On the Electron Self-Energy to Three Loops

in

Quantum Electrodynamics

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Based on joint work w/ C. Duhr, C. Nega, L. Tancredi & S. Weinzierl

arXiv [2408.05154]

High Precision for Hard Processes – Turin, Italy – 10 September 2024

Electron Self-Energy $\hat{\Sigma}(p,m)$ fundamental building block

$$\Pi = \frac{i}{p - m + \hat{\Sigma}(p,m)}$$

Two spinor structures

$$\hat{\Sigma}(p,m) = \Sigma_V(p,m)\,\not{p} + \Sigma_S(p,m)\,m\,\mathbb{I}$$

Compute in perturbation theory

$$\Sigma_\bullet(p,m) = \sum_{\ell=0}^{\infty} \left(\frac{\alpha}{\pi}\,C(\epsilon)\right)^{\ell} \Sigma_\bullet^{(\ell)}(p,m)$$

$\bullet = V, S$

$e = \sqrt{4\pi\alpha} \rightsquigarrow$ bare charge

$m \rightsquigarrow$ bare mass

$C(\epsilon) = \Gamma(1+\epsilon)(4\pi)^{\epsilon}(m^2)^{-\epsilon}$

$\ell = 1$ [Textbooks]

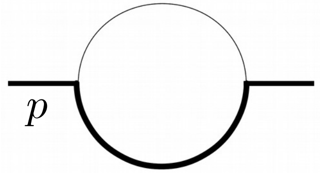
$\ell = 2$ [Sabry; Hoenemann, Tempest & Weinzierl]

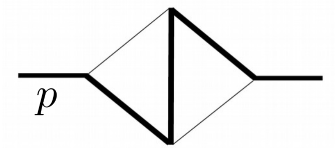
$\ell = 3$ [Duhr, FG, Nega, Tancredi & Weinzierl; In this talk focus Bare Results]

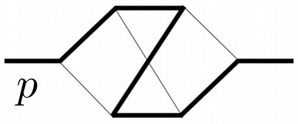
Setup

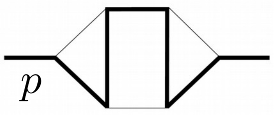
- Generation of Feynman Diagrams [QGRAF; FeynArts]
- Spinor Algebra [FORM; FeynCalc]
- Integration by Parts, reduction to Master Integrals (MIs) [Kira; Reduze; LiteRed]

$\hat{\Sigma}^{(\ell)}$	# Diagrams	# MIs
$\ell = 1$	1	2
$\ell = 2$	3	8
$\ell = 3$	20	51









Evaluation of MIs via Differential Equations

[Kotikov; Remiddi, Gehrmann & Remiddi]

$$\frac{\mathrm{d}\mathbf{I}(x,\epsilon)}{\mathrm{d}x} = \mathbf{A}(x,\epsilon)\,\mathbf{I}(x,\epsilon)$$

$$\epsilon = \frac{4-d}{2}$$

$$x = \frac{p^2}{m^2}$$

Evaluation MIs via Differential Equations in $\epsilon-$factorized form

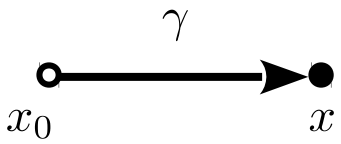
[Henn]

$$\frac{\mathrm{d}\mathbf{J}(x,\epsilon)}{\mathrm{d}x} = \epsilon\,\mathbf{G}(x)\,\mathbf{J}(x,\epsilon)$$

$$\mathbf{G}(x) = \sum_i \mathbf{G}_i \underbrace{\omega_i(x)}_{f_i(x)dx}$$

Solution "trivial" via Path Ordered Exponential $\rightsquigarrow$ Iterated Integrals

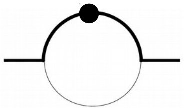
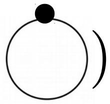
$$\mathbf{J}(x,\epsilon) = \mathbb{P}\left[\epsilon \int_\gamma \mathbf{G}(x)\right] \mathbf{J}_0(\epsilon)$$



$\mathbf{J}_0(\epsilon) \rightsquigarrow$ Boundary Constants

Natural "truncation" in ϵ

E.g.: $\Sigma_\bullet^{(\ell=1)}$

$\mathbf{J} = ($ $\epsilon\, x$  , ϵ  $)$

Differential Forms? dlog forms $\{\omega_i(x)\} = \{\mathrm{d}\log x, \mathrm{d}\log(1-x)\}$

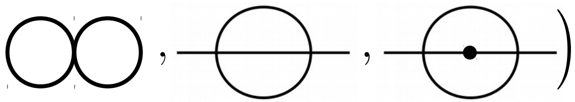
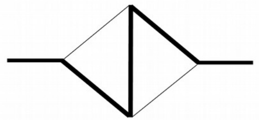
Iterated Integrals? Harmonic Polylogarithms (HPLs) [Remiddi & Vermaseren]

What happens for the integrals needed for $\Sigma_{\bullet}^{(\ell)}$ with $\ell = 2, 3$?

