QCD splitting functions at four loops

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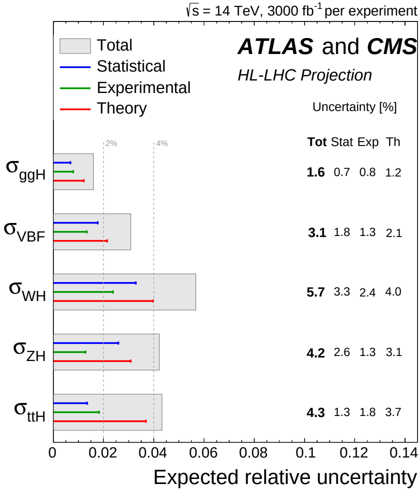
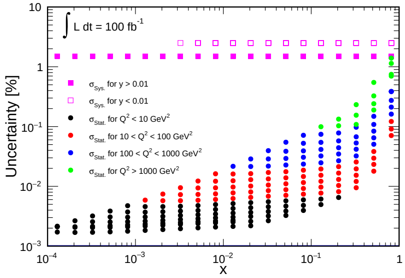
Outline

Outline

High Precision for Hard Processes



(CERN Yellow Reports 2019)



(EIC Yellow Report 2021) Experimental errors O(1%)

- Higgs production at the HL-LHC
- Inclusive DIS at the EIC

QCD at N^{3}LO

	Q [GeV]	$\delta\sigma^{\mathrm{N^3LO}}$	δ(scale)	δ(PDF-TH)
NCDY	100	-2.1%	$^{+0.66\%}_{-0.79\%}$	$\pm$2.5%
CCDY(W^+)	150	-2.0%	$^{+0.5\%}_{-0.5\%}$	$\pm$2.1%
CCDY(W^-)	150	-2.1%	$^{+0.6\%}_{-0.5\%}$	$\pm$2.13%

J. Baglio, C. Duhr, B. Mistlberger, R. Szafron 2022

- N^{3}LO corrections $O(\%)$, scale uncertainties sub-percent.
- δ(PDF-TH): Additional error due missing N^{3}LO PDFs.

$$\delta(\text{PDF-TH}) = \frac{1}{2}\left|\frac{\sigma^{\text{NNLO}}(\text{NNLO PDF}) - \sigma^{\text{NNLO}}(\text{NLO PDF})}{\sigma^{\text{NNLO}}(\text{NNLO PDF})}\right|$$

Scale evolution of the PDFs

Flavour Non-Singlet

$q_{ns}^{\pm} = (f_{\mathrm{i}} \pm f_{\bar{\mathrm{i}}}) - (f_{\mathrm{k}} \pm f_{\bar{\mathrm{k}}})$, $q_{ns}^{\mathrm{v}} = \sum_i f_i - f_{\bar{i}}$

$$\mu^2 \frac{d}{d\mu^2} q_{ns} = \int_x^1 \frac{dz}{z} P_{ns}(z) q_{ns}\left(\frac{x}{z}, \mu^2\right) = P_{ns} \otimes q_{ns}$$

Flavour Singlet

- Quark singlet $q_s(x, \mu^2) = \sum_{\mathrm{i=u,d,s}} (f_{\mathrm{i}} + f_{\bar{\mathrm{i}}})$,
- Gluon distribution $f_g(x, \mu^2)$

$$\mu^2 \frac{d}{d\mu^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix}$$

Towards N^{3}LO splitting functions

$$P_{\rm ij}(\alpha_s, x) = \underbrace{a\, P^{(0)}_{\rm ij}}_{\rm LO} + \underbrace{a^2\, P^{(1)}_{\rm ij}}_{\rm NLO} + \underbrace{a^3\, P^{(2)}_{\rm ij}}_{\rm NNLO} + \underbrace{a^4\, P^{(3)}_{\rm ij}}_{\rm N^3LO},\; a = \frac{\alpha_s}{4\pi}$$

- **Exact results**
  - $P^{(3)}_{\rm ns}$ in the **planar limit** (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017)
  - $P^{(3)}_{\rm ij}$ in the large-n_f limit (Gracey 1994,1998; Davies,Vogt,Ruijl,Ueda,Vermaseren 2016)
  - n_f^2 term in $P^{(3)}_{\rm qq}$ (Gehrmann,von Manteuffel,Sotnikov,Yang 2023)
  - n_f^2 term in $P^{(3)}_{\rm gq}$ (**GF**,Herzog,Moch,Vermaseren,Vogt 2023)
  - Flavour non-singlet: $n_f C_F^3$ term (Gehrmann,von Manteuffel,Sotnikov,Yang 2023)
- **High-precision approximations**
  - $q_{\rm ns}$ evolution to 1%-level (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017).

