Integrated Unitarity
for Scattering Amplitudes

2403.18047 & 2408.06325

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10.09.2024
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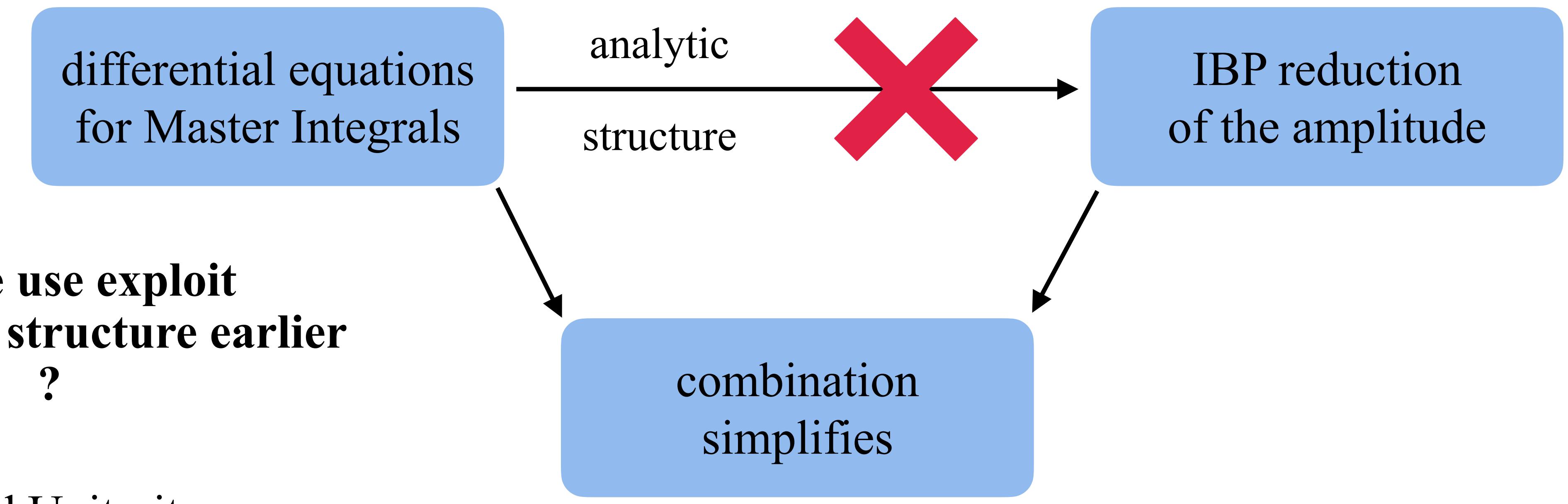
European Research Council
Established by the European Commission

Introducing Integrated Unitarity

standard amplitude workflow



**can we use exploit
the analytic structure earlier
?**

YES : Integrated Unitarity

- dispersion relations : discontinuities algorithmic
- cut canonical differential equations (DEQ) : algorithmic in dimReg

[see Appendix for technical definitions]

- Generalized Unitarity @ integrated level : constrains both Master Integrals (MIs) and their coefficients using cuts
- less subsectors, less MIs, simpler DEQ and IBP system

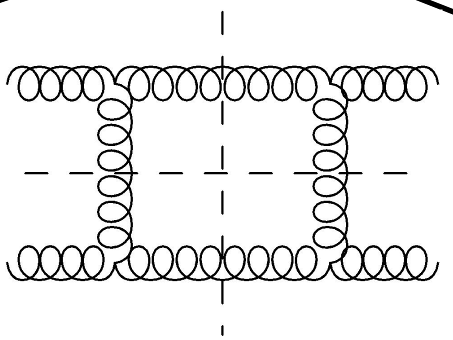
Generalized VS Integrated Unitarity

consider a toy 1-loop amplitude

$$\mathscr{A}^{(1)} \sim \int \frac{d^d k}{(2\pi)^d} \frac{\mathscr{N}(k)}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)} = r_{1234} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)} + \text{subsectors}$$

cut all 4 propagators

$$\text{Cut}_{1234}\,\mathscr{A}^{(1)} \sim \int \frac{d^d k}{(2\pi)^d} \mathscr{A}_1^{(0)}(k)\ \mathscr{A}_2^{(0)}(k)\ \mathscr{A}_3^{(0)}(k)\ \mathscr{A}_4^{(0)}(k)\,\delta^+(\mathscr{D}_1(k))\,\delta^+(\mathscr{D}_2(k))\,\delta^+(\mathscr{D}_3(k))\,\delta^+(\mathscr{D}_4(k)) = r_{1234} \int \frac{d^d k}{(2\pi)^d} \delta^+(\mathscr{D}_1(k))\,\delta^+(\mathscr{D}_2(k))\,\delta^+(\mathscr{D}_3(k))\,\delta^+(\mathscr{D}_4(k))$$



Generalized Unitarity

can compute integral coefficient

$$r_{1234} = \frac{1}{2} \sum_{\text{2 cut solns } k^*} \mathscr{A}_1^{(0)}(k^*)\ \mathscr{A}_2^{(0)}(k^*)\ \mathscr{A}_3^{(0)}(k^*)\ \mathscr{A}_4^{(0)}(k^*)$$

scalar integral remains to be computed

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)}$$

Integrated Unitarity

compute cut amplitude

$$\text{Cut}_{1234}\,\mathscr{A}^{(1)} = r_{1234} \int \frac{d^d k}{(2\pi)^d} \delta^+(\mathscr{D}_1(k))\,\delta^+(\mathscr{D}_2(k))\,\delta^+(\mathscr{D}_3(k))\,\delta^+(\mathscr{D}_4(k))$$

can reconstruct whole amplitude

$$\mathscr{A}^{(1)}(s,u) \sim \int_0^\infty \frac{ds'}{s'-s} \int_0^\infty \frac{du'}{u'-u} \text{Cut}_{1234}\,\mathscr{A}^{(1)}(s',u')$$

Dispersion relation

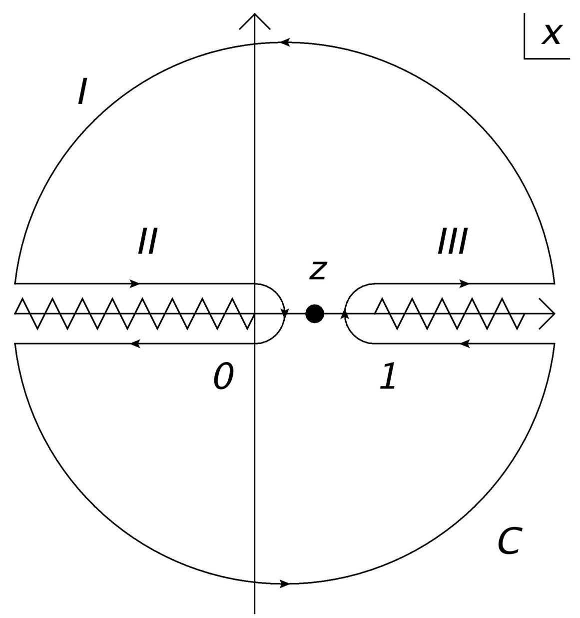
Cauchy's integral formula

$$\mathscr{A}(z) = \frac{1}{2\pi i}\oint_C \frac{\mathscr{A}(x)dx}{x-z}$$

[Cutkosky, Mandelstam, Eden, Landshoff, Olive, Polkinghorne, Remiddi, van Neerven, Kniehl, Sirlin]