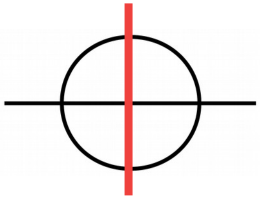
How to obtain an $\epsilon-$factorized basis?

(We assume that such a basis exists)

Paradigmatic example, “Elliptic Sunrise” $(\ell = 2)$ [Laporta & Remiddi]

$\mathbf{I} = ($  $) \subset$ 

Analyze the Maximal Cut

 $= \int \frac{\mathrm{d}z}{\sqrt{P_4(z)}}$

$d = 2$ [loop-by-loop] Baikov rep.

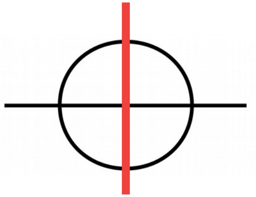
P_4 quartic polynomial

Square Root Quartic $\rightsquigarrow$ Elliptic Curve

$$\{(X,Y) \in \mathbb{C}^2 \,|\, Y^2 = (X - X_1)(X - X_2)(X - X_3)(X - X_4)\}$$

$X_1 = 0$
$X_2 = (1 - \sqrt{x})^2$
$X_3 = (1 + \sqrt{x})^2$
$X_4 = 4$

Maximal Cut annihilated by Picard Fuchs operator (int. over proper cycles) [Primo & Tancredi]

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]$$  $= 0$

$x = \frac{p^2}{m^2}$

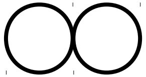
Singular points $x_0 = \{0, 1, 9, \infty\}$

The Ansatz Method $(\ell = 2)$

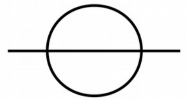
[Adams & Weinzierl]
[Poegel , Wang & Weinzierl]

$d = 2 - 2\epsilon$

$$\mathrm{J}_1 = \epsilon^2$$



$$\mathrm{J}_2 = \epsilon^2 \frac{\pi}{\varpi(x)}$$



$$\mathrm{J}_3 = \frac{1}{2\pi i \epsilon} J(x) \frac{\mathrm{d}\mathrm{J}_2}{\mathrm{d}x} + S_{32}(x) \mathrm{J}_2(x)$$

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon(\ldots) & \epsilon(\ldots) \\ \epsilon(\ldots) & \epsilon^{-1}(\ldots)+\epsilon^0(\ldots)+\epsilon(\ldots) & \epsilon^0(\ldots)+\epsilon(\ldots) \end{pmatrix} \mathbf{J}$$

First two rows ϵ−form (tautological), ϵ^{-1}, ϵ^0 give differential constraints

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right] \varpi(x) = 0$$

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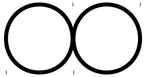
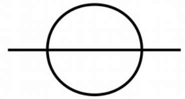
$\mathrm{J}_1 = \epsilon^2$ 

$\mathrm{J}_2 = \epsilon^2 \dfrac{\pi}{\varpi(x)}$ 

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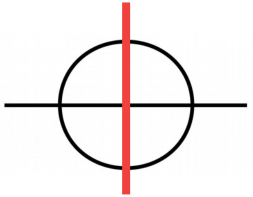
cf.

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]$$  $= 0$

The Ansatz Method $(\ell = 2)$

[Adams & Weinzierl]
[Poegel , Wang & Weinzierl]

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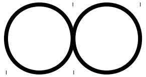
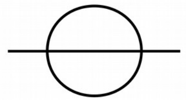
$J_1 = \epsilon^2$ 

$J_2 = \epsilon^2 \dfrac{\pi}{\varpi(x)}$ 

$$J_3 = \frac{1}{2\pi i \epsilon} J(x) \frac{dJ_2}{dx} + S_{32}(x) J_2(x)$$

$$\frac{d\mathbf{J}}{dx} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon(\ldots) & \epsilon(\ldots) \\ \epsilon(\ldots) & \epsilon^{-1}(\ldots)+\epsilon^{0}(\ldots)+\epsilon(\ldots) & \epsilon^{0}(\ldots)+\epsilon(\ldots) \end{pmatrix} \mathbf{J}$$