Outline

Moments of the splitting functions

$$\gamma_{\mathrm{ij}}^{(k)}(N) = -\int_0^1 dx\, x^{N-1}\, P_{\mathrm{ij}}^{(k)}(x)$$

Computational strategies

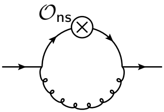
- DIS structure functions: $\gamma_{\mathrm{qq}}^{(3)}(N \leq 12)$; $\gamma_{\mathrm{qg}}^{(3)}, \gamma_{\mathrm{gq}}^{(3)}, \gamma_{\mathrm{gg}}^{(3)}(N \leq 10)$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2021-2023)
- Operator anomalous dimensions (Gross,Wilczek 1974)

$$\mu^2 \frac{d}{d\mu^2}\mathcal{O}_{\mathrm{i}}(N) = \gamma_{\mathrm{ij}}(N)\,\mathcal{O}_{\mathrm{j}}(N),$$

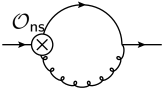
with $\mathcal{O}_{\mathrm{g}}$ ($\mathcal{O}_{\mathrm{q}}$) gluon (quark) operator of twist 2 spin N.

Off-shell renormalisation

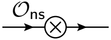
Moments from the poles of Operator Matrix Elements $\mathrm{A_{ij}} = \langle \mathrm{j} | \mathcal{O}_\mathrm{i} | \mathrm{j} \rangle$



$+ 2 \times$



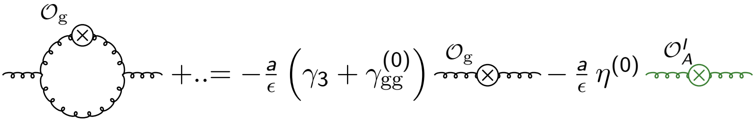
$= -\frac{1}{\epsilon} \left(\gamma_\Psi + \gamma_\mathrm{ns}^{(0)}(N) \right)$



$+ O(\epsilon^0)$

- The calculation of the OMEs $\mathrm{A_{ij}}$ is automated to **4 loops** $\rightarrow$ `Forcer` (Ruijl,Ueda,Vermaseren 2017)
- $\gamma_\mathrm{ns}^{(3)}$ computed up to $N = 16$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017) and extended to $N = 22$ (Moch,Ruijl,Ueda,Vermaseren,Vogt to appear)

Troubles in the singlet sector



- Presence of unphysical *alien* operators (Gross,Wilczek 1974)

$$\mathcal{O}_A^I = \eta\,(\partial^2 A^a - \partial(\partial.A^a))(\partial^{N-2} A^a) + \ldots$$

- $\eta = \frac{a}{\epsilon}\eta^{(0)} + a^2\eta^{(1)} + \ldots$ is the mixing between $\mathcal{O}_g$ and $\mathcal{O}_A^I$
- $\eta^{(0)} = -\frac{C_A}{N(N-1)}$ (Dixon,Taylor 1974; Hamberg,van Neerven 1991)
  $\eta^{(1)}$ (Hamberg,van Neerven 1991)
  $\eta^{(2)}$ (Gehrmann,von Manteuffel,Yang 2023)

Venturing out to two loops

It took over 15 years to show that one needs
(Dixon,Taylor 1974 → Hamberg,van Neerven 1991)

$$\mathcal{O}_{\mathrm{EOM}} = \underbrace{\eta\,(D.F)^a\,\left(\partial^{N-2}A^a\right)}_{\mathcal{O}_A^I} + \underbrace{g\,f^{aa_1a_2}\,(D.F)^a \sum_{\substack{i,j\geq 0\\ i+j=N-3}} \kappa_{ij}^{(1)}\,\left(\partial^i A^{a_1}\right)\left(\partial^j A^{a_2}\right)}_{\mathcal{O}_A^{II}} + ..,$$

$$\mathcal{O}_c = \underbrace{-\eta\,\left(\partial\bar{c}^a\right)\left(\partial^{N-1}c^a\right)}_{\mathcal{O}_c^I} - \underbrace{g\,f^{aa_1a_2}\,\left(\partial\bar{c}^a\right) \sum_{\substack{i,j\geq 0\\ i+j=N-3}} \eta_{ij}^{(1)}\,\left(\partial^i A^{a_1}\right)\left(\partial^j c^{1+a_2}\right)}_{\mathcal{O}_c^{II}} + ..,$$

- $\kappa_{ij}^{(1)}$, $\eta_{ij}^{(1)}$ are 3-point counterterms  $+..\Big|_{\frac{1}{\epsilon}} = -\sum_{i,j}\eta_{ij}^{(1)}$ 

Systematic constructions to four loops

General operator structure - (**GF**,Herzog 2022)

- $\mathcal{O}_{\mathrm{EOM}} \to$ generated by a field redefinition.
- $\mathcal{O}_c \to$ generated by a BRST-type operator.