$$x = -\frac{t}{s}$$

4-point massless



piecewise contour

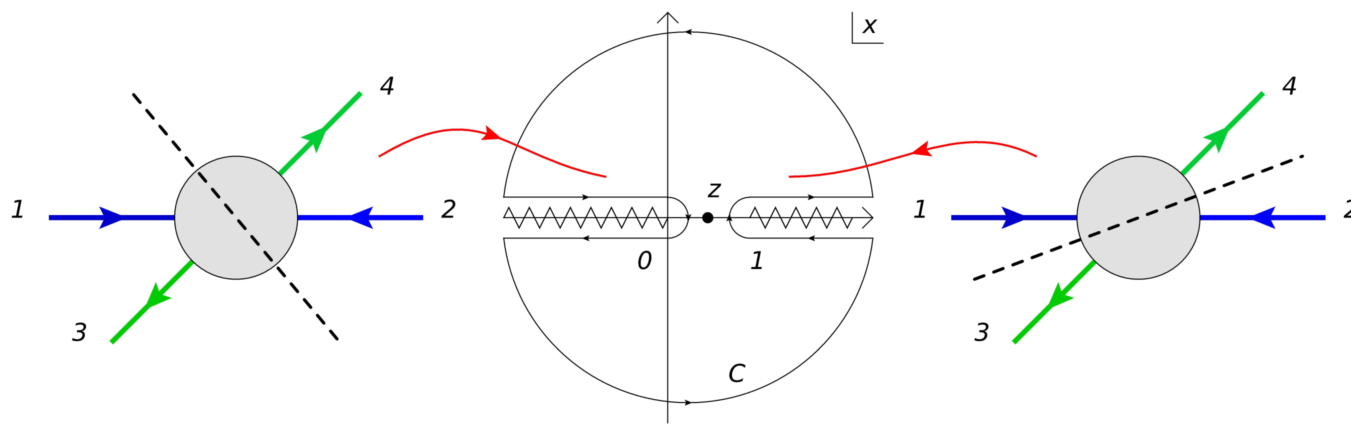
$$\mathscr{A}(z) = c_\infty + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Disc}_0 + \int_1^\infty \mathrm{Disc}_1\right)\frac{\mathscr{A}(x)dx}{x-z}$$

cancel constant from infinite arc

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Disc}_0 + \int_1^\infty \mathrm{Disc}_1\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

unitarity

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$



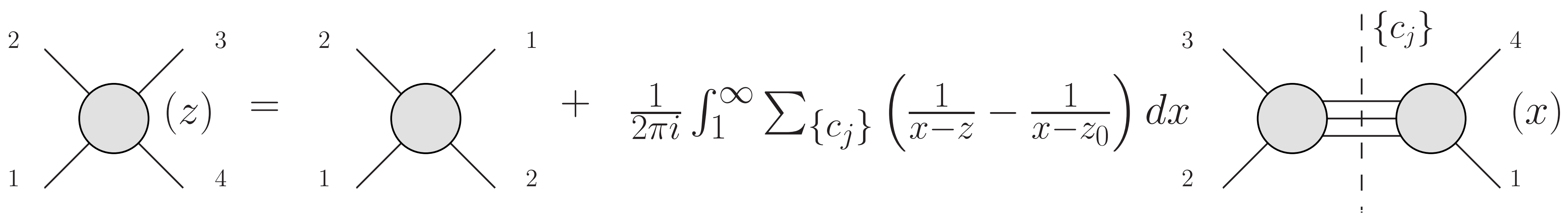
Dispersion relation

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

expressing cuts as phase space integrals

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \sum_{\{c_i\}\in\mathscr{C}_t} \int d\mathrm{PS}_{t,\{c_i\}}\, \mathscr{A}_{t,\{c_i\},L}\, \mathscr{A}^*_{t,\{c_i\},R}\right.$$
$$\left. + \int_1^\infty \sum_{\{c_j\}\in\mathscr{C}_u} \int d\mathrm{PS}_{u,\{c_j\}}\, \mathscr{A}_{u,\{c_j\},L}\, \mathscr{A}^*_{u,\{c_j\},R}\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)dx$$

diagrammatically, in the planar case



Integrated Unitarity : 3 methods

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

A. explicit integration : convergent e.g. for canonical MIs

subtraction terms needed for full amplitude

$$\mathscr{A}(x) \to S(x)\ \mathscr{A}(x) \qquad S(x) = \frac{(1-x)^p x^q}{(x-z_1)^r} \qquad -\mathrm{Res}_{x\to z_1}\mathscr{A}(x)\,S(x)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)$$

B. ansatz reconstruction with 2 discontinuities + # evaluations :

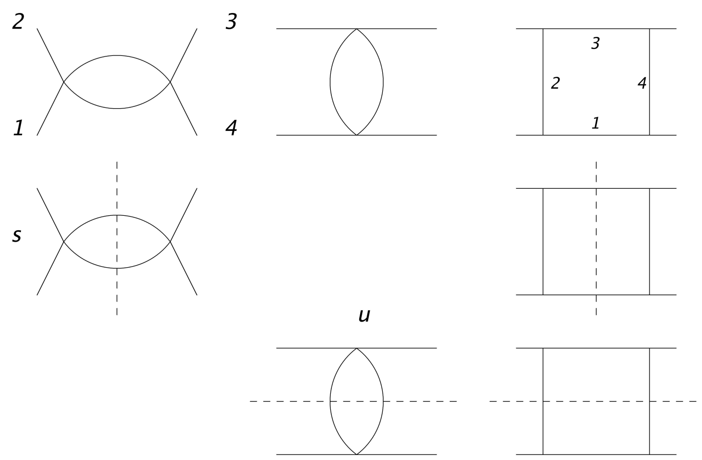
$$\begin{cases} \mathrm{Disc}_0\,\mathscr{A} & = \mathrm{Cut}_t\,\mathscr{A} \\ \mathrm{Disc}_1\,\mathscr{A} & = \mathrm{Cut}_u\,\mathscr{A} \\ \mathscr{A}(z_i) & = \mathscr{A}_i \end{cases} \qquad \mathscr{A}(z) = \epsilon^{\#}\sum_{n\geq 0}\epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} r_{n,\vec{\alpha}}(z)\,G(\overrightarrow{\alpha}, z)$$

unknowns ($r_{n,\vec{\alpha}}$)

C. ansatz reconstruction with 3 discontinuities :

$$\begin{cases} \mathrm{Disc}_0\,\mathscr{A} & = \mathrm{Cut}_t\,\mathscr{A} \\ \mathrm{Disc}_1\,\mathscr{A} & = \mathrm{Cut}_u\,\mathscr{A} \\ \mathrm{Disc}_\infty\,\mathscr{A} & = \mathrm{Cut}_s\,\mathscr{A} \end{cases} \qquad \mathscr{A}(z) = \epsilon^{\#}\sum_{n\geq 0}\epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} r_{n,\vec{\alpha}}(z)\,G(\overrightarrow{\alpha}, z)$$

Integrated Unitarity : example



cut MIs

$$M_i^c \in \{\epsilon(2\epsilon-1)\, I_{1,0,1,0}\,,\, \epsilon(2\epsilon-1)\, I_{0,1,0,1}\,,\, \epsilon^2(x-1)\, I_{1,1,1,1}\}$$

$$\text{Cut}_s M_i^c \in \{\epsilon(2\epsilon-1)\, I_{1,0,1,0;1,3},\ 0,\ \epsilon^2(x-1)\, I_{1,1,1,1;1,3}\}$$

$$\text{Cut}_u M_i^c \in \{0,\ \epsilon(2\epsilon-1)\, I_{0,1,0,1;2,4},\, \epsilon^2(x-1)\, I_{1,1,1,1;2,4}\}$$

$$\text{Cut}_t M_i^c \in \{0,\ 0,\ 0\}$$

cut DEQ

$$\partial_x M_i^c(x,\epsilon) = \epsilon\, A_{ij}^c(x)\, M_j^c(x,\epsilon)$$

$$\partial_x \text{Cut}_s M_i^c(x,\epsilon) = \epsilon\, A_{ij}^{c,s}(x)\, \text{Cut}_s M_j^c(x,\epsilon)$$

$$\partial_x \text{Cut}_u M_i^c(x,\epsilon) = \epsilon\, A_{ij}^{c,u}(x)\, \text{Cut}_u M_j^c(x,\epsilon)$$

$$\partial_x \text{Cut}_t M_i^c(x,\epsilon) = 0$$

$$A^c(x) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\frac{1}{1-x}} & \mathbf{0} \\ \mathbf{\frac{2}{x}} & \mathbf{\frac{2}{x}} + \mathbf{\frac{2}{1-x}} & \frac{1}{x} + \frac{1}{1-x} \end{pmatrix}$$

unknowns

3 methods

$$M_i^c(z) = \epsilon^{-2} \sum_{n\geq 0} \epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} c_{n,\vec{\alpha}}\, G(\vec{\alpha}, z) \qquad \alpha_k \in \{0,1\}$$