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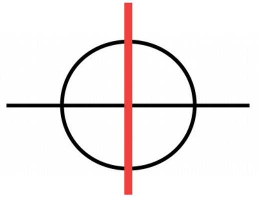


$$= \varpi(x) = (\ldots)\texttt{EllipticK}[\ldots] = \frac{2\pi}{\sqrt{3}} \left(1 + \frac{x}{3} + \frac{5x^2}{27} + \frac{31x^3}{243} + \frac{71x^4}{729} + \mathcal{O}(x^5) \right)$$

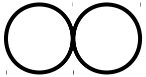
Holomorphic sol. $x_0 = 0$

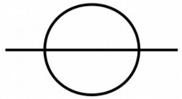
Holomorphic sol. can be obtained at any $x_0 = \{0, 1, 9, \infty\}$

The Ansatz Method $(\ell = 2)$

[Adams & Weinzierl]
[Poegel , Wang & Weinzierl]

$d = 2 - 2\epsilon$

$J_1 = \epsilon^2$ 

$J_2 = \epsilon^2 \dfrac{\pi}{\varpi(x)}$ 

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$$\varpi(x) = \frac{2\pi}{\sqrt{3}} \left(1 + \frac{x}{3} + \frac{5x^2}{27} + \frac{31x^3}{243} + \frac{71x^4}{729} + \mathcal{O}(x^5)\right)$$

$$S_{32}(x) = \frac{(3x^2 - 10x - 9)}{24} \frac{\varpi(x)^2}{\pi^2}$$

$$J(x) = -\frac{x(1-x)(9-x)}{6i\pi} \varpi(x)^2$$

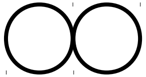
Comments:

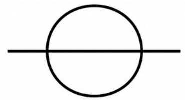
- J_2 normalized by "Maximal Cut" $\rightsquigarrow$ not algebraic in this case
- Transformation depends on $\varpi(x)$ and $\partial_x \varpi(x)$

The Ansatz Method $(\ell = 2)$

[Adams & Weinzierl]
[Poegel , Wang & Weinzierl]

$d = 2 - 2\epsilon$

$$\mathrm{J}_1 = \epsilon^2$$ 

$$\mathrm{J}_2 = \epsilon^2 \frac{\pi}{\varpi(x)}$$ 

$$\mathrm{J}_3 = \frac{1}{2\pi i \epsilon} J(x) \frac{\mathrm{dJ}_2}{\mathrm{d}x} + S_{32}(x) \mathrm{J}_2(x)$$

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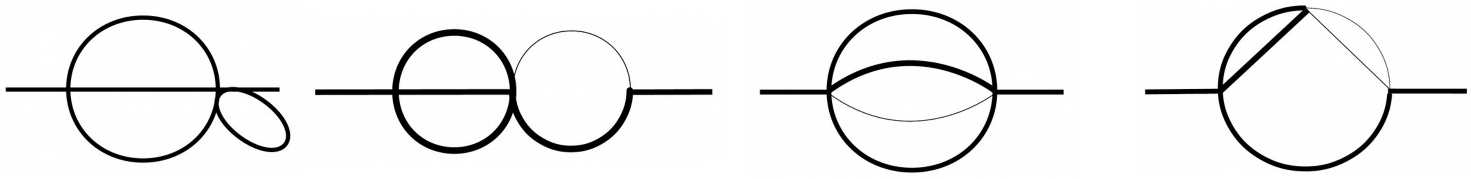
Similar strategy works for other Elliptic Sectors $(\ell = 3)$



$(\ell = 2) \times (\ell = 1)$

3 MIs, follow idea from [Jiang , Wang, Yang & Zhao]

They all share same underlying Elliptic Curve

Alternative, yet equivalent, method [Goerges, Nega, Tancredi & Wagner]

