# Finite Feynman Integrals in Momentum and Parameter Space

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IPhT CEA/Saclay

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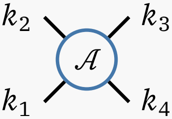
# Motivation



$$= c_1 \text{Master}_1 + \cdots + c_N \text{Master}_N$$

1. IR factorization: $\mathcal{A}^{(L)} = \mathcal{F}^{(L)} + \mathcal{A}_{\text{IR}}^{(<L)}$
   ☞ organize integrals according to their divergence properties
   [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka 2011; Badger, Mogull, Peraro 2016; Henn, Peraro, Stahlhofen, Wasser 2019]

2. Finite integrals are simpler to evaluate
   ☞ (quasi-)finite basis [von Manteuffel, Panzer, Schabinger 2015]

# Questions

1. Classification
   - locally finite
   - evanescently finite: $1/\epsilon$ divergence × $\mathcal{O}(\epsilon)$ integrand
   - evanescent: $\mathcal{O}(\epsilon)$
2. Construction
   - momentum space
   - parameter space
3. Evaluation
   - numerical
   - IBPs, differential equations, ...

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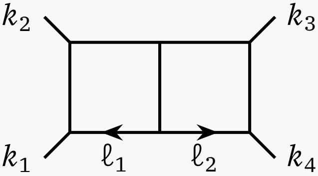
# Constructing finite integrals in momentum space

# Setup

Find Num such that $\int \prod_{i=1}^{\mathrm{L}} \mathrm{d}^4 \ell \, \frac{\mathrm{Num}}{\mathrm{Den}_1 \cdots \mathrm{Den}_{\mathrm{E}}} < \infty$



$$\mathrm{Num} = \mathrm{Poly}\left(\ell_i \cdot \ell_j \,, \ell_i \cdot k_j\right) \text{ whose coefficients are } \mathrm{Rational}\left(k_i \cdot k_j \,, m_i\right)$$

$$\mathrm{Den} = \left(\sum \pm \ell_i \pm k_j\right)^2 - m^2 + i\delta$$

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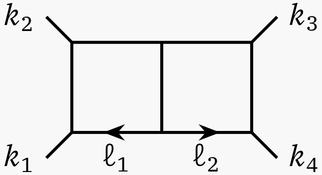

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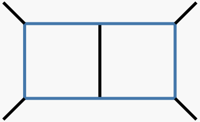
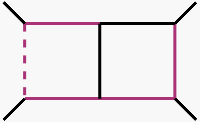
locally finite ~ absolutely convergent: $\int \mathrm{d}x\, f(x) = \int_{f(x)>0} \mathrm{d}x\, |f(x)| - \int_{f(x)<0} \mathrm{d}x\, |f(x)|$

# Strategy

| | | divergent surface | power-counting rule |
|---|---|---|---|
| UV |  | $\left\{ \begin{array}{l} \ell_1 = \infty \\ \ell_1 + \ell_2 = C \end{array} \right\}$ | $\left\{ \begin{array}{l} \ell_1 \to \lambda \ell \\ \ell_2 \to C - \lambda \ell \end{array} \right\}$ |
| IR |  | $\left\{ \begin{array}{l} \ell_1 = k_1 \\ \ell_2 = x k_4 \end{array} \right\}$ | $\left\{ \begin{array}{l} \ell_1 \to k_1 + \lambda^2 \ell_s \\ \ell_2 \to x k_4 + \lambda^2 \eta_4 + \lambda \ell_\perp \end{array} \right\}$ |

Constrain the ansatz: $\text{Num} = c_1 + c_2 \, (\ell_1 \cdot k_1) + \cdots + c_n \left(\ell_2^2\right)^2 (\ell_2 \cdot k_3)$

# IR divergences from Landau equations

$$\forall\, \ell_i \;:\; \frac{\partial}{\partial \ell_i} \sum_e \alpha_e \mathrm{Den}_e = 0$$

