Three-loop Feynman integrals and an application to three-point form factors in N=4 super Yang-Mills

Jungwon Lim

Based on work in collaboration with Thomas Gehrmann, Johannes Henn, Petr Jakubčík, Cesare Carlo Mella, Nikolaos Syrrakos, Lorenzo Tancredi and William J. Torres Bobadilla

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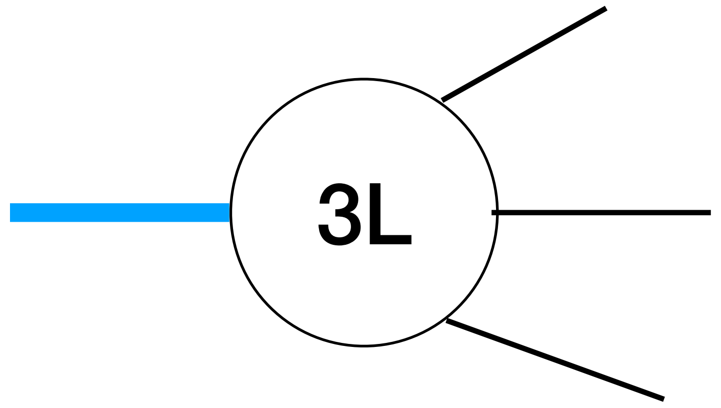