A. explicit integration

$$M_i^c(z) = M_{i,0}^c + \frac{1}{2\pi i}\int_1^\infty \left(\frac{dx}{x-z} - \frac{dx}{x}\right) \text{Cut}_u M_i^c(x)$$

✔

B. ansatz + 2cuts + 1pt

ansatz	computed
$\text{Disc}_0 M_i^c$	$= \text{Cut}_t M_i^c$
$\text{Disc}_1 M_i^c$	$= \text{Cut}_u M_i^c$
$M_i^c(0)$	$= M_{i,0}^c$

✔

C. ansatz + 3cuts

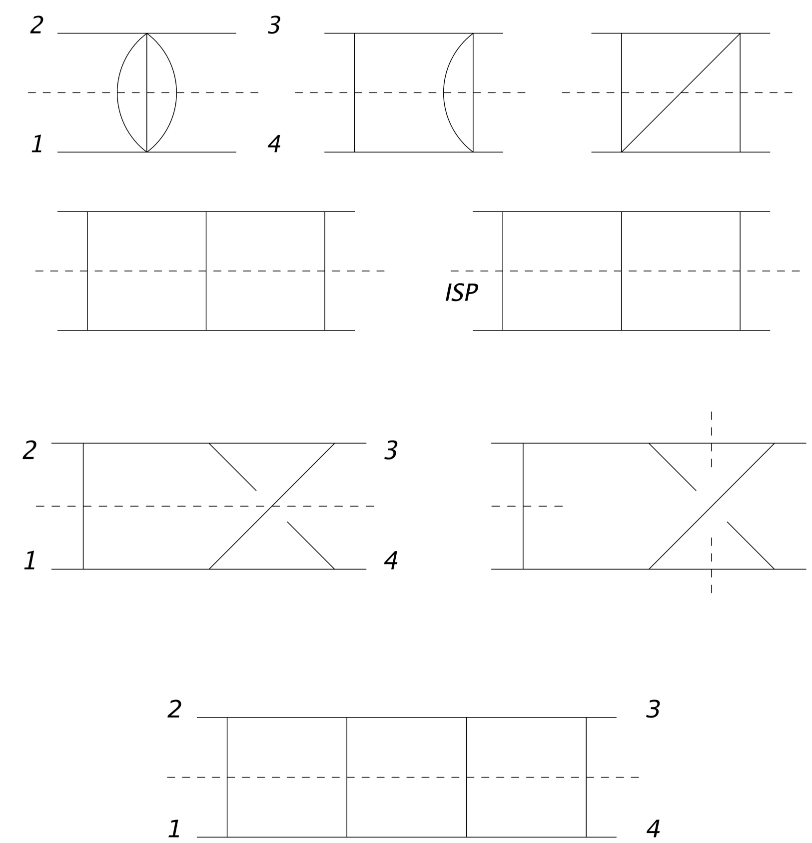
$$\begin{cases} \text{Disc}_0 M_i^c &= \text{Cut}_t M_i^c \\ \text{Disc}_1 M_i^c &= \text{Cut}_u M_i^c \\ \text{Disc}_\infty M_i^c &= \text{Cut}_s M_i^c \end{cases}$$

✔

Integrated Unitarity : further checks

Master Integrals (from DEQ)

- 2-loop planar (#MIs : 8 → 5) ✓
- 2-loop nonplanar (#MIs : 12 → 2+6) ✓
- 3-loop planar ladder (#MIs : 26 → 17) ✓

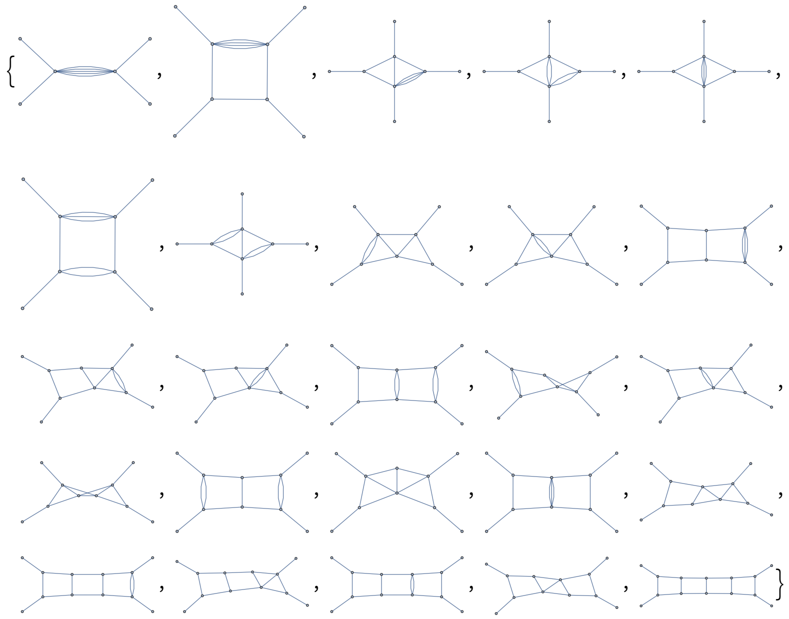


Amplitudes (from form factor method)

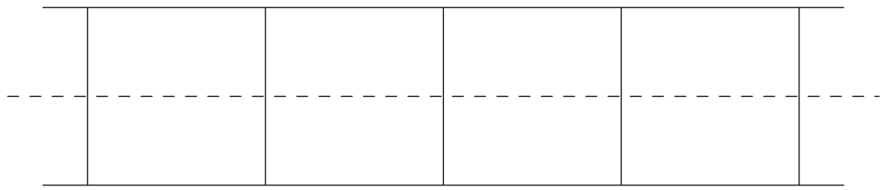
- 1-loop gg → gg (#INTs : /~2 per cut) ✓
- 2-loop gg → gg planar (#INTs : /~8 per cut) ✓
- 2-loop gg → gg nonplanar (#INTs : /~4 per cut) ✓

Application : four-loop ladder

procedure



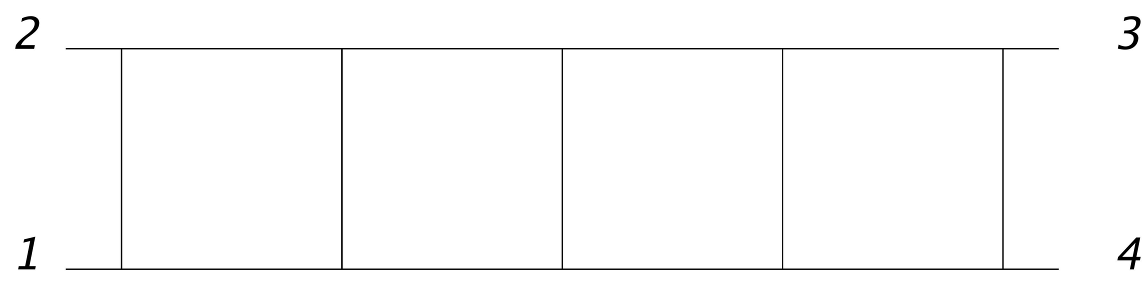
- 22 generalized propagators = 13 denominators + 9 ISPs
- cut u : 5 propagators
- IBP : 59 MIs $\mathrm{Cut}_u M_i^c$ (LiteRed + Kira)
- canonical DEQ : $A_{ij}^c(x) = \frac{a_{ij}}{x} + \frac{b_{ij}}{1-x}$ (CANONICA + MultivariateApart + FiniteFlow)
- canonical general solution : $M_u^c(x,\epsilon) = \mathbb{P}e^{\epsilon \int A^c(x) dx} M_{u,0}^c(\epsilon)$ (PolyLogTools + in-house)
- regularity constraints on BCs $M_{u,0}^c(\epsilon)$: 59 → 5 (in-house)