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# IR divergences from Landau equations

$$\forall\, \ell_i \,:\, \frac{\partial}{\partial \ell_i} \sum_e \alpha_e \mathrm{Den}_e = 0 \xrightarrow{\mathcal{U}(\alpha) \neq 0} \begin{cases} \ell_1 = \dfrac{\alpha_1}{\alpha_1 + \alpha_2} k_1 \\ \ell_2 = \dots \end{cases}$$

$$\forall\, \alpha_e \,:\, \alpha_e \mathrm{Den}_e = 0$$

# IR divergences from Landau equations

$$\begin{tikzcd}
\forall\, \ell_i \;:\; \frac{\partial}{\partial \ell_i} \sum_e \alpha_e \mathrm{Den}_e = 0 \arrow[r, "\mathcal{U}(\alpha) \neq 0"] & \begin{cases} \ell_1 = \dfrac{\alpha_1}{\alpha_1 + \alpha_2} k_1 \\ \ell_2 = \dots \end{cases} \arrow[d] \\
\forall\, \alpha_e \;:\; \alpha_e \mathrm{Den}_e = 0 \arrow[r] & \forall\, \alpha_e \;:\; \alpha_e \frac{\partial}{\partial \alpha_e} \mathcal{F}(\alpha) = 0
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\end{tikzcd}$$

in many cases [Stephen's talk]

can form a subtraction-free combination:

$$\sum_e w_e \left( \alpha_e \frac{\partial}{\partial \alpha_e} \mathcal{F}(\alpha) \right) = 0$$

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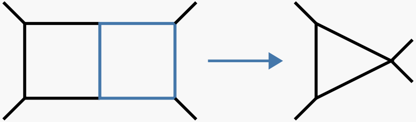
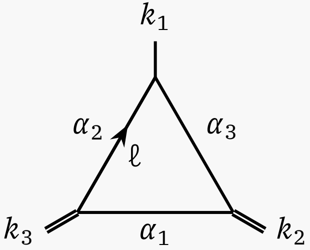
degenerate case: $\mathcal{U}(\alpha) = 0$



# Example



$$\frac{\partial}{\partial \ell}\left(\alpha_1 (\ell + k_3)^2 + \alpha_2 \ell^2 + \alpha_3 (\ell - k_1)^2\right) = 0$$

$$\Rightarrow \quad \ell = \frac{\alpha_3 k_1 - \alpha_1 k_3}{\alpha_1 + \alpha_2 + \alpha_3}$$

$$\forall\, \alpha_e \;: \qquad \alpha_e \frac{\partial}{\partial \alpha_e}\left(m_3^2 \alpha_1 \alpha_2 + m_2^2 \alpha_1 \alpha_3\right) = 0$$

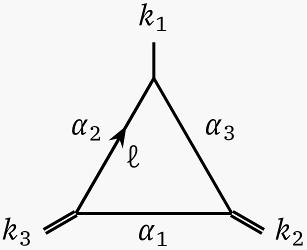
$$\Rightarrow \quad \alpha_1 = 0$$

$$\ell = \frac{\alpha_3}{\alpha_2 + \alpha_3} k_1$$

$$\Rightarrow \quad \alpha_2 = \alpha_3 = 0$$

$$\ell = -k_3$$

# Example



$$\frac{\partial}{\partial \ell}\left(\alpha_1\,(\ell+k_3)^2+\alpha_2\ell^2+\alpha_3\,(\ell-k_1)^2\right)=0$$

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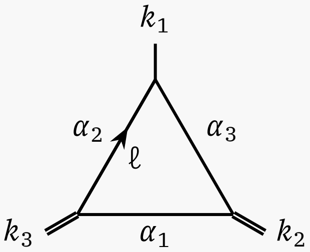
$$\ell=-k_3$$

by power counting

# Example



$$\mathrm{Num} = c_1 + c_2\,(\ell\cdot k_1) + c_3\,(\ell\cdot k_2)$$

$$\mathrm{Num}\,(\ell = x k_1) = 0$$

$$c_1 + x\,(k_1\cdot k_2)\,c_3 = 0$$

$$\Rightarrow\quad c_1 = c_3 = 0$$

$$\Rightarrow\quad \mathrm{Num} = (\ell\cdot k_1)$$

$$\Rightarrow\quad \int \mathrm{d}^4\ell \frac{(\ell\cdot k_1)}{\ell^2\,(\ell-k_1)^2\,(\ell+k_3)^2} < \infty$$