Motivation

- **Phenomenological motivation**

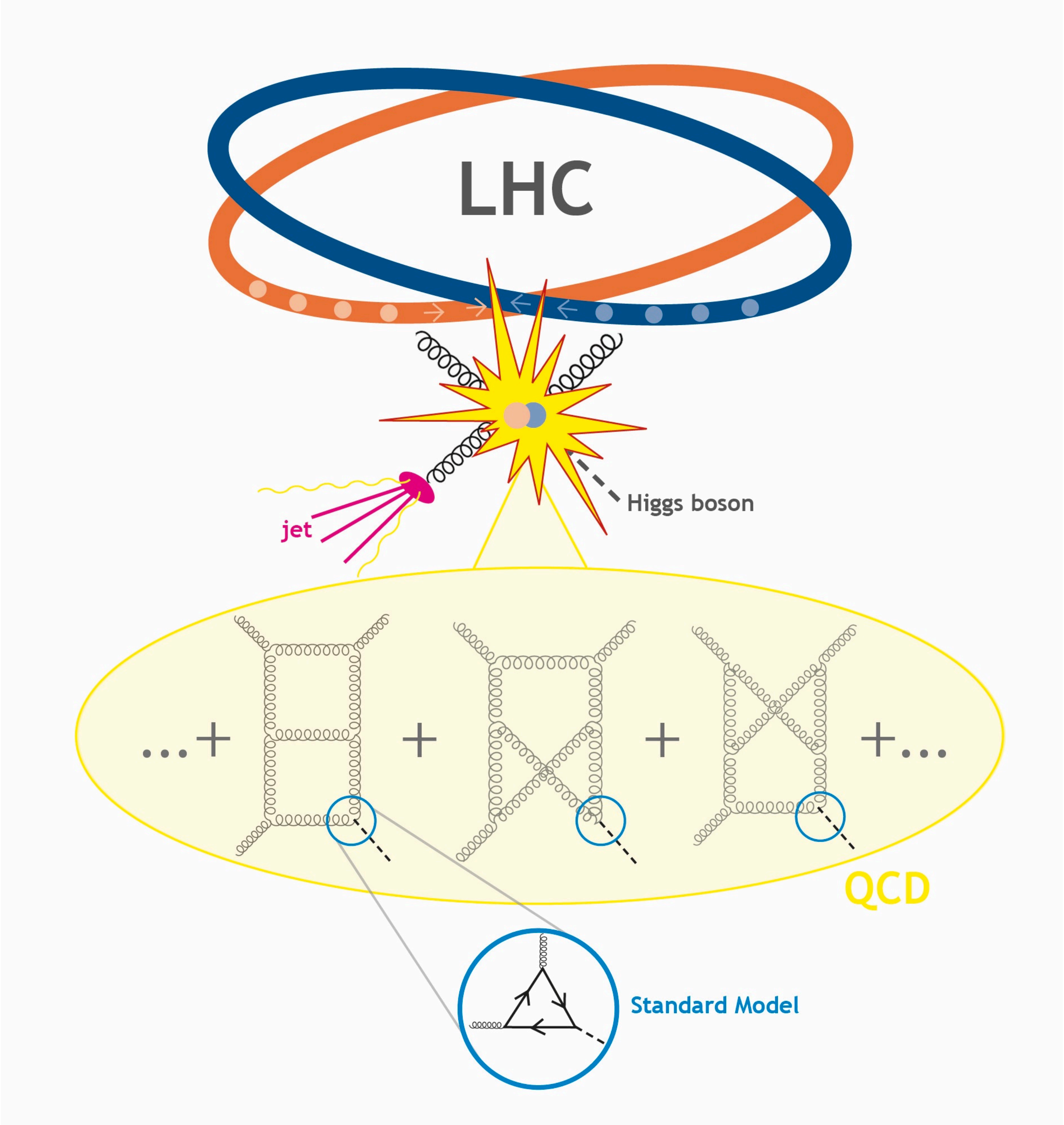
  Higgs/Vector+jet toward N3LO

- **Mathematical motivation**

  - Cluster algebra → Adjacency conditions
  - Antipodal duality

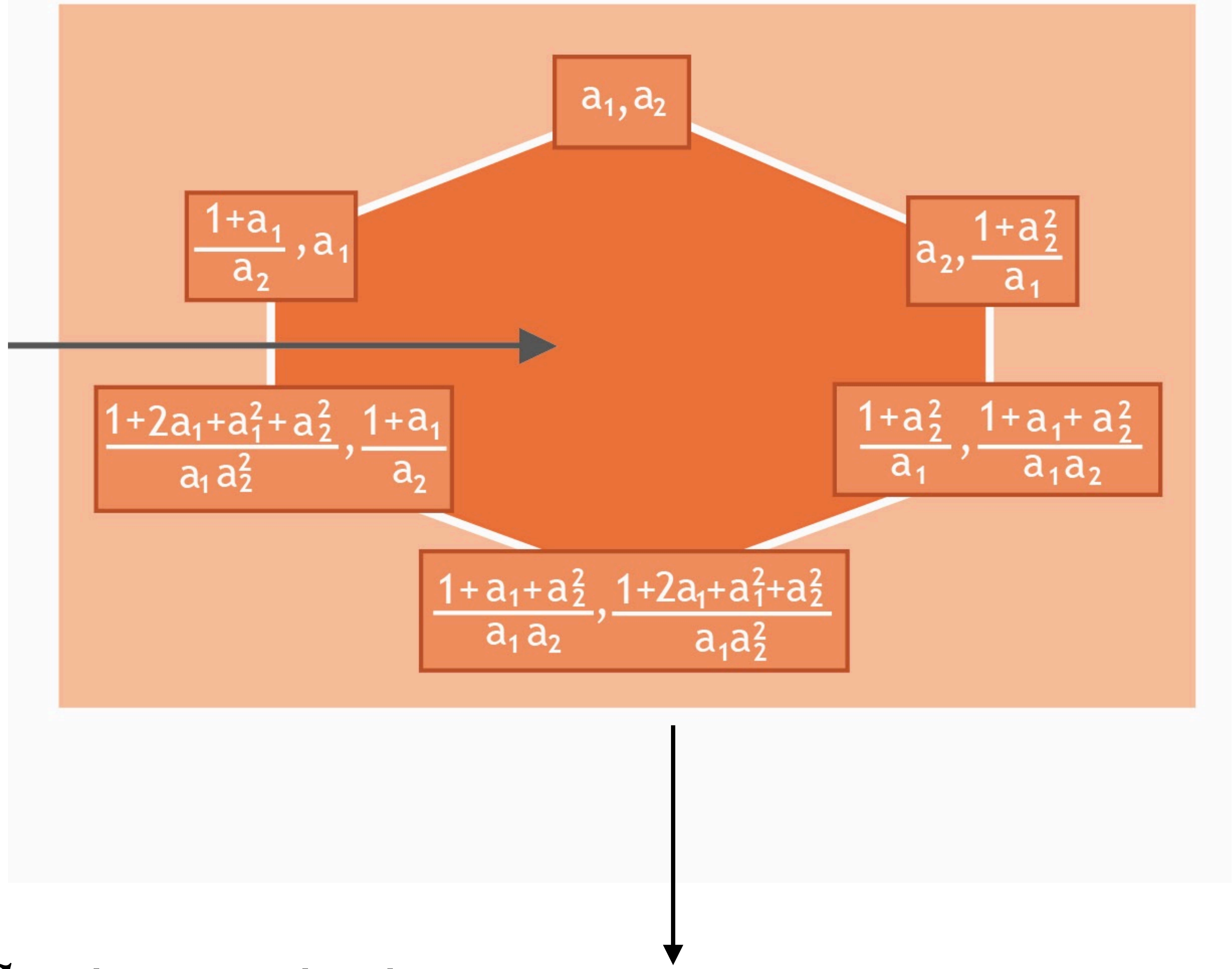
  [Dixon, Gürdoğan, McLeod and Wilhelm 2021]

