Polytope symmetries of Feynman integrals

Leonardo de la Cruz

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Institut de Physique Theorique, CEA/Saclay

High Precision for Hard Processes at the LHC, University of Turin, Italy.

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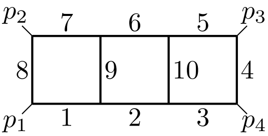
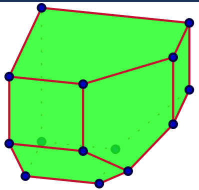
Motivation: Feynman integral + polytopes



Feynman integrals



Newton polytopes

1. Sector decomposition SecDec 3.0, '15
2. Feynman integrals in Lee-Pomeransky representation Mint, '13 and Euler characteristic Bitoun et'al, '18
3. Landau Analysis Stephen's talk
4. Finite integrals Pavel's talk

1. Triangulations of polytopes
2. A-hypergeometric functions
3. Principal A-determinant
4. Interior of the polytopes

Gel'fand-Kapranov-Zelevinsky (GKZ),80, 90's

Today: Linear transformations and polytope symmetries

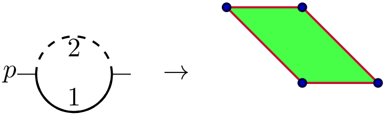
$$ {}_2F_1(a,b,c;x) = (1-x)^{-b}\,{}_2F_1(c-a,b;c;x/(x-1)) $$

Summary

Take home message

Linear transformations of Feynman integrals can be deduced from symmetries of their Newton polytopes. They are inherited from their parent A-hypergeometric Feynman integrals.

Example: Bubble or Gauß ${}_2F_1$



$$I_{\mathcal{G}}(\kappa) = \int_{\mathbb{R}_+^2} \mathrm{d}\eta_2 \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(z_2 + z_2 z_1(m^2+s) + z_1 + m^2 z_1^2)^{d/2}} \leftrightarrow \mathsf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\phi(z_1) = \tfrac{z_2}{z_1},\ \phi(z_2) = z_2 \Rightarrow I_{\mathcal{G}}(c,\kappa) = I_{\mathcal{G}}((c_4, c_2, c_3, c_1), \kappa') = I_{\mathcal{G}}(\mathsf{P}c, \kappa\mathsf{T})$$

Matrices (T, P) encode a symmetry of Newton(A)

Polytopes and their symmetries

- We consider a polynomial with n monomials in N variables

$$f(z) = \sum_{i=1}^{n} c_i z^{a_i}$$

exponent vectors$\in \mathbb{Z}^N$

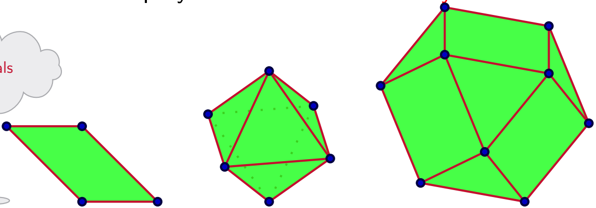
- Newton polytope is the convex hull of its exponent vectors

$$\text{Newton}(f) := \text{ConvexHull}(a_1, \ldots, a_n) \in \mathbb{R}^N$$

- Symmetry groups of polyhedra:
  - Combinatorial
  - Projective symmetries
  - Linear symmetries
- Finding symmetries of polyhedra are useful in daily life

optimizat

guess the Feynman integrals



Feynman integrals in Lee-Pomeransky representation

- Feynman integral in Euclidean space in dim-reg

$$I_F(\alpha) = \int_{\mathbb{R}^L} \left(\prod_{i=1}^{L} \frac{\mathrm{d}^d k_i}{\pi^{d/2}} \right) \frac{1}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}}$$

L loops

N propagators

- Lee and Pomeransky (LP)

$$I_{\mathcal{G}}(\kappa) := I_{\mathsf{LP}}(\alpha)/\xi_\Gamma = \int_{\mathbb{R}_+^N} z^\alpha \mathcal{G}(z)^{-d/2} \mathrm{d}\eta_N, \quad \mathrm{d}\eta_N := \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_N}{z_N}$$

- Sum of Symanzik polynomials and vector of parameters

$$\mathcal{G}(z) := \mathcal{U} + \mathcal{F}, \quad \kappa = -(d/2, \alpha_1, \ldots, \alpha_N)$$

- Feynman integrals are Euler-Mellin integrals Berkesh-Forsgård-Passare, '11
  Pierpaolo's talk

- From LP representation

$$\mathcal{G}(c, z) = \sum_{i=1}^{n} c_i z^{a_i} \longleftrightarrow \mathsf{A} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} \longleftrightarrow \mathsf{Newton}(\mathsf{A})$$

Generalized Feynman integrals

Idea

- Promote c_i to be indeterminate
- Feynman integrals are special points of A-hypergeometric functions[DLC,'19, Klausen, '19]
- Can be evaluated using canonical series [Saito-Sturmfels-Takayama, '00] or triangulations

- New integral is A-hypergeometric

$$I_{\mathcal{G}}(\kappa, c) = \int_{\Omega} \frac{z^{\alpha}}{\mathcal{G}(c,z)^{d/2}} \mathrm{d}\eta_N$$