The construction automatically implies relations e.g.

$$\eta_{ij}^{(1)} = 2\kappa_{ij}^{(1)} + \eta(N) \begin{pmatrix} i+j+1 \\ i \end{pmatrix} = -\sum_{s=0}^{i} (-1)^{s+j} \begin{pmatrix} s+j \\ j \end{pmatrix} \eta_{i-s,j+s}^{(1)} \qquad (1)$$

- **Fixed** N: pick a basis of independent operators
- $\eta_{ij}^{(1)}, \kappa_{ij}^{(1)}$.. computed with R^*-based (Herzog,Ruijl 2017) approach (**GF**,Herzog,Moch,Pelloni,Vogt 2024)
- Agreement with direct 3-loop calculation (Gehrmann,von Manteuffel,Yang 2023)

Results - fixed moments

"Top row" $\gamma_{\rm qq}^{(3)}$ and $\gamma_{\rm qg}^{(3)}$

- $\gamma_{\rm qq}^{(3)}(N \leq 20)$ 2302.07593 - (**GF**,Herzog,Moch,Vogt)
- $\gamma_{\rm qg}^{(3)}(N \leq 20)$ 2307.04158 - (**GF**,Herzog,Moch,Vogt)

"Bottom row" $\gamma_{\rm gq}^{(3)}$ and $\gamma_{\rm gg}^{(3)}$

- $\gamma_{\rm gq}^{(3)}(N \leq 20)$ 2404.09701 - (**GF**,Herzog,Moch,Pelloni,Vogt)
- **NEW $\gamma_{\rm gg}^{(3)}(N \leq 20)$** to appear - (**GF**,Herzog,Moch,Pelloni,Vogt)

$$\gamma_{\rm gg}^{(3)}(N=2) = 654.463 n_f - 245.611 n_f^2 + 0.924991 n_f^3,$$

$$\dots$$

$$\gamma_{\rm gg}^{(3)}(N=20) = 90499.3 - 26132.3 n_f + 1178.50 n_f^2 + 25.6433 n_f^3.$$

Checks

The moments are compared with the results in the literature for

- Large-n_f (Gracey 1994,1998; Davies,Ruijl,Ueda,Vermaseren,Vogt 2016)
- Quartic Casimir contributions (Moch,Ruijl,Ueda,Vermaseren,Vogt 2018)
- ζ_4 terms from the no-π^2 theorem
  (Jamin,Miravitllas 2018; Baikov,Chetyrkin 2018; Kotikov,Teber 2019)
- $\gamma_{\rm qq}^{(3)}(N \leq 12)$ and $\gamma_{\rm qg,gq,gg}^{(3)}(N \leq 10)$ from DIS structure functions
  (Moch,Ruijl,Ueda,Vermaseren,Vogt 2021-2023)
- $\gamma_{\rm qq}^{(3)}\Big|_{n_f^2}$ (Gehrmann,Sotnikov,von Manteuffel,Yang 2023)

Analytic reconstruction

Use moments to constrain an ansatz for $\gamma_{\rm ij}^{(3)}(N)$

- Denominator structure $D_a = 1/(N+a)$
- Harmonic sums $S_{\pm m_1,\dots m_k} = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\dots,m_k}$

Large ansatz: at weight $w = 7 \rightarrow 2 \cdot 3^{w-1} = 1458$ S-sums.

- Restrict to solutions with **integer** coefficients (Lenstra^2,Lovász 1982)!
- Some further constraints e.g. **reciprocity** $N \rightarrow -1 - N$ (Gribov,Lipatov 1972;Blümlein,Ravindran,van Neerven 2001;Chen,Yang,Zhu,Zhu 2020) in the **non-singlet** (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017).

Complete reconstruction

- Coefficients of ζ_k multiply S of lower weight e.g. **new:** $\gamma_{\rm ij}^{(3)}\big|_{\zeta_5}$

$$\gamma_{gg}(N)\big|_{\zeta_5 C_A^4} = C_A^4\Big(-12016/27 - 1120/3\nu - 640/3\nu^2 + 3008/9\eta + 640\eta^2 - 8/27N(N+1) + 1760/3S_1 + 640/3S_1\left\{2\nu - 2\eta - S_1\right\}\Big)$$

$\eta = D_0 - D_1$, $\nu = D_{-1} - D_2$.

- Colour factors with simpler structure $\rightarrow \gamma_{\rm gq}^{(3)}(N)\Big|_{n_f^2}$
  - Only S-sum up to weight 4
  - Simpler diagrams computed with `Forcer` up to $N = 60$
  - Complete reconstruction of $\gamma_{\rm gq}^{(3)}(N)\Big|_{n_f^2}$ and translation to x-space (**GF**,Herzog,Moch,Vermaseren,Vogt 2023)

Outline

Approximate approach

Parameterise $P_{\mathrm{ij}}^{(3)}(x)$ with an ansatz that

- **reproduces** the computed moments
- **matches** the known endpoint behaviour at $x \to 1$ and $x \to 0$

Several collaborations are developing approximate N^{3}LO PDFs
(Cridge,Harland-Lang,McGowan,Thorne 2022;NNPDF 2024)

Recent benchmarking of different approximate evolutions
(Cooper-Sarkar,Cridge,Giuli,Harland-Lang,Hekhorn,Huston,Magni,Moch,Thorne 2024)