➡ Bootstrapping approach

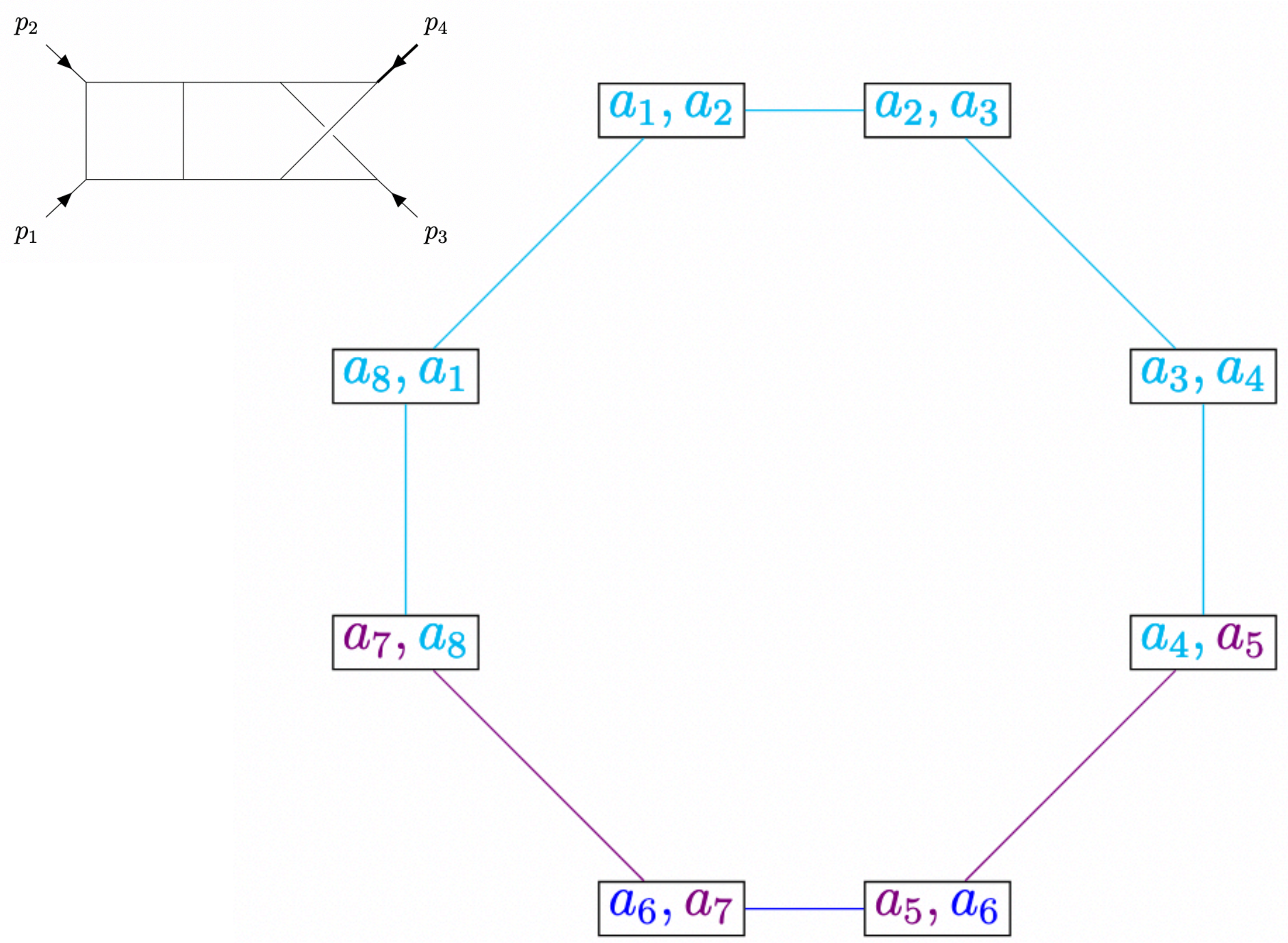


www.scattering-amplitudes.com

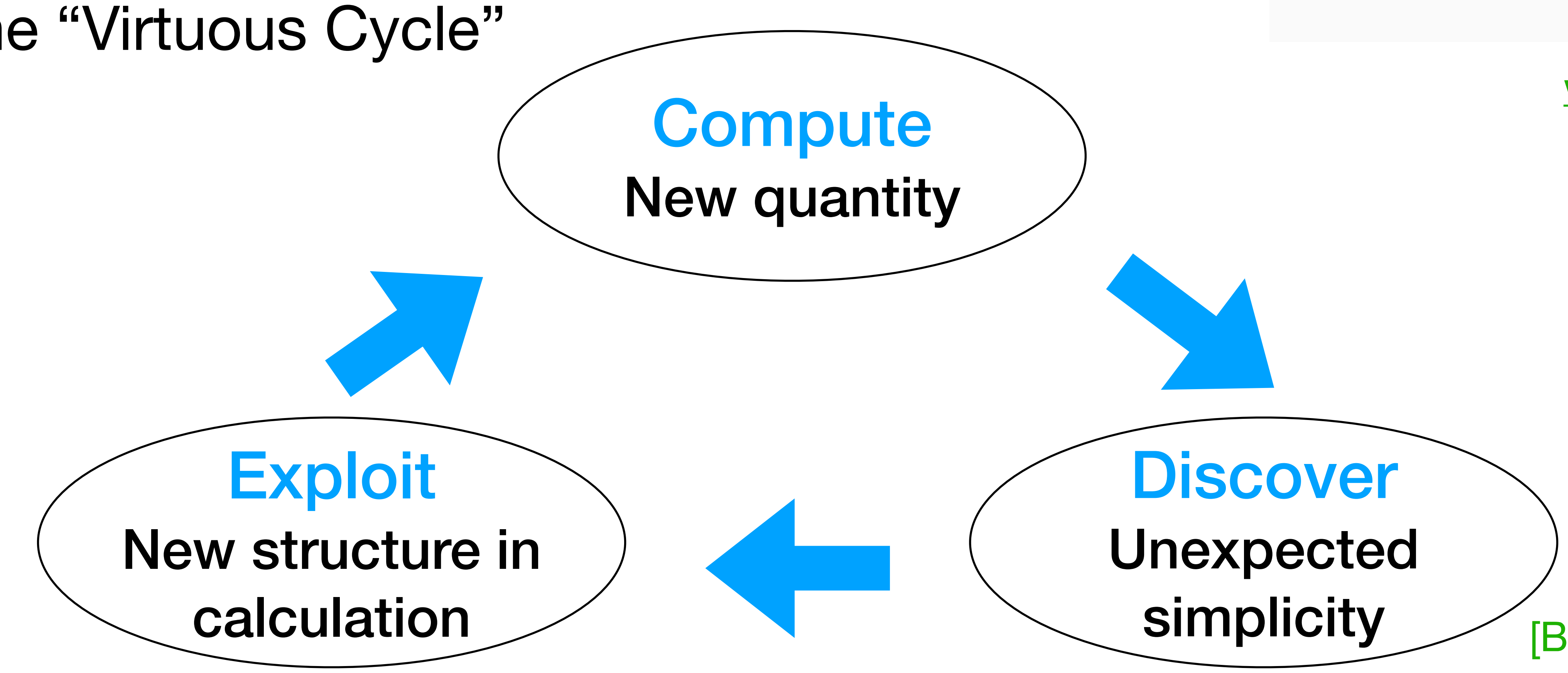
C_2 cluster algebra [Chicherin, Henn, Papathanasiou 2020]