- 5 remaining BCs : weight 7 (AMFlow)
- method B : $\mathrm{Disc}_0 M_i^c = \mathrm{Cut}_t M_i^c = 0 \quad \& \quad \mathrm{Disc}_1 M_i^c = \mathrm{Cut}_u M_i^c$ (in-house)

fixed all HPL coefficients to weight 8

Application : four-loop ladder



$$((1.295358800817136032 \times 10^7 - 2.021550923233216017\0 \times 10^8 i)G(1;x) - (4916.0033092753716635 - 925.8911388065244199i)G(0,1;x) + (10416.080957097\ldots$$

$$+ (907.2273861240918352 + 1689.5773142537034608i)G(0,0,1,1;x) + (1714.4000021984428954 + 4022.5367183722277758i)G(0,1,0,1;x) - (7177.72716969811\ldots$$

$$+ (256.61688851155190596 - 289.39191568279832164i)G(0,0,0,1,1;x) + (540.12423515304436557 - 1005.98142118306083236i)G(0,0,1,0,1;x) - (1197.0705\ldots$$

$$+ (821.40555049238942836 - 654.57695213966287037i)G(1,0,0,0,1;x) - (4298.7335682028809563 - 5622.4715046943673918i)G(1,0,0,1,1;x) - (5357.96830\ldots$$

$$- (24674.39944802677715\7 - 58402.044698271394624i)G(1,1,1,1,1;x) + 16.449340668482264365G(0,0,0,0,0,1;x) - 86.267653283595875335G(0,0,0,0,1,1\ldots$$

$$- 7322.5153838008159945G(0,1,0,1,1,1;x) + 5448.0216294013259576G(0,1,1,0,0,1;x) - 9025.9359952480904821G(0,1,1,0,1,1;x) - 11277.66796231144\ldots$$

$$+ 18563.629255737851411G(1,0,1,1,1,1;x) + 6296.8076078950107988G(1,1,0,0,0,1;x) - 14737.147075345132670G(1,1,0,0,1,1;x) - 25160.91148651047\ldots$$

[illegible]

Further applications

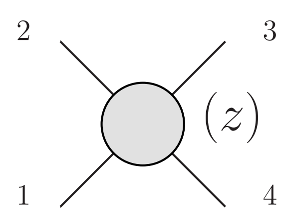
kinematic limits

e.g. $\text{Disc}_1\, \mathscr{A} = \text{Cut}_u\, \mathscr{A}$ alone gives u→0 limit at fixed Log accuracy to any subleading power

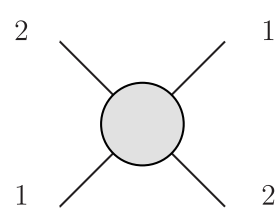
$$\lim_{x\to 1} \mathscr{A}(x) \sim c_{1,1}(x)\, G(1,1,x) + c_{0,1}(x)\, G(0,1,x) + c_{\ldots,0}(x)\, G(\ldots,0,x)$$

Leading Log → $G(1,1,x)$; Next-to-Leading Log → $G(0,1,x)$; suppressed → $G(\ldots,0,x)$

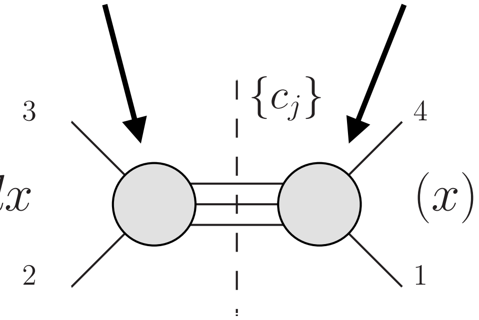
recursive approach



$(z) =$

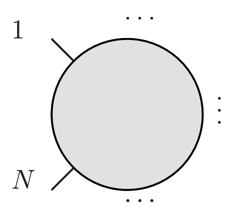


$$+ \frac{1}{2\pi i}\int_1^\infty \sum_{\{c_j\}} \left(\frac{1}{x-z} - \frac{1}{x-z_0}\right) dx$$

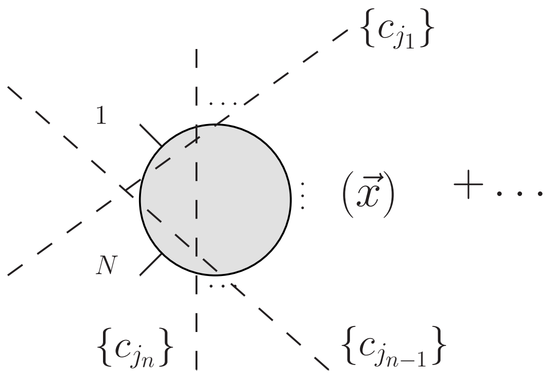


(x)

Multivariate Integrated Unitarity



$$(\vec{z}) = \frac{1}{(2\pi i)^n}\left(\prod_{m=1}^{n}\int_{x^{(m)}_{\text{thres}}}^{\infty} \frac{dx_m}{x_m - z_m} \sum_{\{c_{j_m}\}}\right)$$

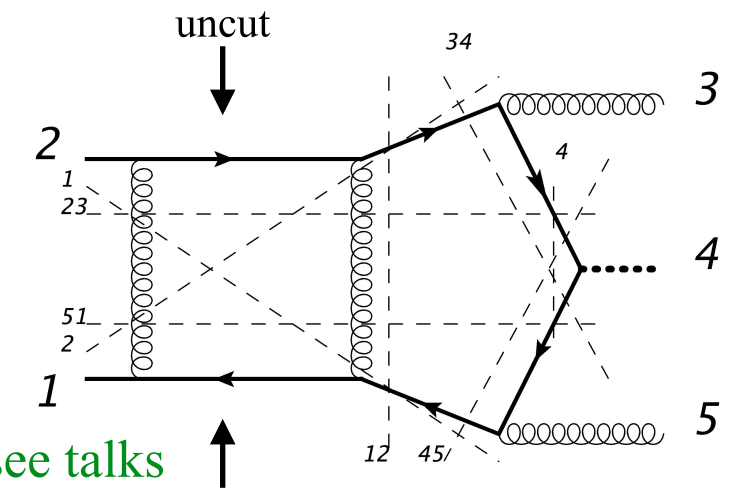


$(\vec{x}) + \ldots$

Outlook into Multivariate Integrated Unitarity

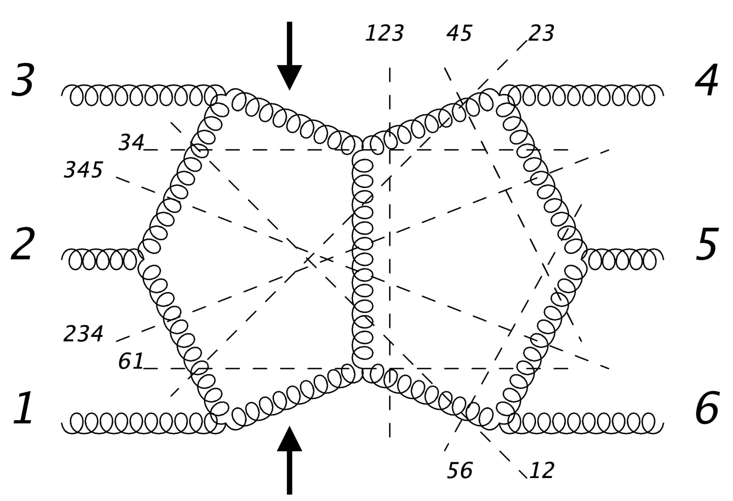
examples

$pp \to t\bar{t}H$ @ 2 loops



[see talks by Anton, Guoxing]

