \implies I(\{k_i\}) = 0, \quad \{k_i\} = \{k_i\}_a \cup \{k_i\}_b$$

Where $c \neq 1$ and $q \neq 0$ is some scaling dimension

Feynman parameter space

$$(\mathcal{U}\mathcal{F})\big(c^{\mathbf{u}}\mathbf{x}\big) = c^q(\mathcal{U}\mathcal{F})(\mathbf{x}), \quad \mathbf{u} \neq n\mathbf{1}, \quad n \in \mathbb{R}$$

Geometrical view

For Δ built from $\mathcal{U} + \mathcal{F}$

$$\dim(\Delta) = \dim(\mathbf{x}) \iff I \text{ scaleful}$$
$$\dim(\Delta) < \dim(\mathbf{x}) \iff I \text{ scaleless}$$

Important consequences:

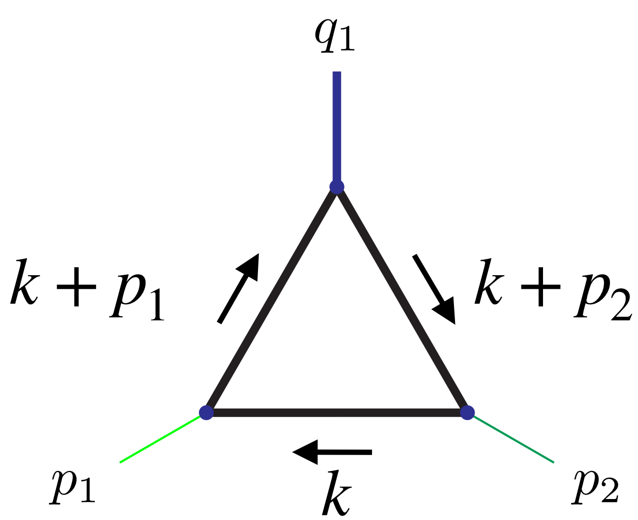
Faces of co-dimension > 1 are scaleless

``Region'' vectors not normal to a facet give scaleless integrals

Overlap contributions i.e. rescaling by two region vectors, are scaleless

Triangle Example

Consider the on-shell limit $p_1^2 \sim p_2^2 \sim \lambda q_1^2$ for $\lambda \to 0$



$$I = i\pi^{D/2} \mu^{4-D} \int \mathrm{d}^D k \frac{1}{(k+p_1)^2 (k+p_2)^2 (k^2)}$$

$$p_1 = (p_1^+, p_1^-, p_1^\perp) \sim Q(\lambda, 1, \lambda^{\frac{1}{2}})$$

$$p_2 \sim Q(1, \lambda, \lambda^{\frac{1}{2}})$$

1) Split integrand up into regions

$$\text{Hard}: k_H^\mu \sim (1,1,1)\, Q$$

$$\text{Collinear to } p_1: k_{J_1}^\mu \sim (\lambda, 1, \lambda^{\frac{1}{2}})\, Q$$

$$\text{Collinear to } p_2: k_{J_2}^\mu \sim (1, \lambda, \lambda^{\frac{1}{2}})\, Q$$

$$\text{Soft}: k_S^\mu \sim (\lambda, \lambda, \lambda)\, Q$$

2) Series expand each region in λ

$$I_H = i\pi^{d/2} \mu^{4-D} \int \mathrm{d}^D k \frac{1}{(k^2 + 2k^+ \cdot p_1^-)(k^2 + 2k^- \cdot p_2^+)(k^2)}$$

$$I_{C_1} = i\pi^{d/2} \mu^{4-D} \int \mathrm{d}^D k \frac{1}{(k+p_1)^2 (2k^- \cdot p_2^+)(k^2)}$$

$$I_{C_2} = i\pi^{d/2} \mu^{4-D} \int \mathrm{d}^D k \frac{1}{(2k^- \cdot p_1^+)(k+p_2)^2 (k^2)}$$

$$I_S = i\pi^{d/2} \mu^{4-D} \int \mathrm{d}^D k \frac{1}{(2k^+ \cdot p_1^- + p_1^2)(2k^- \cdot p_2^+ + p_2^2)(k^2)}$$