G_2 cluster algebra [Aliaj, Papathanasiou 2024]

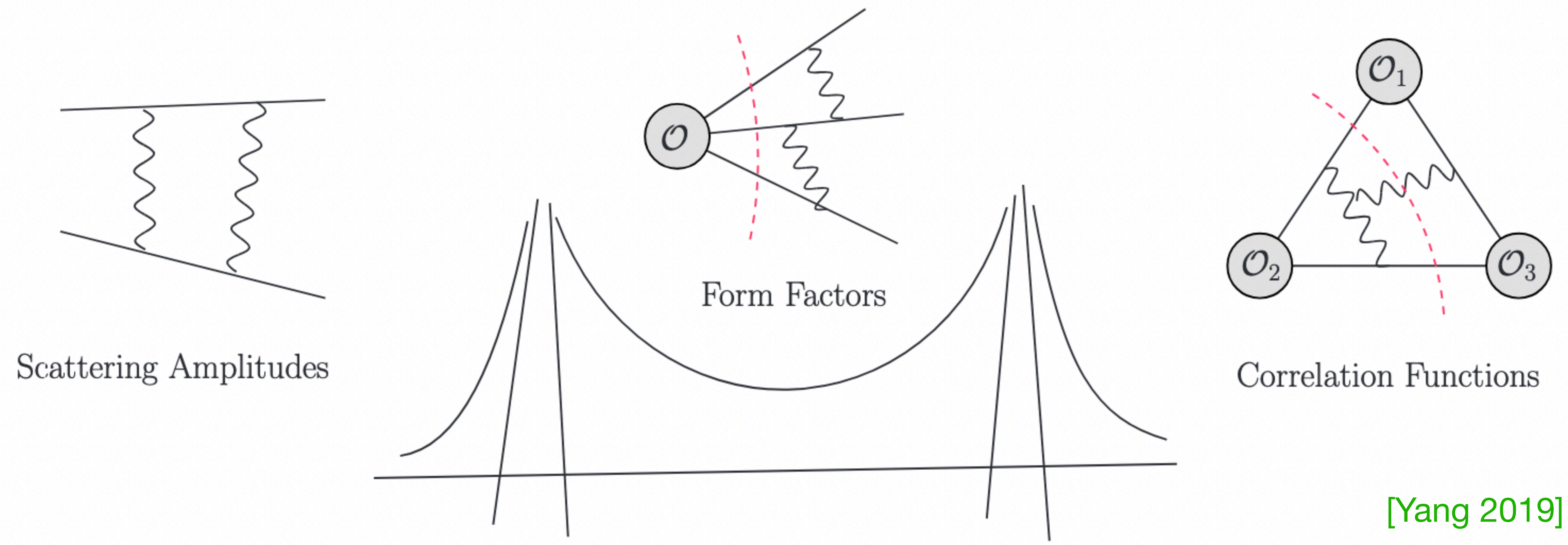


The "Virtuous Cycle"



[Bern's talk in Snowmass 2022]
[McLeod's talk in Galaxy meets QCD 2024]

Three-point form factor to Higgs+jet



[Yang 2019]

Form factors are the matrix elements between on-shell asymptotic states and gauge invariant operators.

Form factors are partially on-shell and partially off-shell, being bridge connecting the amplitudes and correlation functions

$$\mathcal{F}_{\mathcal{O}}(p_1, p_2, p_3, ; q) = \int d^D x e^{-iq\cdot x} \left\langle \Phi_1 \Phi_2 \Phi_3 \,\middle|\, \mathcal{O}(x) \middle|\, 0 \right\rangle$$

[Brandhuber, Travaglini, Yang 2012]

In this talk, we will only focus on form factor of BPS operator $\mathcal{O} = \text{tr}\phi^2, \text{tr}\phi^3$ as they are closely related to Higgs amplitude in QCD.