Differential Equation in $\epsilon-$factorized form

$$\frac{\mathrm{d}\mathbf{J}(x,\epsilon)}{\mathrm{d}x} = \epsilon\,\mathbf{G}(x)\,\mathbf{J}(x,\epsilon) \qquad \mathbf{G}(x) = \sum_i \mathbf{G}_i\,\omega_i(x)$$

$$\omega_i(x) = f_i(x)\mathrm{d}x$$

[Hoenemann, Tempest & Weinzierl]

$$f_i^{(\ell=2)} \in \left\{ \frac{1}{x}, \frac{1}{x-1}, \frac{1}{x-9}, \frac{\pi^2}{x(x-1)(x-9)\varpi(x)^2}, \frac{\varpi(x)}{\pi}, \frac{\varpi(x)}{\pi}\frac{1}{x-1}, \frac{(x+3)^4}{x(x-1)(x-9)}\frac{\varpi(x)^2}{\pi^2} \right\}$$

$$f_i^{(\ell=3)} \in f_i^{(\ell=2)} \bigcup \left\{ \frac{1}{x+3}, \frac{1}{x+1}, \frac{1}{x-2}, \frac{1}{\sqrt{(3+x)(1-x)}}, \frac{1}{\sqrt{(1-x)(9-x)}}, \frac{1}{\sqrt{(1-x)(9-x)}}\frac{1}{x} \right\}$$

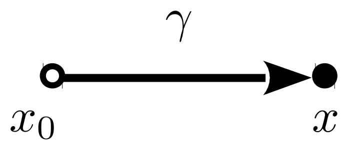
$$\bigcup \left\{ \frac{x-3}{\sqrt{(1-x)(9-x)}}\frac{\varpi(x)}{\pi}, \frac{(x+3)(x-1)}{x(x-9)}\frac{\varpi(x)^2}{\pi^2}, \frac{1}{(x-1)(x-9)}\frac{\varpi(x)^2}{\pi^2} \right\}$$

Always redefine $\varpi(x)$ s.t. f_i locally at most simple pole

Consider $0 < x < 1$, two interesting limits

$x = \frac{p^2}{m^2}$

$$\mathbf{J}(x,\epsilon) = \mathbb{P}\left[\epsilon \int_\gamma \mathbf{G}(x)\right] \mathbf{J}_0(\epsilon)$$



Limit $x \to x_0 = 0$, MIs expected to be finite, collapse to tadpoles (GPLs at sixth roots of unity)

$$\mathbf{J}_0(\epsilon) \rightsquigarrow G(\ldots, a_i, \ldots; 1) \qquad a_i \in \{0, \rho^k\} \quad \text{where} \quad \rho = e^{i\pi/3} \quad \text{with} \quad 0 \leq k < 6$$

Limit $x \to x_0 = 1$, generalized series expansion

$$\mathbf{J}(x,\epsilon) = \sum_n \epsilon^n \mathbf{J}^{(n)}(x) \qquad \mathbf{J}^{(n)}(x) = \sum_{a,b} \mathbf{C}_{a,b}^{(n)} (1-x)^a \log^b(1-x)$$

Match against `AMFlow` at $x = 1 - \delta$ ($\delta \sim 10^{-2}$) and PSLQ

$$\mathbf{J}_0(\epsilon) \rightsquigarrow \pi,\, \zeta(n),\, \mathrm{Li}_n(1/2)$$

Self Energy $\ell = 3$ via iterated integrals (Gauge dependent)

$$\Sigma_\bullet^{(\ell=3)} = \frac{\Sigma_{\bullet,-3}^{(\ell=3)}}{\epsilon^3} + \frac{\Sigma_{\bullet,-2}^{(\ell=3)}}{\epsilon^2} + \frac{\Sigma_{\bullet,-1}^{(\ell=3)}}{\epsilon} + \Sigma_{\bullet,0}^{(\ell=3)} + \mathcal{O}(\epsilon) \qquad \bullet = V, S$$

$\Sigma_{\bullet,-3}^{(\ell=3)} \rightsquigarrow$ const.

$\Sigma_{\bullet,-2}^{(\ell=3)} \rightsquigarrow f_i \in f_i^{(\ell=1)}$

$\Sigma_{\bullet,-1}^{(\ell=3)} \rightsquigarrow f_i \in f_i^{(\ell=2)}$ (i.e. poles in ϵ are described by lower loop kernels)

$\Sigma_{\bullet,0}^{(\ell=3)} \rightsquigarrow$ kernels w/ pos. powers $\varpi^2(x)$ drop out

Let $\varpi(x)$ hol. solution $x_0 = 0$, iterated integrals $\rightsquigarrow$ local series expansion for $\Sigma_\bullet^{(\ell=3)}$

$$\Sigma_\bullet^{(\ell=3)} = (\dots) + (\dots)x + (\dots)x^2 + \mathcal{O}(x^3)$$