# Constructing finite integrals in parameter space

# From momentum to parameter space

$$\int \prod_{i=1}^{\mathrm{L}} \mathrm{d}^{\mathrm{D}} \ell_i \, \frac{\mathrm{Num}\,(\ell, k)}{\mathrm{Den}_1 \cdots \mathrm{Den}_{\mathrm{E}}}$$

$r$ is rank of $\mathrm{Num}(\ell, k)$

$s = \{(k_i \cdot k_j)\,, m_i\}$

↓

$$\Gamma\big(\mathrm{E} - \mathrm{LD}/2 - \lfloor r/2 \rfloor\big) \int_{\mathbb{R}^+} \mathrm{d}^{\mathrm{E}-1}\alpha \; \mathrm{Num}(\alpha; s, \mathrm{D}) \; \mathcal{U}(\alpha)^{\mathrm{E}-(\mathrm{L}+1)\mathrm{D}/2-r} \; \mathcal{F}(\alpha; s)^{\mathrm{LD}/2-\mathrm{E}}$$

- $\mathrm{Num}(\alpha; s, \mathrm{D})$ is a polynomial in $\alpha$s of degree $r\mathrm{L}$
- not every $\mathrm{Num}(\alpha; s, \mathrm{D})$ can be obtained from $\mathrm{Num}(\ell, k)$
- for some $\mathrm{Num}(\ell, k) \neq 0$ we have $\mathrm{Num}(\alpha; s, \mathrm{D}) = 0$

# Convergence in parameter space

$$\int_{\mathbb{R}^+} \mathrm{d}^n\alpha \, \frac{\mathrm{Num}(\alpha)}{\mathrm{Den}(\alpha)} = \sum_x c_x \int_{\mathbb{R}^+} \frac{\mathrm{d}\alpha_1}{\alpha_1} \cdots \frac{\mathrm{d}\alpha_n}{\alpha_n} \frac{\alpha_1^{x_1} \cdots \alpha_n^{x_n}}{\mathrm{Den}(\alpha)}$$

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Euler–Mellin integral [Pierpaolo’s talk]

Newton polytope [Leonardo’s talk]

Theorem: $\int$ converges $\Leftrightarrow (x_1, \dots, x_n) \in \mathrm{Newt}\,(\mathrm{Den})$

[Nilsson, Passare 2013; Berkesch, Forsgård, Passare 2014]

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✗ theorem probes only end-point divergences

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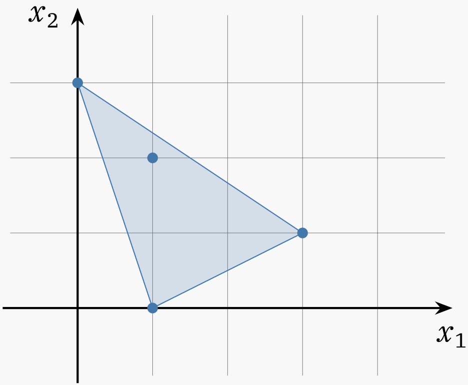
- ✗ theorem applies to single terms of $\mathrm{Num}(\alpha)$
  - ✓ can show there are no cancellations between terms
- ✗ theorem probes only end-point divergences
  - ✓ map inner divergences to the boundary [Stephen's talk]

# Convergence in parameter space



$$\mathrm{Den}(\alpha) = \sum_{x \in X} c_x \alpha^x = \alpha_1 + \alpha_1^3 \alpha_2 + \alpha_1 \alpha_2^2 + \alpha_2^3$$

$$X = \{(1,0),(3,1),(1,2),(0,3)\}$$

$$\mathrm{Newt}(\mathrm{Den}) = \mathrm{conv}(X)$$

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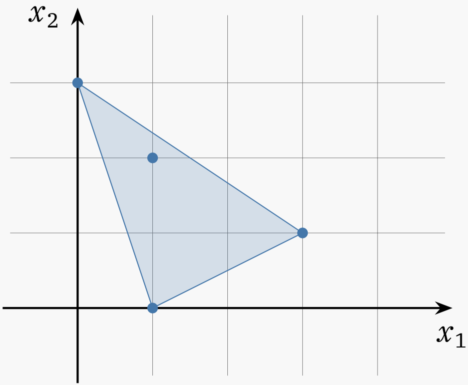