Talk by G. Magni

Example: quark pure singlet $P_{\rm ps} = P_{\rm qq} - P_{\rm ns}^{+}$

Large-x behaviour

$$P_{\rm ps}^{(3)}(x \to 1) = \sum_{j \geq 1} \sum_{k \leq 4} C_{j,k} \, (1-x)^j \log^k(1-x),$$

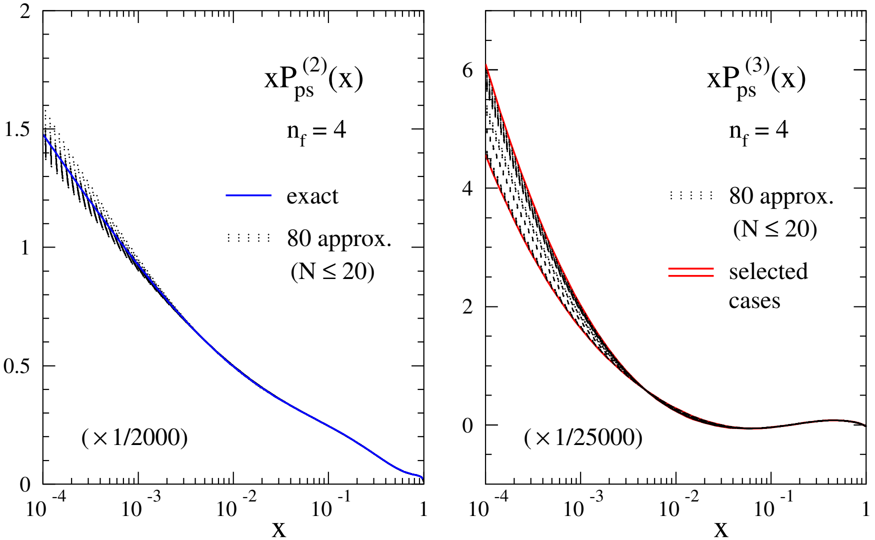
where $C_{1,4}$ and $C_{1,3}$ are known (Soar,Moch,Vermaseren,Vogt 2009)

Small-x behaviour

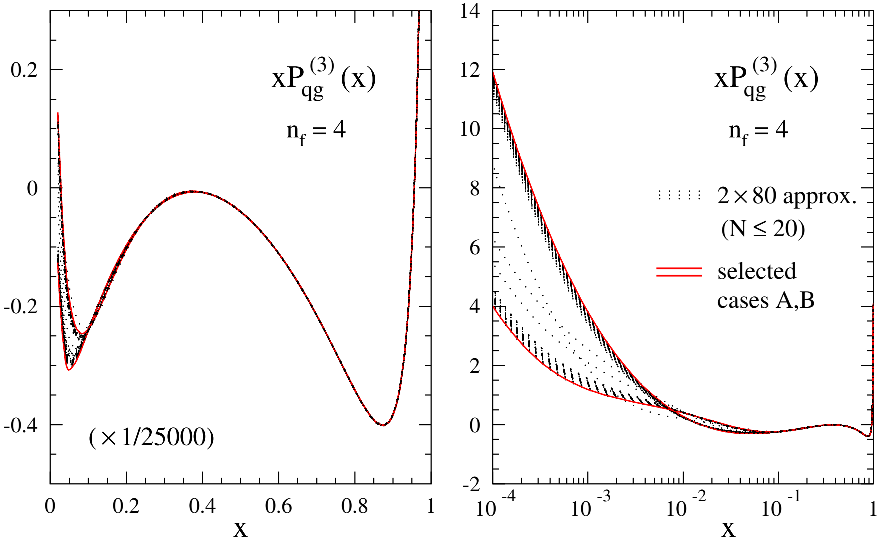
$$P_{\rm ps}^{(3)}(x \to 0) = \sum_{l=1}^{3} E_l \frac{\log^{3-l} x}{x} + \sum_{l=0}^{6} F_l \log^{6-l} x$$

- Leading BFKL logarithm E_1 known (Catani,Hautman 1994)
- Next-to-leading BFKL log in MS-scheme only for $P_{\rm gg}$ (Ciafaloni,Colferai 1999;Ciafaloni,Colferai,Salam,Stasto 2000)
- Non-BFKL logarithm known for $l < 3$ (Davies,Kom,Moch,Vogt 2022)