Self Energy $\ell = 3$ via iterated integrals (Gauge dependent)

$$\Sigma_{\bullet}^{(\ell=3)} = \frac{\Sigma_{\bullet,-3}^{(\ell=3)}}{\epsilon^3} + \frac{\Sigma_{\bullet,-2}^{(\ell=3)}}{\epsilon^2} + \frac{\Sigma_{\bullet,-1}^{(\ell=3)}}{\epsilon} + \Sigma_{\bullet,0}^{(\ell=3)} + \mathcal{O}(\epsilon) \qquad \bullet = V, S$$

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$\Sigma_{\bullet,0}^{(\ell=3)} \rightsquigarrow$ kernels w/ pos. powers $\varpi^2(x)$ drop out

A peek at the result (Gauge choice: $\xi = 0$)

$$\begin{aligned}\Sigma_{S}^{(\ell=3)}\Big|_{x=0} = & -\frac{5}{1152\epsilon^3} + \frac{1241}{3456\epsilon^2} - \left[\frac{3373}{1296} + \frac{\pi^2}{1152} + \frac{\zeta(3)}{4} + \frac{3\sqrt{3}}{16}\mathrm{Cl}_2\left(\tfrac{\pi}{3}\right)\right]\frac{1}{\epsilon} \\ & - \frac{4447957}{311040} - \frac{7031}{576}\zeta(3) + \pi^2\left(\frac{3019}{17280} + \frac{7\log^2(2)}{6} - \frac{3\log^2(3)}{16} + \frac{151\pi^2}{1152}\right) \\ & - \frac{21}{8}\mathrm{Cl}_2\left(\tfrac{\pi}{3}\right)^2 - 28\mathrm{Li}_4\left(\tfrac{1}{2}\right) - \frac{9}{4}\left[\mathrm{Li}_4\left(-\tfrac{1}{3}\right) - 2\mathrm{Li}_4\left(\tfrac{1}{3}\right)\right] - \frac{7\log^4(2)}{6} + \frac{3\log^4(3)}{32} \\ & + \sqrt{3}\left[\frac{691\pi^3}{960} + \frac{22}{5}\mathrm{Cl}_2\left(\tfrac{\pi}{3}\right) + \frac{129}{320}\pi\log^2(3) - \frac{387}{20}\mathrm{Im}\Big(\mathrm{Li}_3\Big(\tfrac{i}{\sqrt{3}}\Big)\Big)\right] + \mathcal{O}(\epsilon)\end{aligned}$$

Self Energy $\ell = 3$ via iterated integrals (Gauge dependent)

$$\Sigma_{\bullet}^{(\ell=3)} = \frac{\Sigma_{\bullet,-3}^{(\ell=3)}}{\epsilon^3} + \frac{\Sigma_{\bullet,-2}^{(\ell=3)}}{\epsilon^2} + \frac{\Sigma_{\bullet,-1}^{(\ell=3)}}{\epsilon} + \Sigma_{\bullet,0}^{(\ell=3)} + \mathcal{O}(\epsilon) \qquad \bullet = V, S$$

$\Sigma_{\bullet,-3}^{(\ell=3)} \rightsquigarrow$ const.

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$\Sigma_{\bullet,-1}^{(\ell=3)} \rightsquigarrow f_i \in f_i^{(\ell=2)}$ (i.e. poles in ϵ are described by lower loop kernels)

$\Sigma_{\bullet,0}^{(\ell=3)} \rightsquigarrow$ kernels w/ pos. powers $\varpi^2(x)$ drop out

A peek at the result (Gauge choice: $\xi = 0$)

$$\begin{aligned}\Sigma_V^{(\ell=3)}\Big|_{x=0} &= \frac{7}{144\epsilon^2} + \left[\frac{5053}{3456} + \frac{\pi^2}{24} - \sqrt{3}\mathrm{Cl}_2\left(\tfrac{\pi}{3}\right)\right]\frac{1}{\epsilon} + \frac{201881}{25920} + \frac{733\zeta(3)}{24} \\
&+ \frac{5\log^4(2)}{3} - \frac{3\log^4(3)}{16} - \pi^2\left[\frac{23}{360} + \frac{5\log^2(2)}{3} - \frac{3\log^2(3)}{8} + \frac{1459\pi^2}{8640}\right] \\
&+ \frac{15}{4}\mathrm{Cl}_2\left(\tfrac{\pi}{3}\right)^2 + 40\mathrm{Li}_4\left(\tfrac{1}{2}\right) + \frac{9}{2}\left[\mathrm{Li}_4\left(-\tfrac{1}{3}\right) - 2\mathrm{Li}_4\left(\tfrac{1}{3}\right)\right] \\
&- \sqrt{3}\left[\frac{1103\pi^3}{2160} + \frac{8197}{480}\mathrm{Cl}_2\left(\tfrac{\pi}{3}\right) + \frac{23}{80}\pi\log^2(3) - \frac{69}{5}\mathrm{Im}\Big(\mathrm{Li}_3\Big(\tfrac{i}{\sqrt{3}}\Big)\Big)\right] + \mathcal{O}(\epsilon)\end{aligned}$$