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$$\alpha_i \to \rho^{w_i}\alpha_i \qquad \alpha^x \sim \rho^{w\cdot x} \qquad \rho \to \infty$$

$$\lim_{\rho\to\infty} \mathrm{Den} = ? \qquad w\cdot x \to \max_{x\in X}$$

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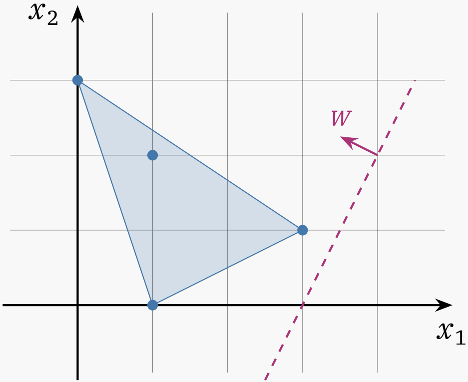

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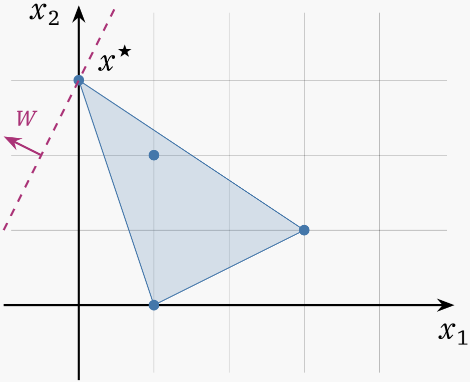

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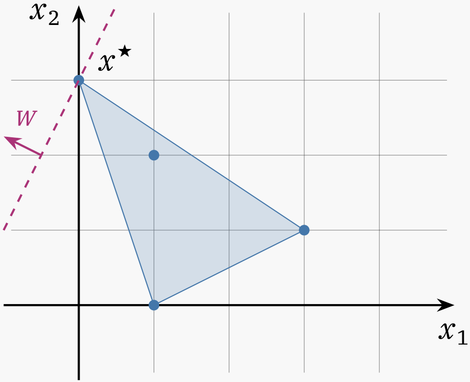

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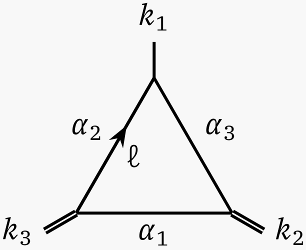
$\alpha^x \subset$ Num cannot become dominant (for any choice of $w$) if $x$ lies inside Newt

# Example



$$\mathrm{Num}(\ell) = c_1 + c_2\,(\ell\cdot k_1) + c_3\,(\ell\cdot k_2) \;\;\rightarrow\;\; \int_{\mathbb{R}^+} \frac{\mathrm{d}\alpha_1}{\alpha_1}\frac{\mathrm{d}\alpha_2}{\alpha_2}\frac{\mathrm{Num}(\alpha)}{\mathcal{U}^2(\alpha)\mathcal{F}(\alpha)}$$

$$\begin{aligned}\mathrm{Num}(\alpha) = {}& 2c_1\alpha_1\alpha_2^2 + \left[2c_1 + \left(m_3^2 - m_2^2\right)c_3\right]\alpha_1\alpha_2 \\ &+ \left[2c_1 + \left(m_3^2 - m_2^2\right)c_2 + \left(m_2^2 + m_3^2\right)c_3\right]\alpha_1^2\alpha_2\end{aligned}$$

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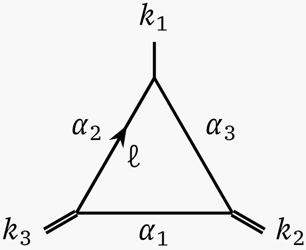