There are many evidence that the form factor support maximal transcendentally principle (MTP), which conjecture that N=4 SYM captures the maximally transcendental part of QCD

Differential equation method

Building canonical differential equations

[Henn 2013]

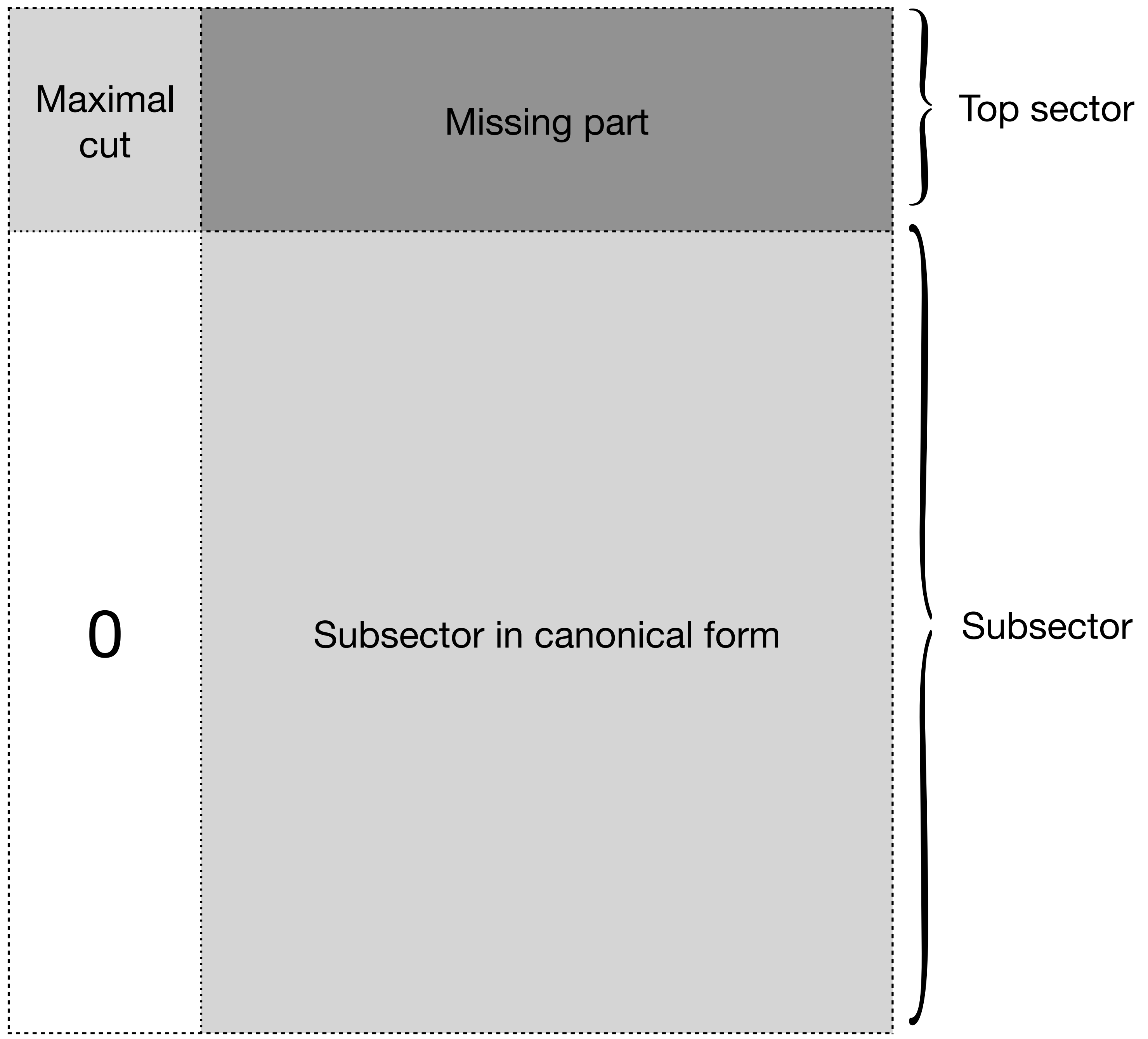
Good choice of basis for Feynman integrals can significantly simplify the computation of differential equation.

$$d\vec{f}\left(\vec{x},\epsilon\right) = \epsilon\left(d\tilde{A}\right)\vec{f}\left(\vec{x};\epsilon\right), \text{ with } \tilde{A} = \left[\sum_k A_k \log \alpha_k(x)\right]$$

- Subsector : DlogBasis, Mapping from other families, loop-by-loop approach

[Wasser 2022] [Flieger, Torres Bobadilla, 2022]

- Topsector : Matrix rotation, loop-by-loop approach



By FiniteFlow, Kira

[Peraro, 2019] [Klappert, Lange, Maierhöfer, Usovitsch, 2020]

Iterated integrals

$\vec{f}\left(\vec{x}, \epsilon\right) = \mathbb{P}\exp\left[\epsilon \int_\gamma d\tilde{A}\right]\vec{f}_0(\epsilon)$ where $\vec{f}_0(\epsilon)$ is a boundary vector

$\gamma^*\left(\omega_i\right) = k_i(t)\,dt$ function k_i are defined by pulling back the 1-form ω_i to the interval [0,1]

An ordinary line integral is given by $\int_\gamma \omega_1 = \int_{[0,1]} \gamma^*\left(\omega_1\right) = \int_0^1 k_1\left(t_1\right) dt_1$

Iterated integral of $\omega_1 \ldots \omega_n$ along γ is defined by $\int_\gamma \omega_1 \ldots \omega_n = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} k_1\left(t_1\right) dt_1 \ldots k_n\left(t_n\right) dt_n$

[Chen 1977]

If the alphabet is rational functions , one can write the answer in terms of Goncharov polylogarithms

$$G(\vec{a}_n; z) \equiv G(\vec{a}_1, \vec{a}_{n-1}; z) \equiv \int_0^z \frac{dt}{t - a_1} G(\vec{a}_{n-1}; t)$$