Analysis follows:
Becher, Broggio, Ferroglia 14

Triangle Example

3-5) Integrate each expansion over the whole integration domain, discard scaleless, sum

$$I_H = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_{C_1} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_{C_2} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_2^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_S = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{P_2^2 P_1^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{P_2^2 P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I = I_H + I_{C_1} + I_{C_2} + I_S = \frac{1}{Q^2}\left(\ln\frac{Q^2}{P_2^2}\ln\frac{Q^2}{P_1^2} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right)$$

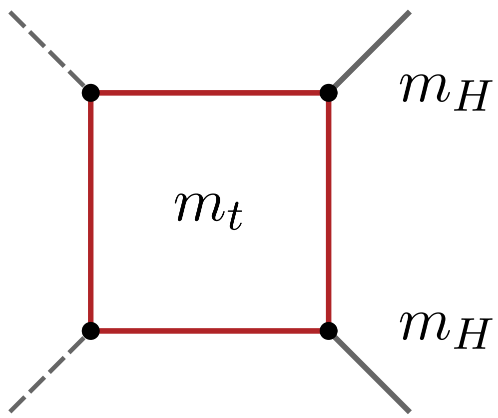
This reproduces the expected result, but why does this work (and does it always)?

1) How did we **find all the regions**?

2) Did we not **double-count** when integrating over the whole domain ?

pySecDec: EBR Box Example

Example: 1-loop massive box expanded for small $m_t^2 \ll s, |t|$



Requires the use of analytic regulators

Can regulate spurious singularities by adjusting propagators powers

$$G_4 = \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2 - m_t^2]^{\delta_1}[(k+p_1)^2 - m_t^2]^{\delta_2}[(k+p_1+p_2)^2 - m_t^2]^{\delta_3}[(k-p_4)^2 - m_t^2]^{\delta_4}}$$

Can keep $\delta_1, \ldots, \delta_4$ symbolic or $\delta_1 = 1 + n_1/2, \delta_2 = 1 + n_1/3, \ldots$ and take $n_1 \to 0^+$

Output region vectors:

$\mathbf{v}_1 = (0,0,0,0,1)$

$\mathbf{v}_2 = (-1,-1,0,0,1)$

$\mathbf{v}_3 = (0,0,-1,-1,1)$

$\mathbf{v}_4 = (-1,0,0,-1,1)$

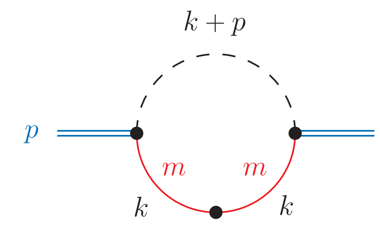
$\mathbf{v}_5 = (0,-1,-1,0,1)$

Result: s = 4.0, t = -2.82843, $m_t^2 = 0.1, m_h^2 = 0$)

$$I = -1.30718 \pm 2.7 \cdot 10^{-6} + (1.85618 \pm 3.0 \cdot 10^{-6})\, i$$

$$+ \mathcal{O}\left(\epsilon, n_1, \frac{m_t^2}{s}, \frac{m_t^2}{t}\right)$$

Transform the expression for the full integral:



$$F = \int_{k\in D_h} \mathrm{D}k\, I + \int_{k\in D_s} \mathrm{D}k\, I = \sum_i \int_{k\in D_h} \mathrm{D}k\, T_i^{(h)} I + \sum_j \int_{k\in D_s} \mathrm{D}k\, T_j^{(s)} I$$

$$= \sum_i \left(\int_{k\in\mathbb{R}^d} \mathrm{D}k\, T_i^{(h)} I - \sum_j \int_{k\in D_s} \mathrm{D}k\, T_j^{(s)} T_i^{(h)} I \right) + \sum_j \left(\int_{k\in\mathbb{R}^d} \mathrm{D}k\, T_j^{(s)} I - \sum_i \int_{k\in D_h} \mathrm{D}k\, T_i^{(h)} T_j^{(s)} I \right)$$

The expansions commute: $\boxed{T_i^{(h)} T_j^{(s)} I = T_j^{(s)} T_i^{(h)} I \equiv T_{i,j}^{(h,s)} I}$