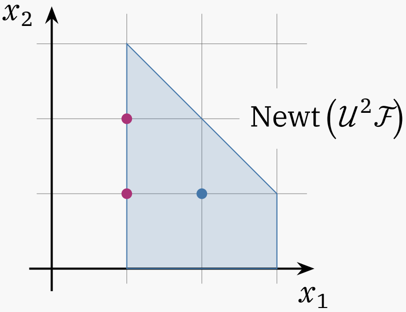

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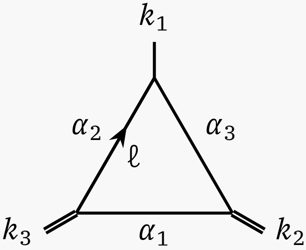
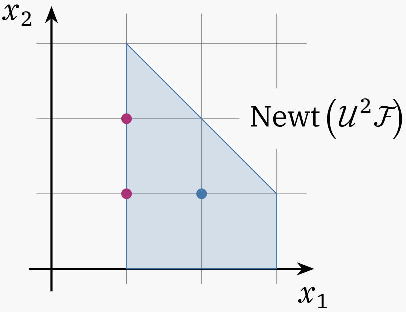


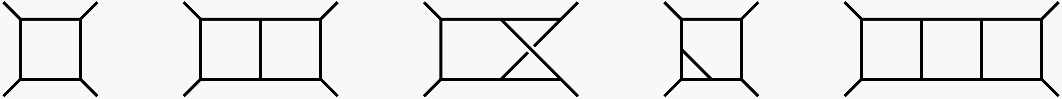


$$\mathrm{Num}(\ell) = c_1 + c_2\,(\ell\cdot k_1) + c_3\,(\ell\cdot k_2) \;\rightarrow\; \int_{\mathbb{R}^+} \frac{\mathrm{d}\alpha_1}{\alpha_1}\frac{\mathrm{d}\alpha_2}{\alpha_2}\frac{\mathrm{Num}(\alpha)}{\mathcal{U}^2(\alpha)\mathcal{F}(\alpha)}$$

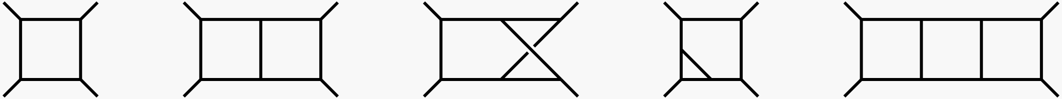
$$\begin{aligned}\mathrm{Num}(\alpha) &= 2c_1\alpha_1\alpha_2^2 + \left[2c_1 + \left(m_3^2 - m_2^2\right)c_3\right]\alpha_1\alpha_2 \\ &\quad + \left[2c_1 + \left(m_3^2 - m_2^2\right)c_2 + \left(m_2^2 + m_3^2\right)c_3\right]\alpha_1^2\alpha_2\end{aligned} \;\Rightarrow\; \begin{aligned} &c_1 = c_3 = 0 \\ &\mathrm{Num}(\ell) = (\ell\cdot k_1)\end{aligned}$$

# Momentum space vs. parameter space

# Results

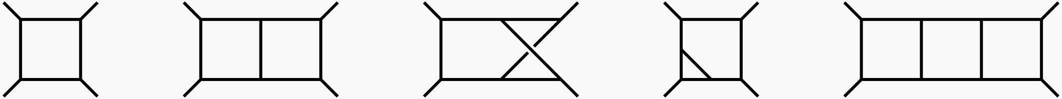


# Results



momentum loc. finite ⊂ parameter loc. finite

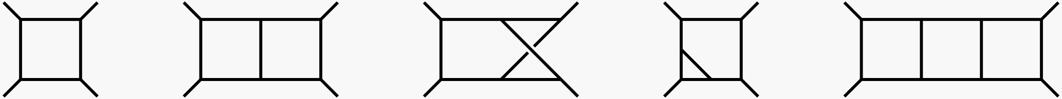
# Results



$$\begin{array}{c}\text{momentum}\\ \text{loc. finite}\end{array} \subset \begin{array}{c}\text{parameter}\\ \text{loc. finite}\end{array}$$