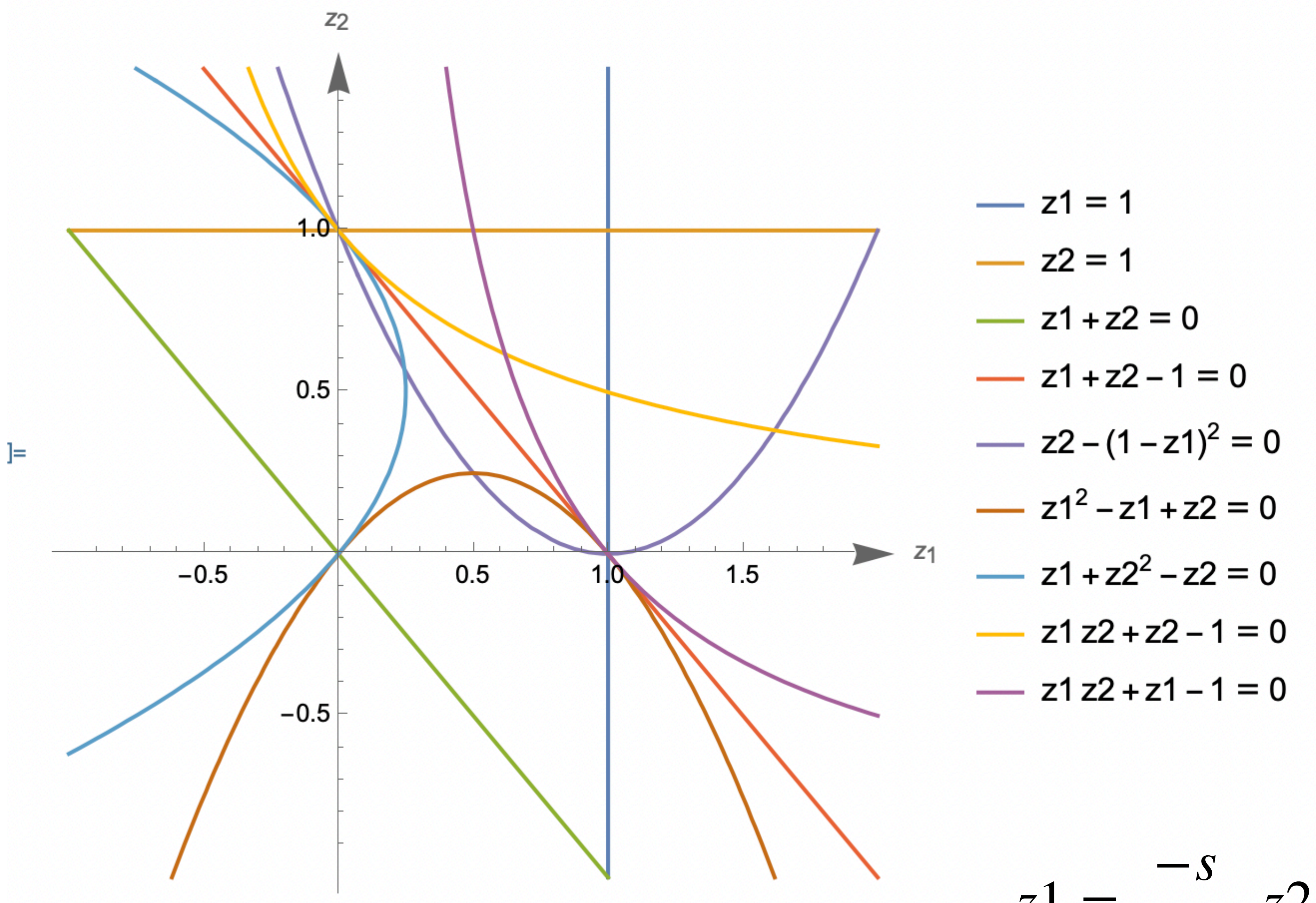
with $G\left(a_1; z\right) = \int_0^z \frac{dt}{t - a_1}$ and $G(\vec{0}_n; z) \equiv \frac{1}{n!}\log^n(z)$

Function space of 3loop integrals

$$\vec{\alpha} = \{p_4^2, s, t, p_4^2 - s - t, p_4^2 - s, p_4^2 - t, s + t, \frac{(p_4^2 - s - t)s - R}{(p_4^2 - s - t)s + R}, \frac{st - R}{st + R}, p_4^4 - t\left(p_4^2 + s\right), p_4^4 - s\left(p_4^2 + t\right), \; t^2 + p_4^2(s - t), s^2 - p_4^2(s - t), -p_4^2 t + \left(p_4^2 - s\right)^2\}$$