$gg \to gggg$ @ 2 loops



properties

- many cut propagators → simpler IBPs
- iterative Cauchy formula → multivariate complex analysis
- Landau singularities → nontrivial analytic structure

  [Helmer, Papathanasiou, Tellander 2402.14787] [Correia 2212.06157]

- genealogical constraints → possible simplifications

  [Hannesdottir, Lippstreu, McLeod, Polackova 2406.05943]

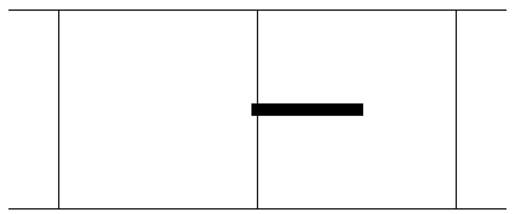
ongoing : 4-point 1-mass

[**PB**, collaborators in the group of Lorenzo Tancredi]

Analytic structure at 2 loops

5-point

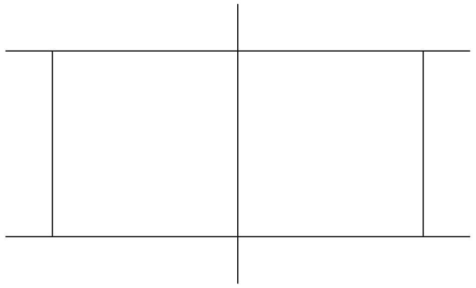
[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2306.15431]



[see talks by Federico, Jungwon]

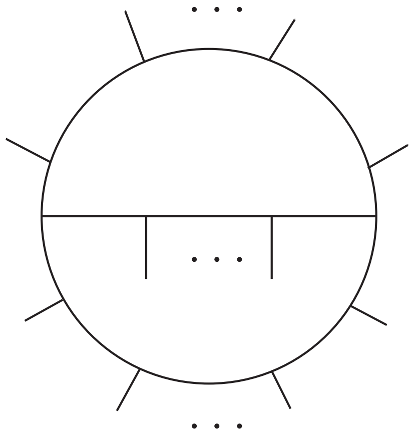
6-point

[Henn, Matijašić, Miczajka, Peraro, Xu, Zhang 2403.19742]



...

n-point



when does it **end** ?

a.k.a. the finite basis problem

The finite basis problem

1-loop n-point : linearly related to subsectors of 5-point

$$\mathcal{I}_{1,n} = \sum_{i_1=1}^{n} \frac{\mathcal{N}_{\text{tadpole},i_1}}{\mathcal{D}_{i_1}} + \sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} \frac{\mathcal{N}_{\text{bubble},i_1,i_2}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}} + \sum_{\substack{i_1,i_2,i_3=1\\ i_j\neq i_m}}^{n} \frac{\mathcal{N}_{\text{triangle},i_1,i_2,i_3}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} + \sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_j\neq i_m}}^{n} \frac{\mathcal{N}_{\text{box},i_1,i_2,i_3,i_4}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}\mathcal{D}_{i_4}} + \sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_j\neq i_m}}^{n} \frac{\mathcal{N}_{\text{pentagon},i_1,i_2,i_3,i_4,i_5}}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}\mathcal{D}_{i_4}\mathcal{D}_{i_5}},$$

beyond 1-loop : new integrals with each new leg ?

- 1-loop : “no” i.e. up to 5-point [Passarino, Veltman, Ossola, Papadopoulos, Pittau]
- 2-loop massless planar in d=4-2ϵ : “also no” i.e. up to 11 denominators [Gluza, Kajda, Kosower 1009.0472]
- 2-loop in d=d_0-2ϵ : “also no” [Kleiss, Malamos, Papadopoulos, Verheyen 1206.4180]
- 2-loop in d=4 : “also no” i.e. up to 8 denominators [Feng, Huang 1209.3747]
- L-loop in d=4 : “also no" [Bourjaily, Herrmann, Langer, Trnka 2007.13905]
- **L-loop in d=d_0-2ϵ : “also no"** [**PB**, Tong-Zhi Yang 2408.06325]
  (for integer d_0 & propagator powers)

2 loops point-by-point

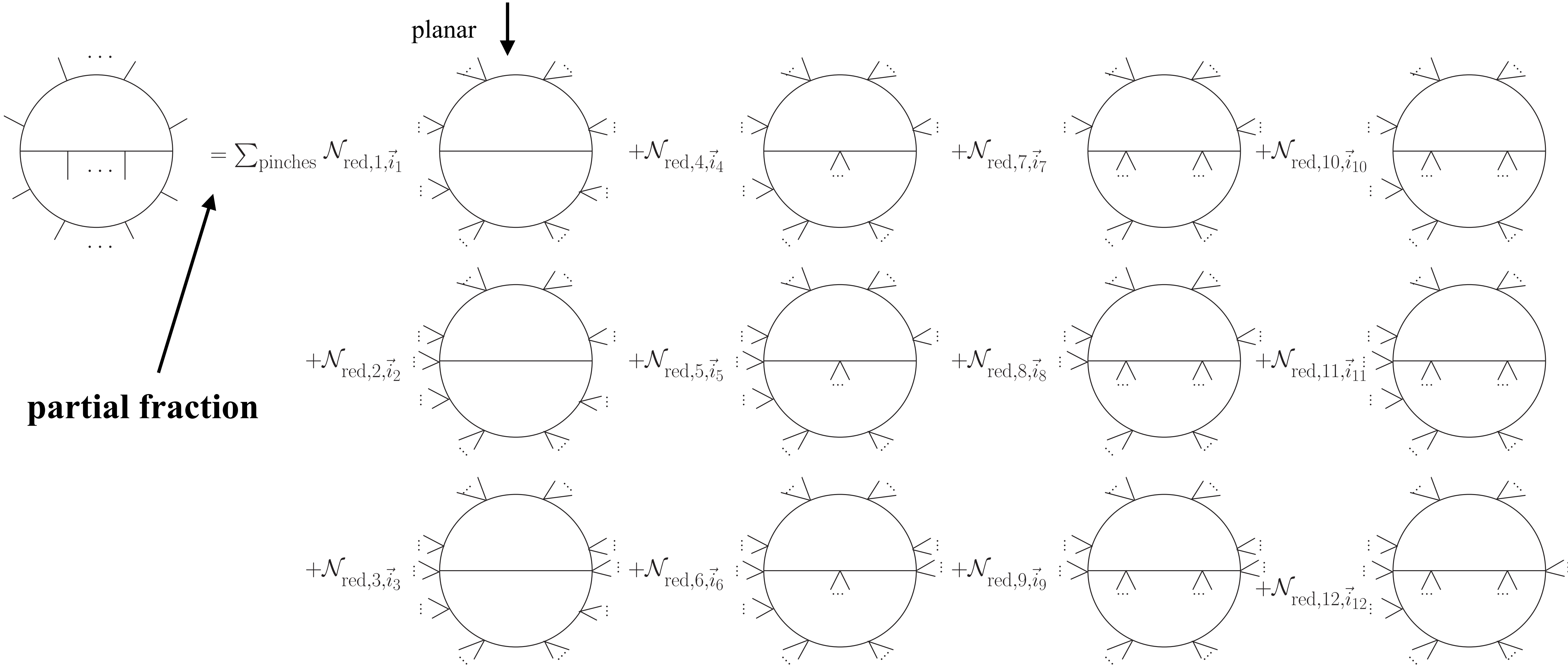
- independent external momenta $\{p_i\}$: (n-1) in Conventional Dimensional Regularization scheme (CDR)
  only 4 in 't Hooft-Veltman scheme (tHV)
- scalar products $\{k_i \cdot k_j\,, \, k_i \cdot p_j\}$ involving loop momenta $\{k_1, k_2\}$: **at most 11** in tHV
- generalized propagators : denominators & Irreducible Scalar Products (ISPs)

kinematics	denominators	CDR ISPs	**tHV ISPs**	generalized CDR propagators	**generalized tHV propagators**
2-loop 4-point	7	2	2	9	9
2-loop 5-point	8	3	3	11	**11**
2-loop 6-point	9	4	2	13	**11**
2-loop 7-point	10	5	1	15	**11**
2-loop 8-point	11	6	**0**	17	**11**
2-loop 9-point	12	7	**0**	19	**11**

only 11 independent ↑ $\Rightarrow$ **partial fraction**

Finite basis topologies at 2 loops

[Gluza, Kajda, Kosower 1009.0472]



lower dimensions : less denominators e.g. 7 in d=2-2ϵ

higher loops : 12p@3L, 17p@4L, ... (loop-by-loop approach analogous but many more topologies)