- Gel'fand-Kapranov-Zelevinsky system $H_{\mathsf{A}}(\kappa)$

$$\left(\partial^u - \partial^v\right) I_{\mathcal{G}}(\kappa, c) = 0, \quad \text{where} \quad \mathsf{A}u = \mathsf{A}v,$$

$$\left(\sum_{j=1}^{n} a_{ij}\theta_j - \kappa_i\right) I_{\mathcal{G}}(\kappa, c) = 0, \quad i = 1, \ldots, N+1$$

- Generic parameters: $\mathrm{rank}(H_{\mathsf{A}}(\kappa)) = \mathrm{Vol}(\mathrm{Newton}(\mathsf{A}))$

Mathematical methods

Forsgård-Matusevich-Sobieska (FMS) theorem, 2017

- Suppose the A-hypergeometric fuction $F(\kappa, c)$ has a transformation

$$F(\kappa, c) = R(\kappa)F(\mathsf{T}\kappa, c\mathsf{P})$$

matrix multiplication

- Suppose above is valid for generic parameters κ and $R(\kappa)$ independent of c
- $\Rightarrow$ $\mathsf{TA} = \mathsf{AP}$ and (T, P) encodes a polytope symmetry of Newton(A)

- The converse is not always true

Idea: Use FMS theorem as a generator of symmetries

- Candidate permutation $\mathsf{P}_{\mathsf{try}}$ satisfy ?

$$\mathsf{TAP}_{\mathsf{try}} - \mathsf{A} = 0$$

- If the system has a solution we have found a symmetry
- Brute force implementation of this idea requires to solve $n!$ equations
- Brute force can be done with FiniteFlow[Peraro, 19'], we solved up to $n = 11!$ systems of equations

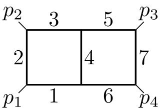
Simplifications in the case of Feynman integrals

- When the converse is true $R(\kappa) = \det(\mathsf{T}) = 1 \rightarrow$ Feynman integrals
- It is easier to compute symmetries of lattice polytopes by constructing the Normal form of A
- PALP (by Kreuze-Skarke) uses normal form to provide candidate permutations (basically choose an ordering of vertices) see Grinis-Kasprzyk, '13
- $n > 11$ PALP normal form

Symmetry finder for Feynman integrals

1. Compute LP polynomial $\mathcal{G}$
2. Construct A and put it in PALP normal form to obtain candidate permutations
3. Solve $\mathsf{TAP}_{\mathsf{try}} - \mathsf{A} = 0$ for T

Example: on-shell double box



$$\mathcal{G}(z) = t z_2 z_4 z_7 + s\left(z_{123} z_5 z_6 + z_{1567} z_1 z_3 + z_{35} z_{16} z_4\right) + z_{123} z_{567} + z_4 z_{123567}$$

- Polynomial has $n = 26$ terms
- Only 8 permutations out $26!$ are symmetries
- 8 pairs (T, P) including identity form a group

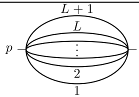
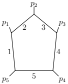
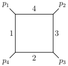
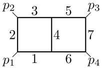
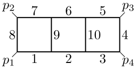
Some pairs change kinematic dependence of integral

$$I_{\mathcal{G}}(\kappa) = \int_{\mathbb{R}^7_+} \mathrm{d}\eta_7 \, \frac{z_1^{-\alpha_{12}+\beta} z_2^{\alpha_2} z_3^{-\alpha_{23}+\beta} z_4^{\alpha_{1234567}-2\beta} z_5^{-\alpha_{57}+\beta} z_6^{-\alpha_{67}} z_7^{\alpha_7}}{\mathcal{G}_{\sigma_1}^{\beta}},$$

$$\begin{aligned}\mathcal{G}_{\sigma_1}(z) = {} & t z_2 z_4 z_7 + s z_4\left(z_{123567} + z_6 z_{35} + z_1 z_{35}\right) \\ & + z_{23}\left(z_{567} + z_5 z_6\right) + z_1\left(z_{567} + z_5 z_{36} + z_3 z_{67}\right)\end{aligned}$$

Results for higher loops and legs

Results

Massive bananas	on-shell n-gons	On-shell ladders		
				
$(L+1)!$	2 for $n \geq 5$	72	8	4

Finite ladders

- Set $d = 6$
- We found

$$\mathcal{I}_{\text{ladder}}(\mathsf{T}\kappa, c\mathsf{P}) = \mathcal{I}_{\text{ladder}}(\kappa, c\mathsf{P})$$

- Finite ladders are invariant under T
- But ladders "ran out" of symmetries at 3-loops ...
- With numerators there are more possibilities

Summary and Outlook

Summary

- Feynman integral inherit symmetries from their parent A-hypergeometric functions
- These are controlled by math's theorem Forsgård-Matusevich-Sobieska, 17'
- We can apply it to compute symmetries as linear algebra problem

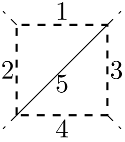
$$\mathsf{TAP}_{\mathsf{try}} - \mathsf{A} = 0$$

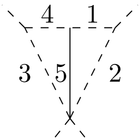
PALP

- We computed symmetries of massive bananas, ladders, and n-gons

Outlook

- Relation to Pak's algorithm: this is also a polytope symmetry!





Thank you!