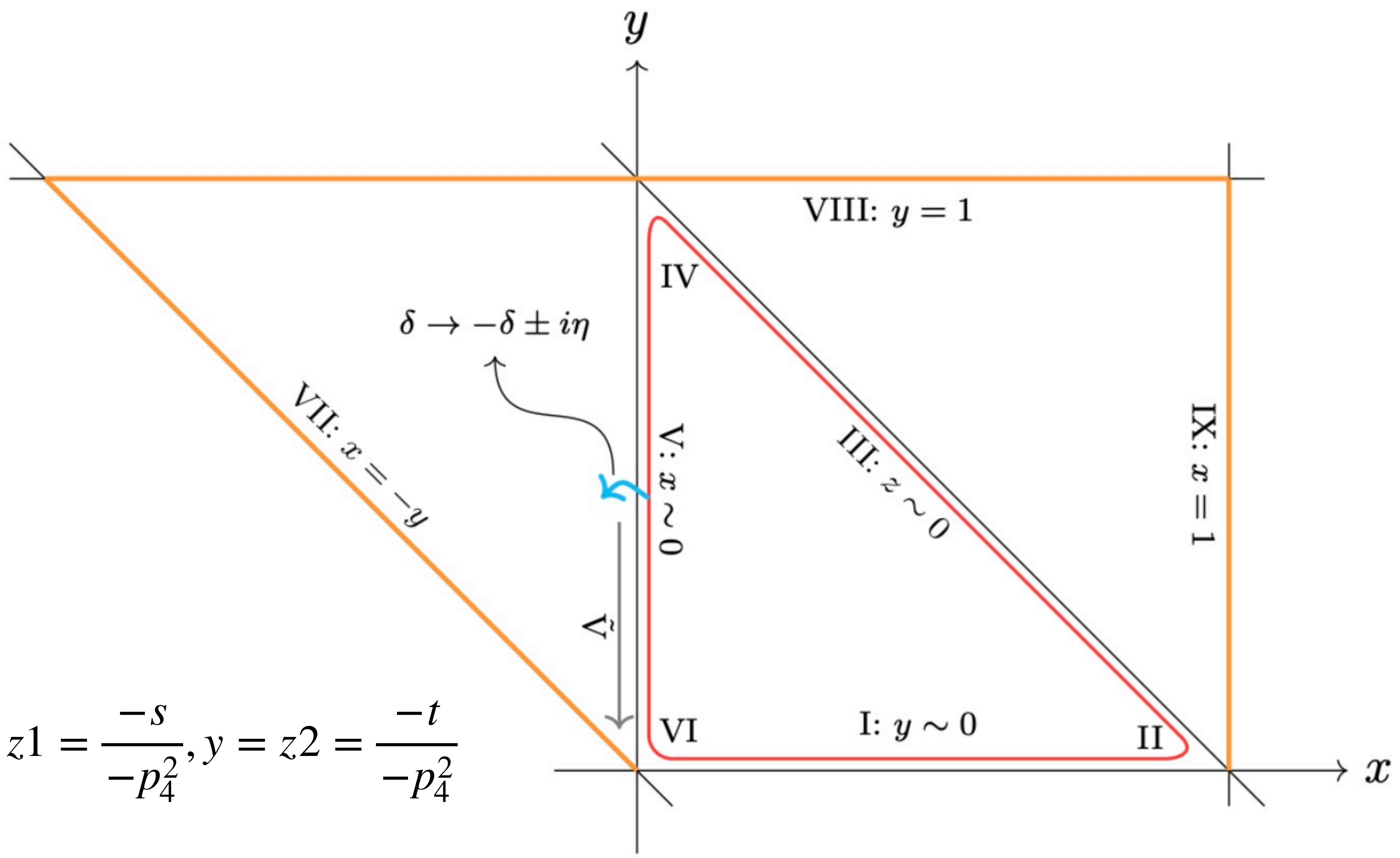
with $R = \sqrt{-p_4^2 s (p_4^2 - s - t) t}$

➡ **21 letters (including kinematic crossings)**



$$z1 = \frac{-s}{-p_4^2}, z2 = \frac{-t}{-p_4^2}$$

Fixing boundary constants



$$x = z1 = \frac{-s}{-p_4^2}, y = z2 = \frac{-t}{-p_4^2}$$

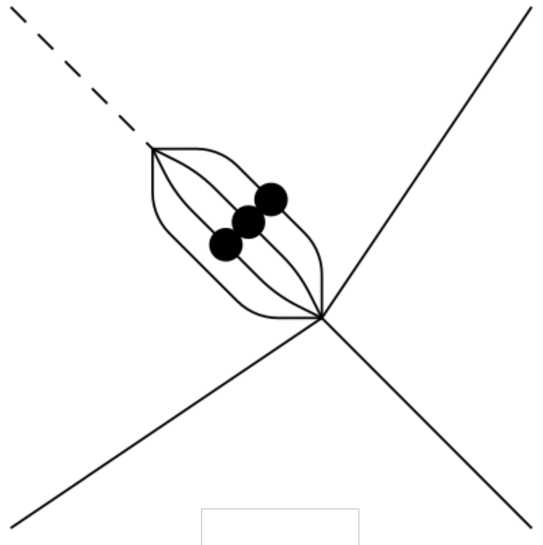
segment	start-/end-point	$x(\delta, t)$	$y(\delta, t)$
I	$P_1 \to P_2$	t	δ
II	$P_2 \to P_3$	$1 - \delta\left[(1-t)^2 + t^2\right]$	δt^2
III	$P_3 \to P_4$	$1 - t - \delta$	t
IV	$P_4 \to P_5$	$\delta(1-t)^2$	$1 - \delta\left[(1-t)^2 + t^2)\right]$
V	$P_5 \to P_6$	δ	$1 - t$
VI	$P_6 \to P_7$	δt^2	$\delta(1-t)^2$
VII	$P_7 \to P_8$	$-t(1-\delta)$	t
VIII	$P_8 \to P_9$	$(-1+t) \cup t$	$1 - \delta$
IX	$P_9 \to P_{10}$	$1 - \delta$	$1 - t$