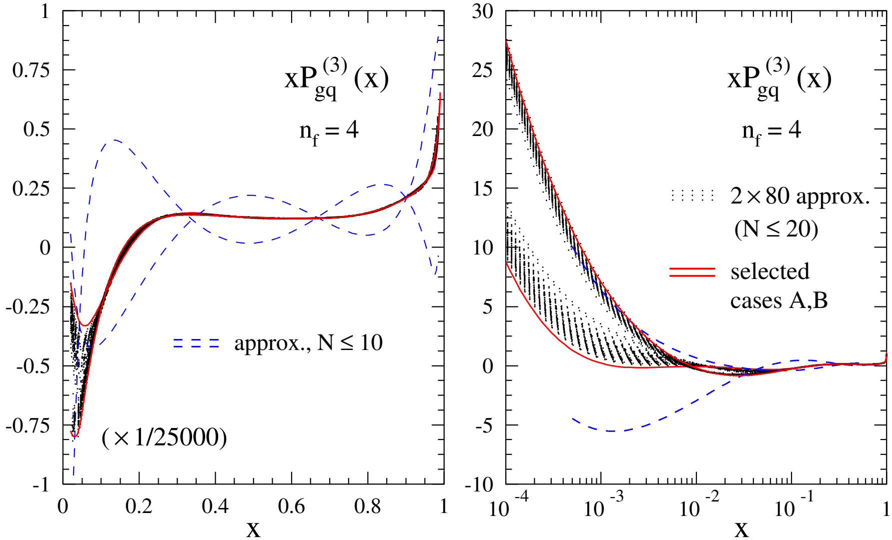
Approximate $P_{\rm ps}^{(3)}(x)$



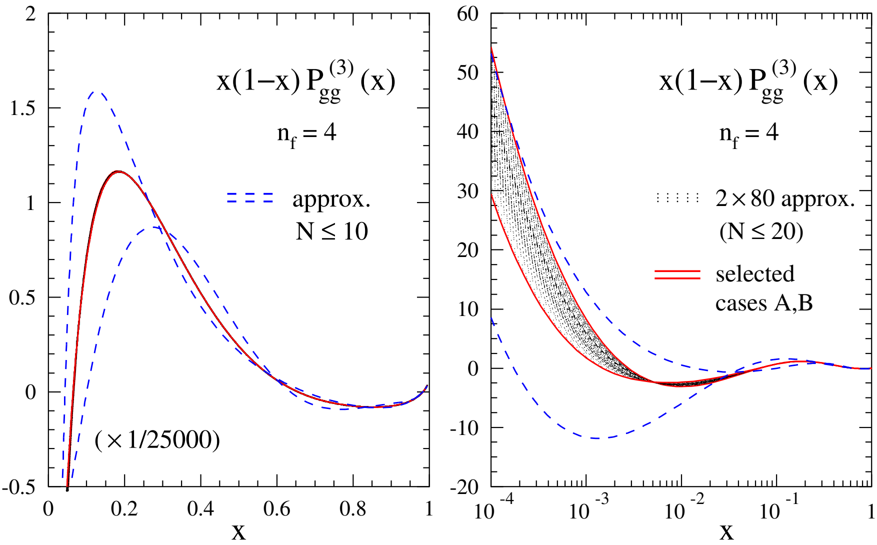
Approximate $P^{(3)}_{qg}(x)$



Approximate $P_{gq}^{(3)}(x)$



NEW: Approximate $P_{gg}^{(3)}(x)$



Evolution of the parton densities

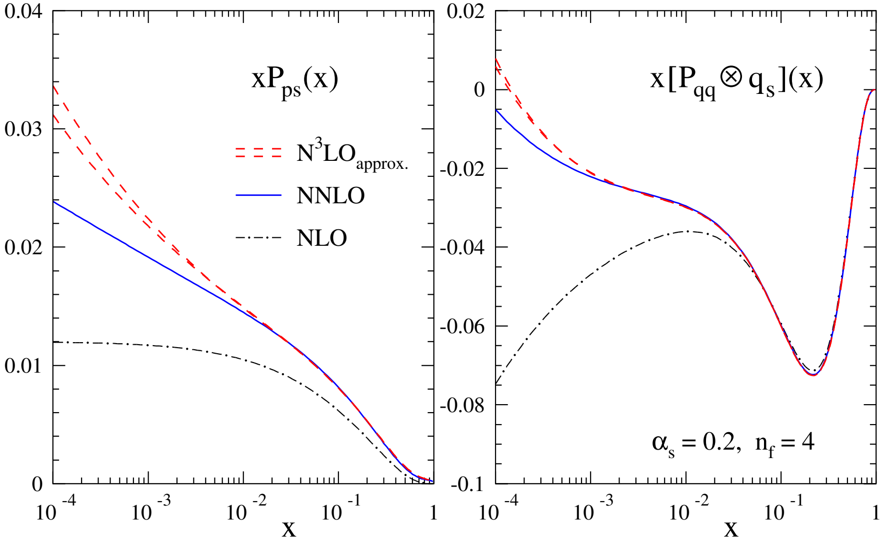
The splitting functions enter the PDF evolution via **convolution**

$$\dot{f}_{\mathrm{i}} \equiv \mu^2 \frac{dq_{\mathrm{i}}}{d\mu^2} = \sum_j \int_x^1 \frac{dz}{z} \underbrace{P_{\mathrm{ij}}(z)}_{\text{approx.}} f_{\mathrm{j}}\left(\frac{x}{z}\right), \quad \mathrm{j} = \mathrm{q}, \mathrm{g}$$