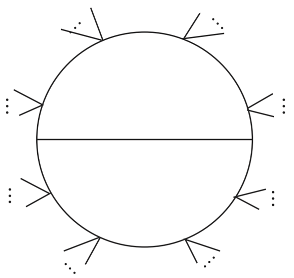
Immediate application : improving the IBP efficiency beyond 5-point

17 CDR generalized propagators

$$\{k_1-k_2, k_1, k_2, k_1+p_1, k_1+p_{12}, k_1+p_{123}, k_1+p_{1234}, k_2+p_{1234}, k_2+p_{12345}, k_2+p_{123456}, k_2+p_{1234567}, k_1-p_5, k_1-p_6, k_1-p_7, k_2-p_1, k_2-p_2, k_2-p_3\}$$

momentum decomposition

$$p_{j>4} = \sum_{i=1}^{4} p_i \frac{\det(\{p_1,\ldots,\hat{p}_i,p_j,\ldots,p_4\}\cdot\{p_1,\ldots,p_4\})}{\det(\{p_1,\ldots,p_4\}\cdot\{p_1,\ldots,p_4\})}$$



$\Rightarrow$ 11 tHV generalized propagators

$$\{k_1-k_2, k_1, k_2, k_1+p_1, k_1+p_{12}, k_1+p_{123}, k_1+p_{1234}, k_2+p_{1234}, k_2+\sum_{j=1}^{4} p_j\, z_{9,j}, k_2+\sum_{j=1}^{4} p_j\, z_{10,j}, k_2+\sum_{j=1}^{4} p_j\, z_{11,j}\}$$

	CDR	tHV	R_t
D2	31.4m/3126MI/13.9G	6.8m/2368MI/1.88G	4.6
D3	51.5m/3302MI/18.19G	10.1m/2368MI/2.9G	5.1
N3	115.2m/4497MI/25.7G	5.7m/2358MI/2.59G	20.2
N4	321.3m/6742MI/56.4G	7.6m/2368MI/4.3G	42.3
N5	908.9m/9779MI/137G	12.5m/2368MI/7.35G	72.7
N6	-	20.1m/2368MI/10.34G	-

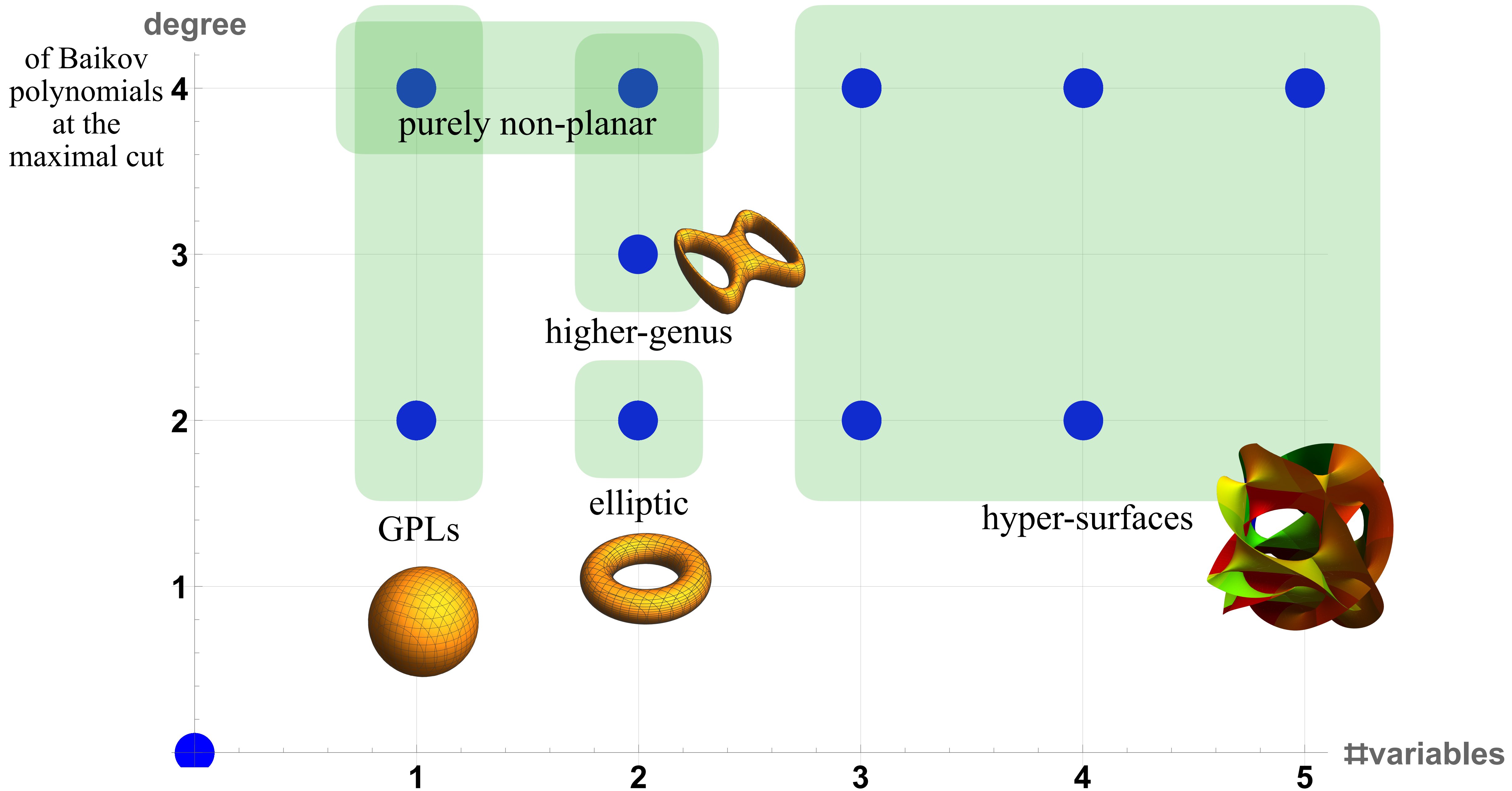
- no cut propagators
- one numerical probe
- one finite field
- with FIRE6