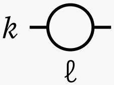
$$\begin{array}{c}\text{parameter}\\ \text{loc. finite}\end{array} = \begin{array}{c}\text{momentum}\\ \text{loc. finite}\end{array} + \begin{array}{c}\text{weakly}\\ \text{UV finite}\end{array} + \begin{array}{c}\text{total}\\ \text{derivatives}\end{array}$$

# Results



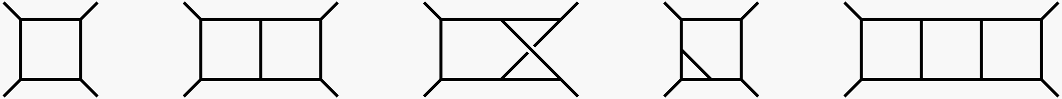
momentum loc. finite $\subset$ parameter loc. finite

parameter loc. finite $=$ momentum loc. finite $+$ **weakly UV finite** $+$ total derivatives



$$\left[(\ell\cdot k)^2\right] = \int \mathrm{d}^2\ell\,\frac{(\ell\cdot k)^2}{(\ell^2-m^2)\left((\ell+k)^2-m^2\right)} \overset{\ell\to\infty}{\sim} k_\mu k_\nu \underbrace{\int \mathrm{d}^2\ell\,\frac{\ell^\mu\ell^\nu}{(\ell^2)^2}}_{\propto\,\eta^{\mu\nu}} = 0$$

# Results

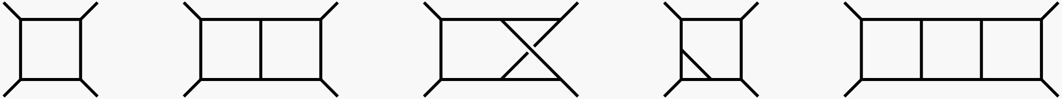


$$\begin{matrix}\text{momentum}\\ \text{loc. finite}\end{matrix} \subset \begin{matrix}\text{parameter}\\ \text{loc. finite}\end{matrix}$$

$$\underbrace{\begin{matrix}\text{parameter}\\ \text{loc. finite}\end{matrix}}_{\text{IBP reduce}} = \underbrace{\begin{matrix}\text{momentum}\\ \text{loc. finite}\end{matrix} + \begin{matrix}\text{weakly}\\ \text{UV finite}\end{matrix}}_{\text{IBP reduce}} + \begin{matrix}\text{total}\\ \text{derivatives}\end{matrix}$$

compare as linear spaces
over D-independent coefficients

# Results



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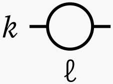
# UV power counting (seemingly) doesn't work



$$\left[(\ell\cdot k)\right] = \int \mathrm{d}^2\ell \frac{(\ell\cdot k)}{(\ell^2 - m^2 + i\delta)\left((\ell+k)^2 - m^2 + i\delta\right)}$$

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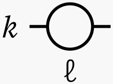

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$$= \int \mathrm{d}\ell_+ \mathrm{d}\ell_- \frac{\ell_+ k_- + \ell_- k_+}{\left(\ell_+ \ell_- - m^2 + i\delta\right)\left((\ell_+ + k_+)(\ell_- + k_-) - m^2 + i\delta\right)} \to \infty$$

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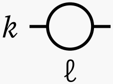

$$\left[(\ell \cdot k)\right] = \int \mathrm{d}^2\ell \, \frac{(\ell \cdot k)}{\left(\ell^2 - m^2 + i\delta\right)\left((\ell + k)^2 - m^2 + i\delta\right)}$$

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Weinberg's theorem assumes Euclidean denominators:

$$\frac{1}{q_0^2 - \vec{q}^2 - m^2 + i\delta} \to \frac{1}{q_0^2 + \vec{q}^2 + m^2}$$

# Absolute convergence with Zimmermann's $i\delta$

$$\begin{tikzcd}
\dfrac{1}{q_0^2 - \vec{q}^{\,2} - m^2 + i\delta_F} \arrow[d] \\
\begin{array}{c}\text{momentum repr.}\\ \text{with } i\delta_F\end{array} \arrow[d, "\times" marking] \\
\begin{array}{c}\text{parametric repr.}\\ \text{with } i\delta_F\end{array}
\end{tikzcd}$$