Self Energy $\ell = 3$ via iterated integrals (Gauge dependent)

$$\Sigma_{\bullet}^{(\ell=3)} = \frac{\Sigma_{\bullet,-3}^{(\ell=3)}}{\epsilon^3} + \frac{\Sigma_{\bullet,-2}^{(\ell=3)}}{\epsilon^2} + \frac{\Sigma_{\bullet,-1}^{(\ell=3)}}{\epsilon} + \Sigma_{\bullet,0}^{(\ell=3)} + \mathcal{O}(\epsilon) \qquad \bullet = V, S$$

$\Sigma_{\bullet,-3}^{(\ell=3)} \rightsquigarrow$ const.

$\Sigma_{\bullet,-2}^{(\ell=3)} \rightsquigarrow f_i \in f_i^{(\ell=1)}$

$\Sigma_{\bullet,-1}^{(\ell=3)} \rightsquigarrow f_i \in f_i^{(\ell=2)}$ (i.e. poles in ϵ are described by lower loop kernels)

$\Sigma_{\bullet,0}^{(\ell=3)} \rightsquigarrow$ kernels w/ pos. powers $\varpi^2(x)$ drop out

A peek at the result (Gauge choice: $\xi = 0$)

$$\begin{aligned}\Sigma_S^{(\ell=3)}\Big|_{x\to 1^-} &= \frac{1}{x-1}\left[\frac{27}{16}\log^2(1-x) + \left(-\frac{27}{16\epsilon} - \frac{27}{8}\right)\log(1-x) + \frac{27}{32\epsilon^2} + \frac{27}{16\epsilon} + \frac{9}{2} + \frac{9\pi^2}{32}\right] \\ &- \frac{3}{4}\log^3(1-x) + \left(\frac{9}{8\epsilon} + \frac{125}{128}\right)\log^2(1-x) \\ &+ \log(1-x)\left(-\frac{9}{8\epsilon^2} - \frac{17}{128\epsilon} - \frac{3433}{768} - \frac{53\pi^2}{32} + \frac{3}{4}\pi^2\log(2) - \frac{9\zeta(3)}{8}\right) \\ &- \frac{5}{1152\epsilon^3} + \frac{7}{108\epsilon^2} + \frac{1}{\epsilon}\left(\frac{8117}{5184} - \frac{95\pi^2}{288} + \frac{5}{24}\pi^2\log(2) - \frac{9\zeta(3)}{16}\right) \\ &+ \frac{162239}{31104} - \frac{768617\pi^2}{103680} + \frac{3139}{288}\pi^2\log(2) - \frac{13237\zeta(3)}{576} + \frac{997\pi^4}{5760} \\ &- \frac{29\log^4(2)}{48} - \frac{7}{12}\pi^2\log^2(2) - \frac{29\mathrm{Li}_4\left(\frac{1}{2}\right)}{2} - \frac{\pi^2\zeta(3)}{16} + \frac{5\zeta(5)}{16} + \mathcal{O}(\epsilon)\,.\end{aligned}$$

Self Energy $\ell = 3$ via iterated integrals (Gauge dependent)

$$\Sigma_{\bullet}^{(\ell=3)} = \frac{\Sigma_{\bullet,-3}^{(\ell=3)}}{\epsilon^3} + \frac{\Sigma_{\bullet,-2}^{(\ell=3)}}{\epsilon^2} + \frac{\Sigma_{\bullet,-1}^{(\ell=3)}}{\epsilon} + \Sigma_{\bullet,0}^{(\ell=3)} + \mathcal{O}(\epsilon) \qquad \bullet = V, S$$

$\Sigma_{\bullet,-3}^{(\ell=3)} \rightsquigarrow$ const.