↓

improves numerical AMFlow evaluation of Feynman integrals

Forthcoming application : spanning the space of special functions at 2 loops

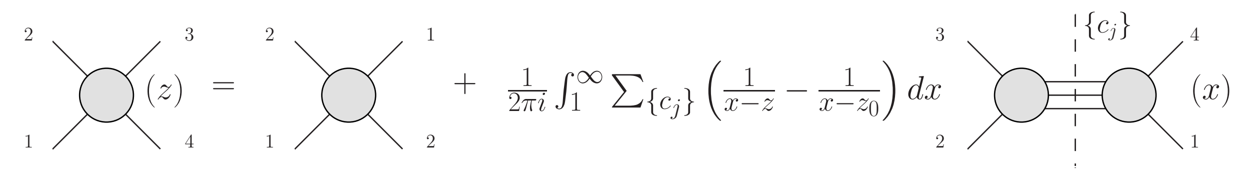
all 84 subsectors of the 12 finite basis topologies



roots appear when integrating out the variables

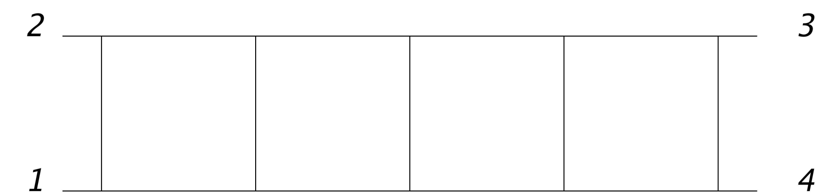
Conclusions

can **exploit** the analytic structure of amplitudes



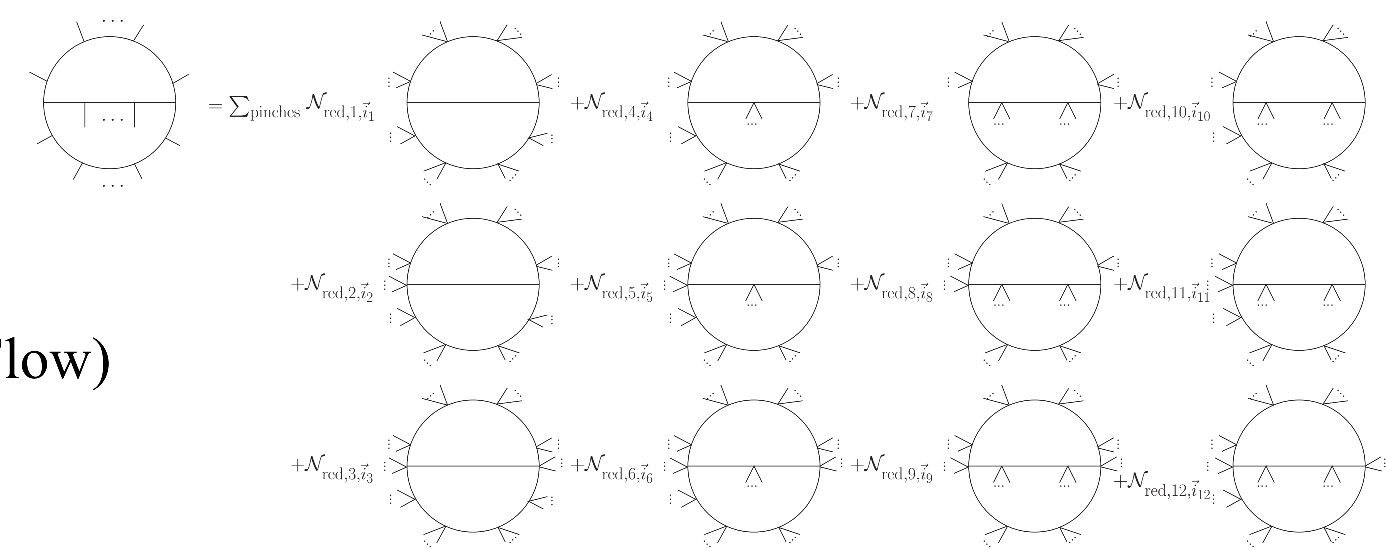
- **Integrated Unitarity** ~ Generalized Unitarity @ integrated level
- allows algorithmic usage of dispersion relations in dimReg
- beneficial for IBP computational complexity
- requires understanding of analytic structure

trade-off



can **constrain** the analytic structure of QCD amplitudes

- **finite basis exists** at any fixed loop order
- explicit reduction to lower points at 2 loops
- improves IBP efficiency (also numerical via AMFlow)
- upper bound on special functions spectrum



THANK YOU

Appendix

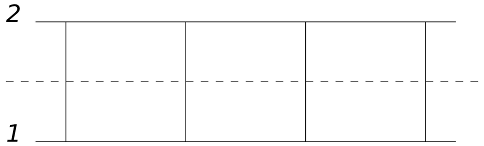
Cuts

- integral definition : $I_{\{n_i\}} = \int \left(\prod_{l=1}^{L} D^d k_l \right) \prod_{i=1}^{N} \mathscr{D}_i^{-n_i}$ $\qquad D^d k_l = e^{\epsilon \gamma_E} \frac{d^d k_l}{i \pi^{d/2}}$

- cut propagator : $\frac{1}{\mathscr{D} + i\varepsilon} \to 2\pi i \, \delta^+(\mathscr{D})$ $\qquad \delta^+(q^2) = \delta(q^2)\, \theta(q_0)$

- cut integral : $\mathrm{Cut}_u I_{\{n_i\}} = \sum_{\{c_j\} \in \mathscr{C}_u} I_{\{n_i\};\{c_j\}} = \sum_{\{c_j\} \in \mathscr{C}_u} \int \left(\prod_{l=1}^{L} D^d k_l \right) \left(\prod_{i \notin \{c_j\}} \mathscr{D}_i^{-n_i} \right) \prod_{m \in \{c_j\}} \delta_{1,n_m} 2\pi i \, \delta^+(\mathscr{D}_m)$

- less subsectors : $/2^C$



- Integration-By-Parts identities (IBPs) : [Laporta 0102033]

$$\int \left(\prod_{l=1}^{L} D^d k_l \right) \frac{\partial}{\partial k_l^\mu} \left(q^\mu \prod_{i=1}^{N} \mathscr{D}_i^{-n_i} \right) = 0$$

[Chetyrkin, Tkachov 1981]

$$p_{j\,\mu}\, p_{l\,\nu} \left(p_n^\nu \frac{\partial}{\partial p_{n,\mu}} - p_n^\mu \frac{\partial}{\partial p_{n,\nu}} \right) I_{\{n_i\}} = 0$$

[Gehrmann, Remiddi 9912329]

- Differential Equations (DEQ) : $\partial_{x_n} M_i(\vec{x}, \epsilon) = A_{ij}(\vec{x}, \epsilon)\, M_j(\vec{x}, \epsilon)$ $\xrightarrow{\text{canonical}}$ $\partial_{x_n} M_i^c(\vec{x}, \epsilon) = \epsilon\, A_{ij}^c(\vec{x})\, M_j^c(\vec{x}, \epsilon)$

[Kotikov 1991] [Henn 1304.1806]

Master Integrals (MIs)

$\to$ can solve cut integrals with DEQ

Discontinuities

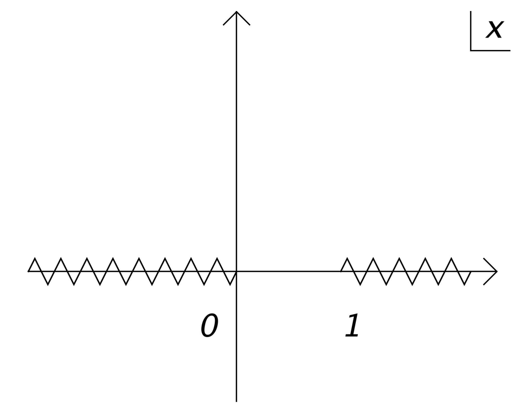
- discontinuity :

$$\mathrm{Disc}_0 f(x) = f(x + i\epsilon) - f(x - i\epsilon)$$

- from now on, focus on specific kinematics :

$$x = -\frac{t}{s}$$

4-point massless



- branch cuts :

$t > 0$	$u > 0$	$s > 0$
$x < 0$	$x > 1$	pushed to ∞

- Harmonic Polylogarithms (HPLs) :
  [Remiddi, Vermaseren 9905237]