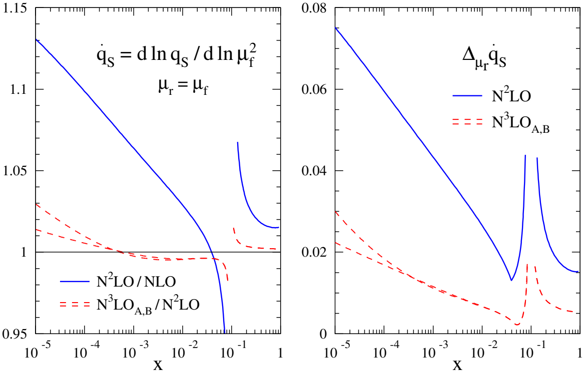
Note the interplay between $P(z \sim x \to 0)$ and $f\left(\frac{x}{z} \to 1\right)$.
$P(z \sim x \to 0)$ has the largest uncertainty
$f\left(\frac{x}{z} \to 1\right)$ is **suppressed**. Model PDFs (Moch,Vermaseren,Vogt 2004)

$$\begin{aligned} x\,g(x) &= 1.6\,x^{-0.3}\,(\mathbf{1-x})^{\mathbf{4.5}}\,(1-0.6x^{0.3}), \\ x\,q_{\mathrm{s}}(x) &= 0.6\,x^{-0.3}\,(\mathbf{1-x})^{\mathbf{3.5}}\,(1+5.0x^{0.8}) \end{aligned}$$

Example: $P_{\mathrm{qq}} \otimes q_{\mathrm{s}}$



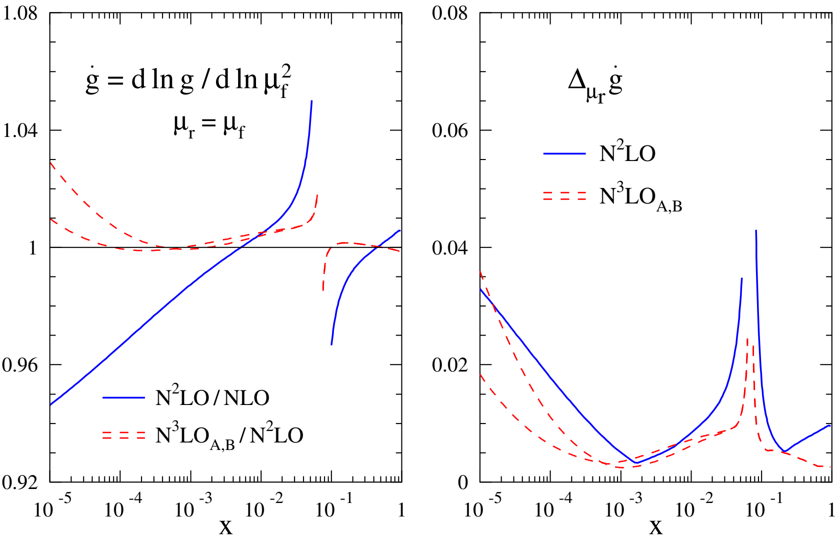
Scale evolution of the quark distribution



Stability under variations of the renormalisation scale

$$\Delta_{\mu_r} \dot{q}_\mathrm{s} = \frac{1}{2} \frac{\max[\dot{q}_\mathrm{s}(\mu_r^2 = \lambda \mu_f^2)] - \min[\dot{q}_\mathrm{s}(\mu_r^2 = \lambda \mu_f^2)]}{\mathrm{average}[\dot{q}_\mathrm{s}(\mu_r^2 = \lambda \mu_f^2)]}, \qquad \lambda = \frac{1}{4} \ldots 4$$

NEW: Scale evolution of the gluon density



Outline

Conclusion

- We computed the moments of **all** the four-loop splitting functions up to $N = 20$.
- We constructed approximate N^{3}LO splitting functions
- Highly precise parameterisation of the **scale evolution** of the parton densities.
  - N^{3}LO corrections below $(2 \pm 1)\%$ up to $x = 10^{-5}$.
  - Renormalisation scale variations $\Delta\dot{q} \sim \Delta\dot{g} \approx 2\%$ for $x \gtrsim 10^{-5}$.
- Can we compute the four-loop splitting functions exactly?

Outlook: all-N results

Our method seems to rely on fixing a value for N.
S. Van Thurenhout noticed **all-N** structure of the *aliens* mixing at leading order

$$\kappa_{ij}^{(1)}, \eta_{ij}^{(1)} \sim \eta(N) \left[a_1 (-1)^j + a_2 \begin{pmatrix} i+j+1 \\ i \end{pmatrix} + a_3 \begin{pmatrix} i+j+1 \\ j \end{pmatrix} \right]$$

This generates an anstaz for the mixing of the **higher point** aliens.