$\Sigma_{\bullet,-2}^{(\ell=3)} \rightsquigarrow f_i \in f_i^{(\ell=1)}$

$\Sigma_{\bullet,-1}^{(\ell=3)} \rightsquigarrow f_i \in f_i^{(\ell=2)}$ (i.e. poles in ϵ are described by lower loop kernels)

$\Sigma_{\bullet,0}^{(\ell=3)} \rightsquigarrow$ kernels w/ pos. powers $\varpi^2(x)$ drop out

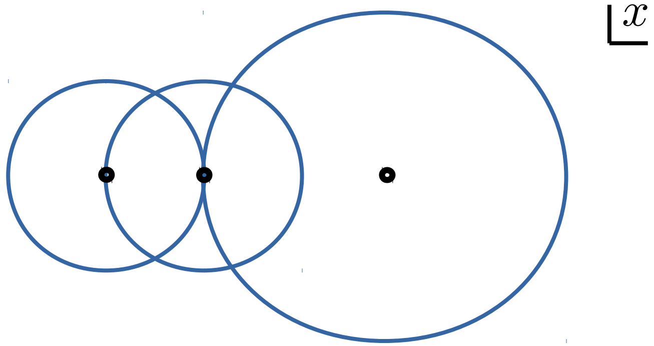
A peek at the result (Gauge choice: $\xi = 0$)

$$\begin{aligned}\Sigma_V^{(\ell=3)}\Big|_{x\to 1^-} &= \frac{27}{16}\log^3(1-x) - \frac{27\log^2(1-x)}{32\epsilon} + \left(-\frac{27}{32\epsilon} - \frac{27}{8} + \frac{9\pi^2}{32}\right)\log(1-x) \\ &+ \frac{7}{144\epsilon^2} + \frac{1}{\epsilon}\left(\frac{5179}{3456} - \frac{139\pi^2}{576} + \frac{7}{12}\pi^2\log(2) - \frac{7\zeta(3)}{8}\right) \\ &+ \frac{5815}{20736} - \frac{6827\pi^2}{12960} - \frac{5149\zeta(3)}{576} + \frac{373}{96}\pi^2\log(2) + \frac{433\pi^4}{3456} - \frac{83\log^4(2)}{144} \\ &- \frac{11}{18}\pi^2\log^2(2) - \frac{83}{6}\mathrm{Li}_4\left(\tfrac{1}{2}\right) + \frac{5\zeta(5)}{16} + \mathcal{O}(\epsilon)\,,\end{aligned}$$

How to obtain results for $x \in \mathbb{R} + i0$?

Redo calculation at each $x_0 \in \{-3, -1, 0, 1, 2, 9, \infty\}$, fix BCs via matching against `AMFlow`

Local series expansion, radius of convergence $\rightsquigarrow$ nearest singularity in DEQ (conservative)



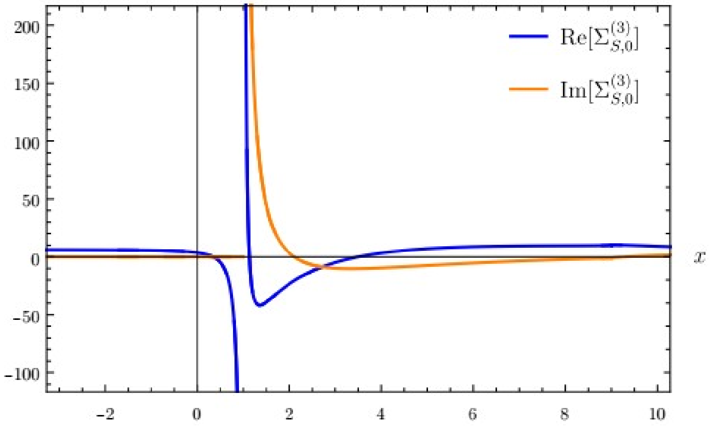
Use matching in the overlapping region as consistency check

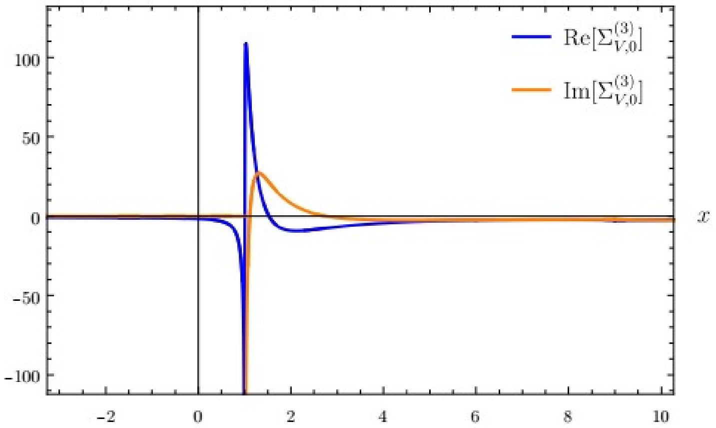
Technical details:

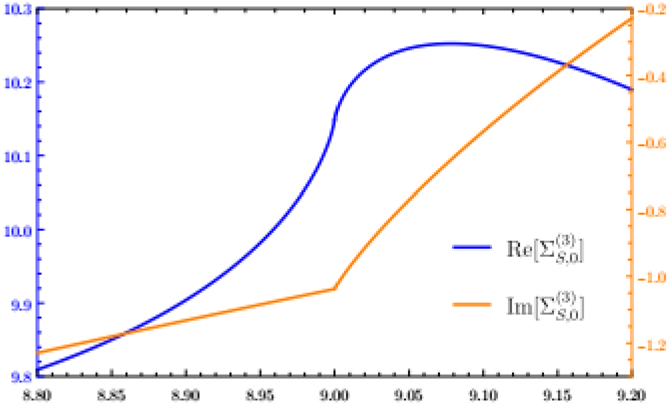
- Moebius transformation to increase radius of convergence
- Analytic continuation by Feynman $i\epsilon$

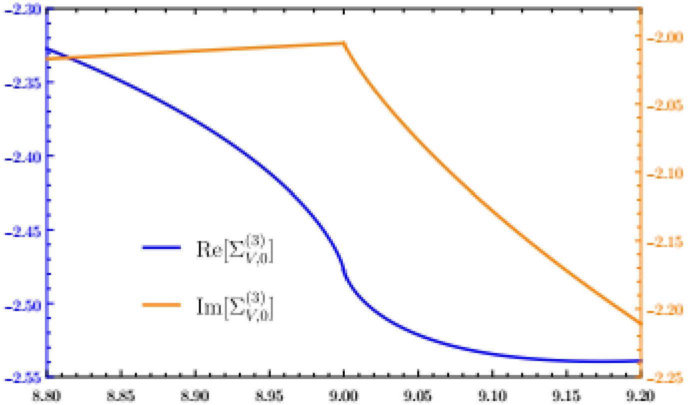
$$\sqrt{x_0 - x} \to -i\sqrt{x - x_0}, \qquad \log(x_0 - x) \to \log(x - x_0) - i\pi$$

How to obtain results for $x \in \mathbb{R} + i0$?









Matching and convergence of the Series (results for $\mathcal{O}(x_0 - x)^{101}$)

$$\Sigma_{S,0}^{(\ell=3)}(1/2) = -4.654743138598507 \qquad (x_0 = 0)$$

$$\Sigma_{S,0}^{(\ell=3)}(1/2) = -4.654743138598507 \qquad (x_0 = 1)$$

$x = 1/2$	
Truncation order	Partial Sum
10	-4.627232359131039
20	-4.654707119839225
30	-4.654743097377922
40	-4.654743138553728
50	-4.654743138598460
60	-4.654743138598507
...	-4.654743138598507
99	-4.654743138598507

Matching and convergence of the Series (results for $\mathcal{O}(x_0 - x)^{101}$)

$$\Sigma_{S,0}^{(\ell=3)}(1/2) = -4.654743138598507 \qquad (x_0 = 0)$$

Bernoulli Change of Variables (improved convergence)

$$x = 1 - e^{-z} \qquad\qquad z = -\log(1-x)$$

$x = 1/2$	
Truncation order	Partial Sum
10	-4.627232359131039
20	-4.654707119839225
30	-4.654743097377922
40	-4.654743138553728
50	-4.654743138598460
60	-4.654743138598507
...	-4.654743138598507
99	-4.654743138598507

$z = \log(2)$	
Truncation order	Partial Sum
10	-4.654743032579379
20	-4.654743138598507
30	-4.654743138598507
40	-4.654743138598507
50	-4.654743138598507
60	-4.654743138598507
...	-4.654743138598507
99	-4.654743138598507

for Li_2, GPLs/HPLs: ['t Hooft & Veltman; Ghermann & Remiddi; GINAC; CHAPLIN] for "Elliptic Sunrise": [Pozzorini & Remiddi]

Conclusions

We discussed analytical and numerical aspects of the computation of $\Sigma_{\bullet}^{(\ell=3)}$ in QED

- Differential Equations in $\epsilon-$factorized form
- Elliptic Kernels and Iterated Integrals
- Local Series Expansions and Bernoulli Change of variables

Techniques developed for Master Integrals "mature" to tackle realistic examples beyond Polylogarithms