$\Rightarrow$ **Identity:** $F = \underbrace{\sum_i \int \mathrm{D}k\, T_i^{(h)} I}_{\boldsymbol{F^{(h)}}} + \underbrace{\sum_j \int \mathrm{D}k\, T_j^{(s)} I}_{\boldsymbol{F^{(s)}}} - \underbrace{\sum_{i,j} \int \mathrm{D}k\, T_{i,j}^{(h,s)} I}_{\boldsymbol{F^{(h,s)}}}$

All terms are integrated over the whole integration domain $\mathbb{R}^d$ as prescribed for the expansion by regions $\Rightarrow$ location of boundary Λ between D_h, D_s is irrelevant.

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The general formalism (details)

Identities as in the examples are generally valid, under some conditions.

Consider

- a (multiple) integral $F = \int \mathrm{D}k\, I$ over the domain D (e.g. $D = \mathbb{R}^d$),
- a set of N regions $R = \{x_1, \dots, x_N\}$,
- for each region $x \in R$ an expansion $T^{(x)} = \sum_j T_j^{(x)}$ which converges absolutely in the domain $D_x \subset D$.

Conditions

- $\bigcup_{x \in R} D_x = D \qquad [D_x \cap D_{x'} = \emptyset\ \forall x \neq x']\,.$
- Some of the expansions commute with each other. Let $R_\mathsf{c} = \{x_1, \dots, x_{N_\mathsf{c}}\}$ and $R_\mathsf{nc} = \{x_{N_\mathsf{c}+1}, \dots, x_N\}$ with $1 \leq N_\mathsf{c} \leq N$. Then: $T^{(x)} T^{(x')} = T^{(x')} T^{(x)} \equiv T^{(x,x')}\ \forall x \in R_\mathsf{c}\,,\ x' \in R\,.$
- Every pair of non-commuting expansions is invariant under some expansion from R_c: $\forall x'_1, x'_2 \in R_\mathsf{nc},\ x'_1 \neq x'_2,\ \exists\, x \in R_\mathsf{c} : T^{(x)} T^{(x'_2)} T^{(x'_1)} = T^{(x'_2)} T^{(x'_1)}\,.$
- $\exists$ regularization for singularities, e.g. dimensional (+ analytic) regularization. $\hookrightarrow$ All expanded integrals and series expansions in the formalism are well-defined.

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The general formalism (2)

Under these conditions, the following **identity** holds: $[F^{(x,\ldots)} \equiv \sum_{j,\ldots} \int \mathrm{D}k\, T^{(x,\ldots)}_{j,\ldots} I]$

$$F = \sum_{x\,\in\,R} F^{(x)} - \sum_{\{x'_1,x'_2\}\subset R}^{\langle R_c+1\rangle} F^{(x'_1,x'_2)} + \ldots - (-1)^n \sum_{\{x'_1,\ldots,x'_n\}\subset R}^{\langle R_c+1\rangle} F^{(x'_1,\ldots,x'_n)} + \ldots + (-1)^{N_c} \sum_{x'\,\in\,R_{nc}} F^{(x',x_1,\ldots,x_{N_c})}$$

where the sums run over subsets $\{x'_1,\ldots\}$ containing at most one region from R_{nc}.

Comments

- This identity is exact when the expansions are summed to all orders. ✓
  Leading-order approximation for $F \rightsquigarrow$ dropping higher-order terms.
- It is independent of the regularization (dim. reg., analytic reg., cut-off, infinitesimal masses/off-shellness, ...) as long as all individual terms are well-defined.
- Usually regions & regularization are chosen such that multiple expansions $F^{(x'_1,\ldots,x'_n)}$ $(n \geq 2)$ are scaleless and vanish.
  [✓ if each $F_0^{(x)}$ is a *homogeneous* function of the expansion parameter with *unique scaling*.]
- If $\exists\, F^{(x'_1,x'_2,\ldots)} \neq 0 \rightsquigarrow$ relevant overlap contributions ($\rightarrow$ "zero-bin subtractions").
  They appear e.g. when avoiding analytic regularization in SCET. e.g. Manohar, Stewart '06; Chiu, Fuhrer, Hoang, Kelley, Manohar '09; ...

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