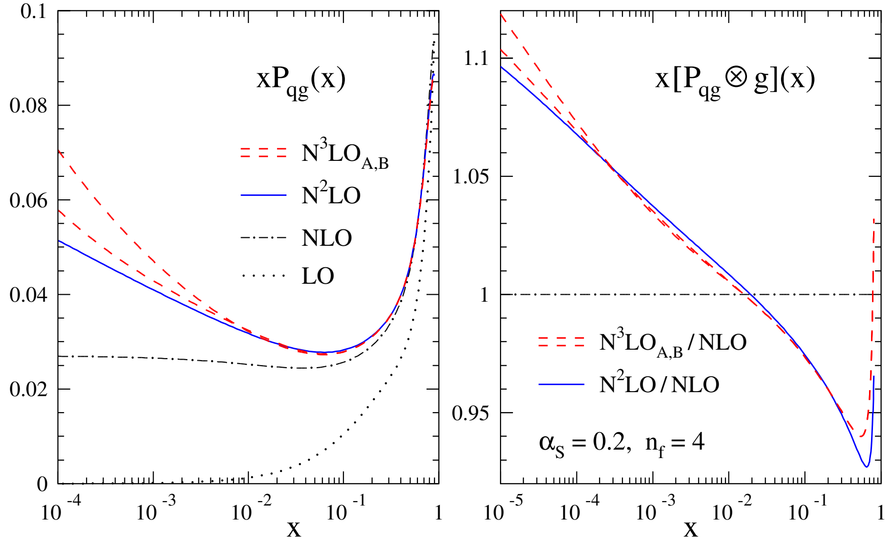
- **NEW:**

Solutions for all the required leading-order mixing constants from symmetry constraints (GF,Herzog,Moch,**Van Thurenhout** 2024).

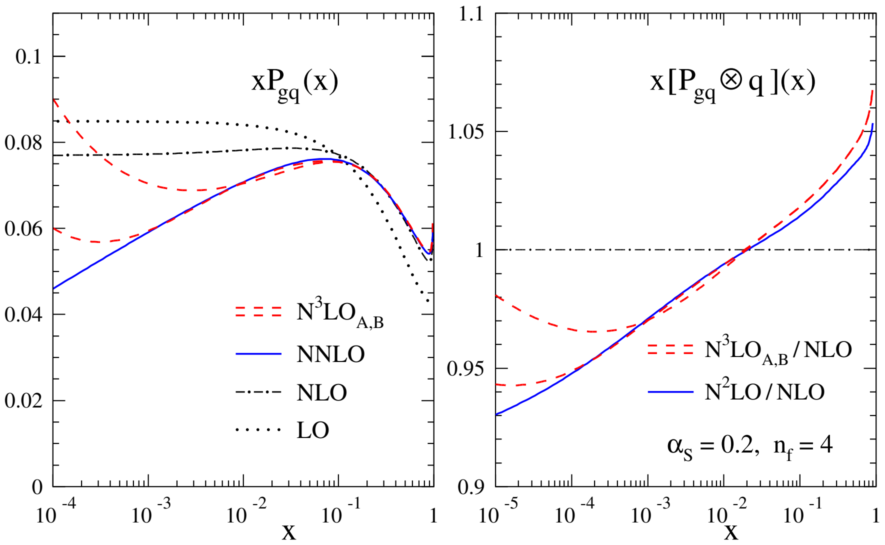
First step towards an all-N generalisation of our approach.

Thank you!

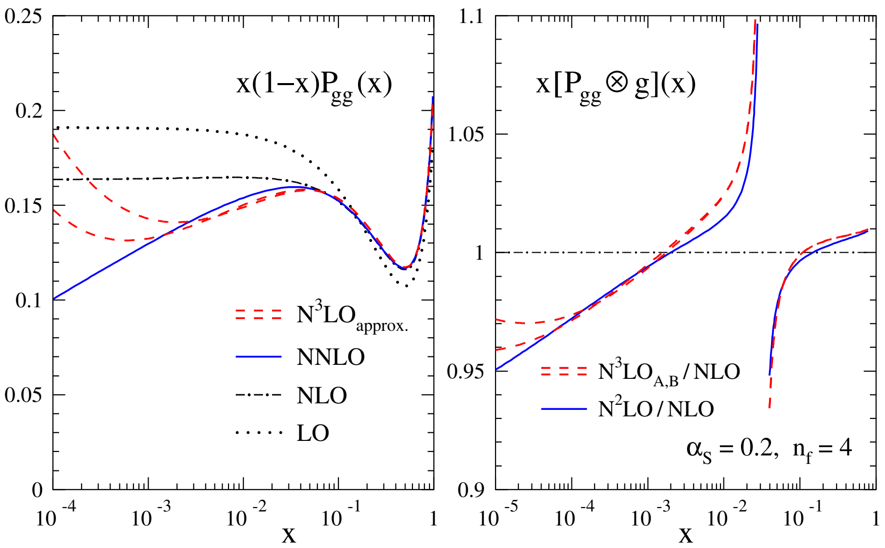
Quark evolution: gluon-to-quark contribution



Gluon evolution: quark-to-gluon contribution



Gluon evolution: gluon-to-gluon contribution



Gluonic aliens

$$\mathcal{L} = -\frac{1}{4} F^{a\,\mu\nu} F^{a}_{\mu\nu} + c_{\rm g} \mathcal{O}_{\rm g} \qquad \mathcal{L} \to \mathcal{L} + \underbrace{(D.F)^a \mathcal{G}^a}_{\mathcal{O}_{\mathsf{EOM}}}$$

$$\hookrightarrow A^a_\mu \to A^a_\mu + \Delta_\mu \, \mathcal{G}^a(A, \partial A, \dots) \nearrow$$