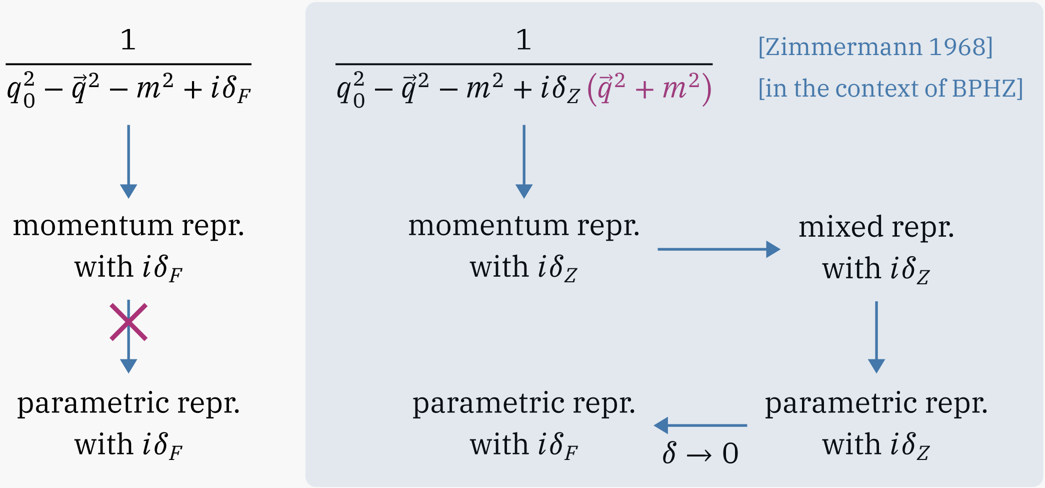
$$\begin{tikzcd}
\dfrac{1}{q_0^2 - \vec{q}^{\,2} - m^2 + i\delta_Z\left(\vec{q}^{\,2} + m^2\right)} \arrow[d] & \\
\begin{array}{c}\text{momentum repr.}\\ \text{with } i\delta_Z\end{array} \arrow[r] & \begin{array}{c}\text{mixed repr.}\\ \text{with } i\delta_Z\end{array} \arrow[d] \\
\begin{array}{c}\text{parametric repr.}\\ \text{with } i\delta_F\end{array} & \begin{array}{c}\text{parametric repr.}\\ \text{with } i\delta_Z\end{array} \arrow[l, "\delta \to 0"']
\end{tikzcd}$$

[Zimmermann 1968]

[in the context of BPHZ]

[Lowenstein, Zimmermann 1975; Lowenstein, Speer 1976]

# Absolute convergence with Zimmermann's $i\delta$



$$\frac{1}{q_0^2 - \vec{q}^2 - m^2 + i\delta_F}$$

momentum repr.
with $i\delta_F$

parametric repr.
with $i\delta_F$

$$\frac{1}{q_0^2 - \vec{q}^2 - m^2 + i\delta_Z\,(\vec{q}^2 + m^2)}$$

[Zimmermann 1968]
[in the context of BPHZ]

momentum repr.
with $i\delta_Z$

mixed repr.
with $i\delta_Z$

parametric repr.
with $i\delta_Z$

$\delta \to 0$

parametric repr.
with $i\delta_F$

[Lowenstein, Zimmermann 1975; Lowenstein, Speer 1976]

# Summary

1. We can systematically construct locally finite Feynman integrals both in momentum and parameter space

2. Parameter-space locally finite $\nRightarrow$ momentum-space locally finite

3. Momentum-space locally finite $\Rightarrow$ parameter-space locally finite $\Rightarrow$ composed of "interior" monomials:
   - can cross-check momentum-space results
   - can expand in powers of $\epsilon$ directly under the integral sign
   - can compute numerically (term by term $\Rightarrow$ numerical stability)

# Summary

1. We can systematically construct locally finite Feynman integrals both in momentum and parameter space
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Thank you!