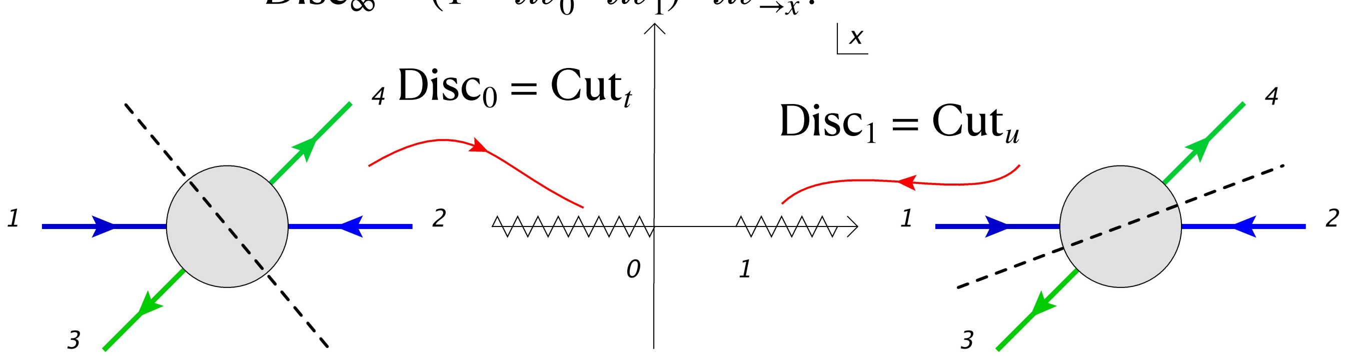
$$G(\alpha_n, \ldots, \alpha_1; x) = \int_0^x \frac{dz}{z - \alpha_n} G(\alpha_{n-1}, \ldots, \alpha_1; z)$$

$$\alpha_k \in \{0,1\}$$

- discontinuities of HPLs algorithmic from monodromy matrices :
  [Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

$$\begin{aligned} \mathrm{Disc}_0 &= (1 - \mathcal{M}_0) \cdot \mathcal{M}_{\to x}\,, \\ \mathrm{Disc}_1 &= -\,(1 - \mathcal{M}_1) \cdot \mathcal{M}_{\to x}\,, \\ \mathrm{Disc}_\infty &= (1 - \mathcal{M}_0 \cdot \mathcal{M}_1) \cdot \mathcal{M}_{\to x}\,. \end{aligned}$$

- unitarity :



$(\mathrm{Disc}_\infty = \mathrm{Cut}_s)$

Monodromies

following :

[Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

- Harmonic Polylog : $G(\alpha_n, \ldots, \alpha_1; x)$ $\qquad \alpha_k \in \{0,1\}$
- vector of derivatives : $$\mathscr{V}_i = \begin{cases} 1 & \text{if } i = 0\,, \\ (-1)^{\#\text{ of } 1}\, G(\alpha_{n+1-i}, \ldots, \alpha_n, x) & \text{if } n \geq i > 0 \end{cases}$$
- connection matrix : $$\omega_{ij} = \frac{dx}{x - \alpha_{n-i}}\,\delta_{i+1,j} \qquad \text{s.t.} \qquad d\mathscr{V} = \mathscr{V}\cdot\omega$$
- variation matrix : $\mathscr{M}_\gamma = \mathscr{P}\, e^{\int_\gamma \omega}$ $\qquad$ collects all $n+1$ solutions for $\mathscr{V}$
- general solution : $$\left(\mathscr{M}_{\to x}\right)_{ij} = \sum_{k=0}^{n} (-1)^{\#\text{ of } 1}\, G(\alpha_{n-i}, \ldots, \alpha_{n-i-k+1}, x)\,\delta_{i+k,j}$$ $$G(x) = 1$$
- monodromy matrices : $$\mathscr{M}_0 = \mathscr{M}_{\circlearrowleft_0}\,,$$ $$\mathscr{M}_1 = \mathscr{M}_{\to 1}\,\mathscr{M}_{\circlearrowleft_1}\,\mathscr{M}_{\to 1}^{-1}\,.$$
- discontinuities : $$\mathrm{Disc}_0 = (1 - \mathscr{M}_0)\cdot\mathscr{M}_{\to x}\,,$$ $$\mathrm{Disc}_1 = -\,(1 - \mathscr{M}_1)\cdot\mathscr{M}_{\to x}\,,$$ $$\mathrm{Disc}_\infty = (1 - \mathscr{M}_0\cdot\mathscr{M}_1)\cdot\mathscr{M}_{\to x}\,.$$

Ansatz matching

- example ansatz : $c_{1,1}\,G(1,1;x)+c_{1,0}\,G(1,0;x)+c_{0,1}\,G(0,1;x)+c_{0,0}\,G(0,0;x)+c_1\,G(1;x)+c_0\,G(0;x)+c$
- impose e.g. $\mathrm{Disc}_0=0$: $c_{1,1}\,G(1,1;x)+0+c_{0,1}\,G(0,1;x)+0+c_1\,G(1;x)+0+c$
- $\mathrm{Disc}_1=2\pi i\,(2\,G(1;x)+3\,G(0;x)+5\,\pi i)$ $=c_{1,1}\,(2\pi i\,(-G(1;x)+i\pi))+c_{0,1}\,(-2\pi i\,G(0,x))+c_1\,(-2\pi i)+0$
- now only constant unconstrained : $-2\,G(1,1;x)-3\,G(0,1;x)-7\,i\pi\,G(1;x)+c$
- impose fixed value e.g. ζ_2 at x=0 : $-2\,G(1,1;x)-3\,G(0,1;x)-7\,i\pi\,G(1;x)+\zeta_2$

Explicit reduction coefficients

$$\mathcal{I}_{L,n} = \prod_{i=1}^{D(n,L)} \frac{1}{\mathcal{D}_i} = \sum_{\substack{i_1,\ldots,i_A=1 \\ i_j \neq i_m}}^{D(n,L)} \frac{c_{i_1,\cdots,i_A}}{\mathcal{D}_1 \cdots \hat{\mathcal{D}}_{i_1} \hat{\mathcal{D}}_{i_2} \cdots \hat{\mathcal{D}}_{iA} \cdots \mathcal{D}_{D(n,L)}}$$

with

$$c_{i_1,\cdots,i_A} = \frac{(-B_{i_1,\cdots,i_A})^A}{B_{0,\cdots,i_A} B_{i_1,0,\cdots,i_A} \cdots B_{i_1,\cdots,0}}$$

$$A = D(n,L) - D(N(L,d_0),L)$$

$$B_{i_1,\cdots,i_A} = \begin{vmatrix} \alpha_{D(N(L,d_0),L)+1,i_1} & \cdots & \alpha_{D(N(L,d_0),L)+1,i_A} \\ \alpha_{D(N(L,d_0),L)+2,i_1} & \cdots & \alpha_{D(N(L,d_0),L)+2,i_A} \\ \vdots & \vdots & \vdots \\ \alpha_{D(n,L),i_1} & \cdots & \alpha_{D(n,L),i_A} \end{vmatrix}$$

$$\mathcal{D}_{i>D(N(L,d_0),L)} = \alpha_{i,0} + \sum_{j=1}^{D(N(L,d_0),L)} \alpha_{i,j} \mathcal{D}_j$$

$$\alpha_{i,i} = -1$$

$$\alpha_{i,j>D(N(L,d_0),L)} = 0$$

partial fractions implemented in Mathematica package **Apart** [Feng 1204.2314]

Leading singularity

no ISPs $\Rightarrow$ **maximal cut** localizes [Bosma, Sogaard, Zhang 1704.04255]

true for finite basis topologies t

e.g. at 2 loops

$$I^{(t)}_{\text{max}} = \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \prod_{i=1}^{11} 2\pi i\, \delta(\mathscr{D}_i^{(t)}) = c(d) \frac{G^{(5-d)/2}}{B_t^{(7-d)/2}}$$

with

$$c(d) = \frac{1}{4^d \pi^{9/2} \Gamma((d-4)/2)\Gamma((d-5)/2)}$$

$$G = \det(\{p_1, \ldots, p_4\} \cdot \{p_1, \ldots, p_4\})$$

$$B_t = \det(\{k_1, k_2, p_1, \ldots, p_4\} \cdot \{k_1, k_2, p_1, \ldots, p_4\})\,|_{\mathscr{D}_i^{(t)}=0\,,\; i=1,\ldots,11}$$

leading singularity for higher-loop finite basis topologies

$$\mathscr{S}_{t,L} = \frac{\sqrt{G}}{\left(\sqrt{B_{t,L}}\right)^{L+1}}$$