To linear order in $\Delta_{\mu_1} \dots \Delta_{\mu_N}$

Gluonic aliens

$$\mathcal{L} = -\frac{1}{4} F^{a\,\mu\nu} F^{a}_{\mu\nu} + c_{\rm g} \mathcal{O}_{\rm g} \qquad \mathcal{L} \to \mathcal{L} + \underbrace{(D.F)^a \mathcal{G}^a}_{\mathcal{O}_{\rm EOM}}$$

$$A^a_\mu \to A^a_\mu + \Delta_\mu \, \mathcal{G}^a(A, \partial A, \dots)$$

To linear order in $\Delta_{\mu_1} \dots \Delta_{\mu_N}$

$$\begin{aligned}
\mathcal{O}_{\rm EOM} = (D.F)^a \Big[& \underbrace{\eta\, \partial^{N-2} A^a}_{\mathcal{O}^{I}_{A}} + g f^{a a_1 a_2} \sum_{i_1+i_2=N-3} \underbrace{\kappa_{i_1 i_2} (\partial^{i_1} A^{a_1})(\partial^{i_2} A^{a_2})}_{\mathcal{O}^{II}_{A}} \\
& + g^2 \sum_{\substack{i_1+i_2+i_3 \\ N-4}} \underbrace{\left(\kappa^{(1)}_{i_1 i_2 i_3} f^{a a_1 z} f^{a_2 a_3 z} + \kappa^{(2)}_{i_1 i_2 i_3} d_4^{a a_1 a_2 a_3} + \kappa^{(3)}_{i_1 i_2 i_3} d_{\widehat{4ff}}^{a a_1 a_2 a_3} \right) (\partial^{i_1} A^{a_1})..(\partial^{i_3} A^{a_3})}_{\mathcal{O}^{III}_{A}} \\
& + g^3 \sum_{\substack{i_1+..+i_4 \\ N-5}} \underbrace{\left(\kappa^{(1)}_{i_1 \dots i_4} (f\,f\,f)^{a a_1 a_2 a_3 a_4} + \kappa^{(2)}_{i_1 \dots i_4} d_{4f}^{a a_1 a_2 a_3 a_4} \right) (\partial^{i_1} A^{a_1})..(\partial^{i_4} A^{a_4})}_{\mathcal{O}^{IV}_{A}} + O(g^4) \Big]
\end{aligned}$$

Generalised gauge invariance

New gauge transformation $A_\mu^a \to A_\mu^a + \delta_\omega A_\mu^a + \tilde{\delta}_\omega A_\mu^a$

$$\begin{aligned} \delta_\omega A_\mu^a &= D_\mu^{ab}\omega^b, \\ \tilde{\delta}_\omega A_\mu^a &= -\Delta_\mu \left(\delta_\omega \mathcal{G}^a - g\, f^{abc}\mathcal{G}^b\omega^c\right) \end{aligned}$$

Generalised BRST transformation $A_\mu^a \to A_\mu^a + \lambda s'(A_\mu^a)$, $s' = s + \tilde{s}$

$$\begin{aligned} s(A_\mu^a) &= D_\mu^{ab} c^b, \\ \tilde{s}(A_\mu^a) &= -\Delta_\mu \left(s(\mathcal{G}^a) - g\, f^{abc}\mathcal{G}^b c^c\right), \end{aligned}$$

are **nihilpotent**.

$$(s')^2 = 0.$$

Ghost aliens

$$\widetilde{\mathcal{L}} = \underbrace{\mathcal{L}_0 + c_{\rm g}\,\mathcal{O}_{\rm g}^{(N)} + \mathcal{O}_{\rm EOM}^{(N)}}_{\mathcal{L}_{\rm GGI}} + \mathbf{s'}\left[\bar{c}^a\left(\partial^\mu A^a_\mu - \frac{\xi_L}{2} b^a\right)\right]$$

- $s\left[\bar{c}^a\left(\partial^\mu A^a_\mu - \frac{\xi_L}{2} b^a\right)\right]$ generates gauge fixing and ghost terms.
- $\tilde{s}\left[\bar{c}^a\left(\partial^\mu A^a_\mu - \frac{\xi_L}{2} b^a\right)\right]$ generates **ghost** aliens

$$\mathcal{O}_G^{(N)} = \bar{c}^a \partial\left(s\left(\mathcal{G}^a\right) - g\, f^{abc}\, \mathcal{G}^b\, c^c\right)$$

- The **couplings** in $\mathcal{O}_G^{(N)}$ are related to those in $\mathcal{O}_{\rm EOM}^{(N)}$ by BRST.
- Further anti-BRST relations impose more constraints

Spin N	2	4	6	8	10	12	14	16
# aliens	1	2	5	12	25	50	87	140