At each segment, your "effective" alphabet is only rational. $\vec{\alpha}_t = \{t, t - \frac{1}{2}, t - 1, t - \frac{e^{i\pi/4}}{\sqrt{t}}, t - \frac{e^{-i\pi/4}}{\sqrt{t}}\}$

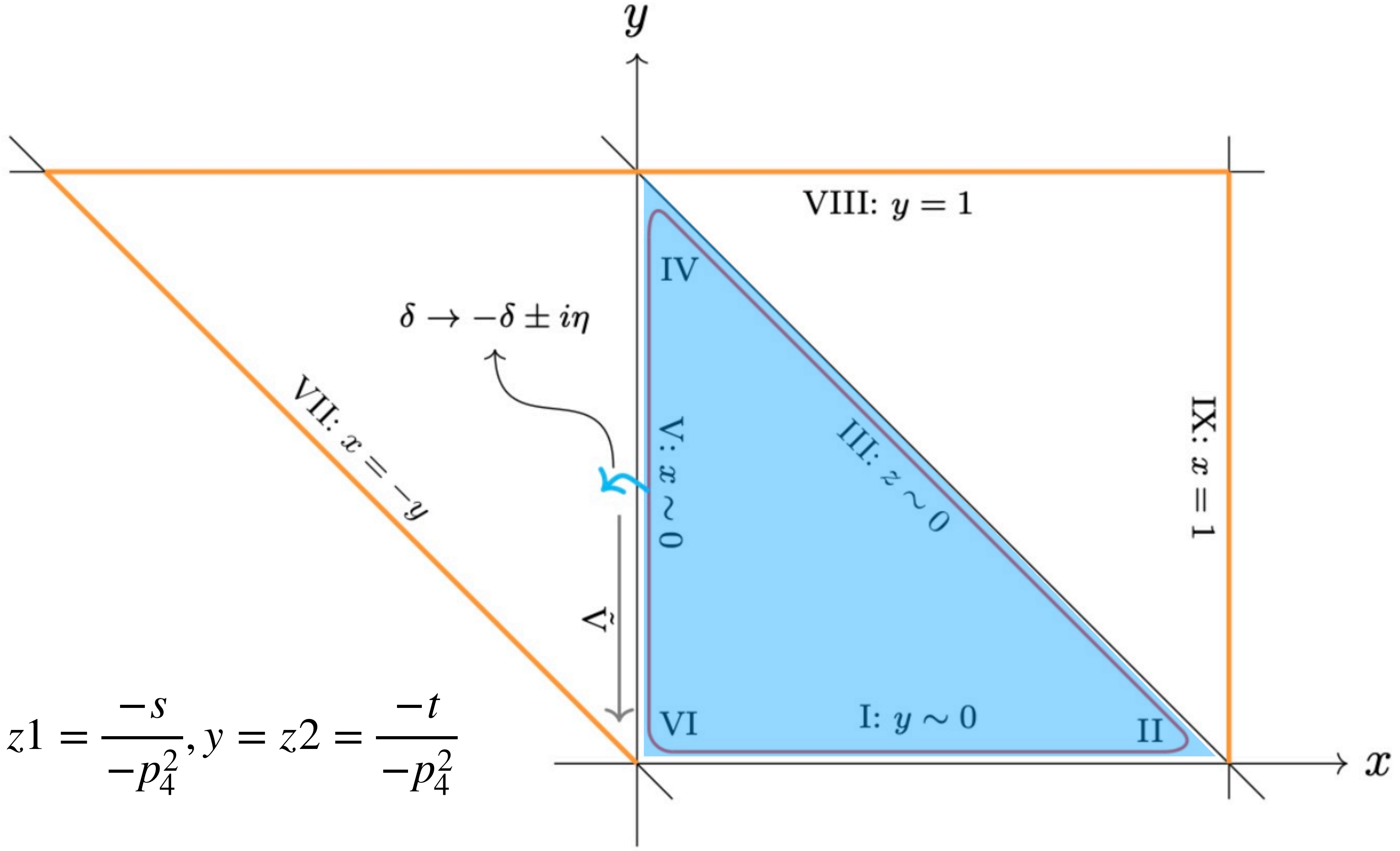
At each segment, one can impose the constraint with the information of singularities.

By matching, one can relate the boundary vector in one segment to another.

➡ Fix all the boundary constants up to one integral.



Fixing boundary constants



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VI	$P_6 \to P_7$	δt^2	$\delta(1-t)^2$
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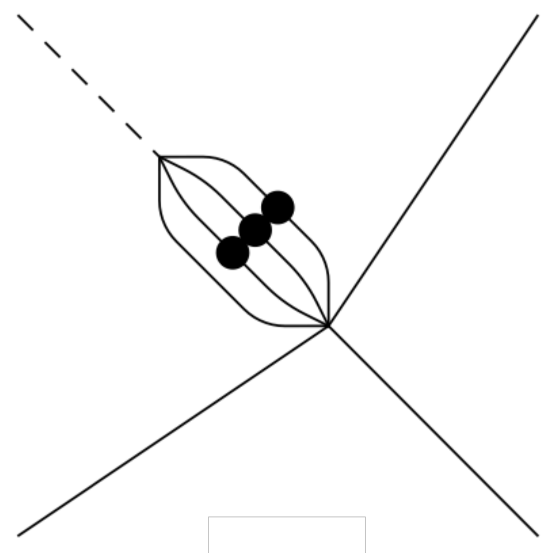
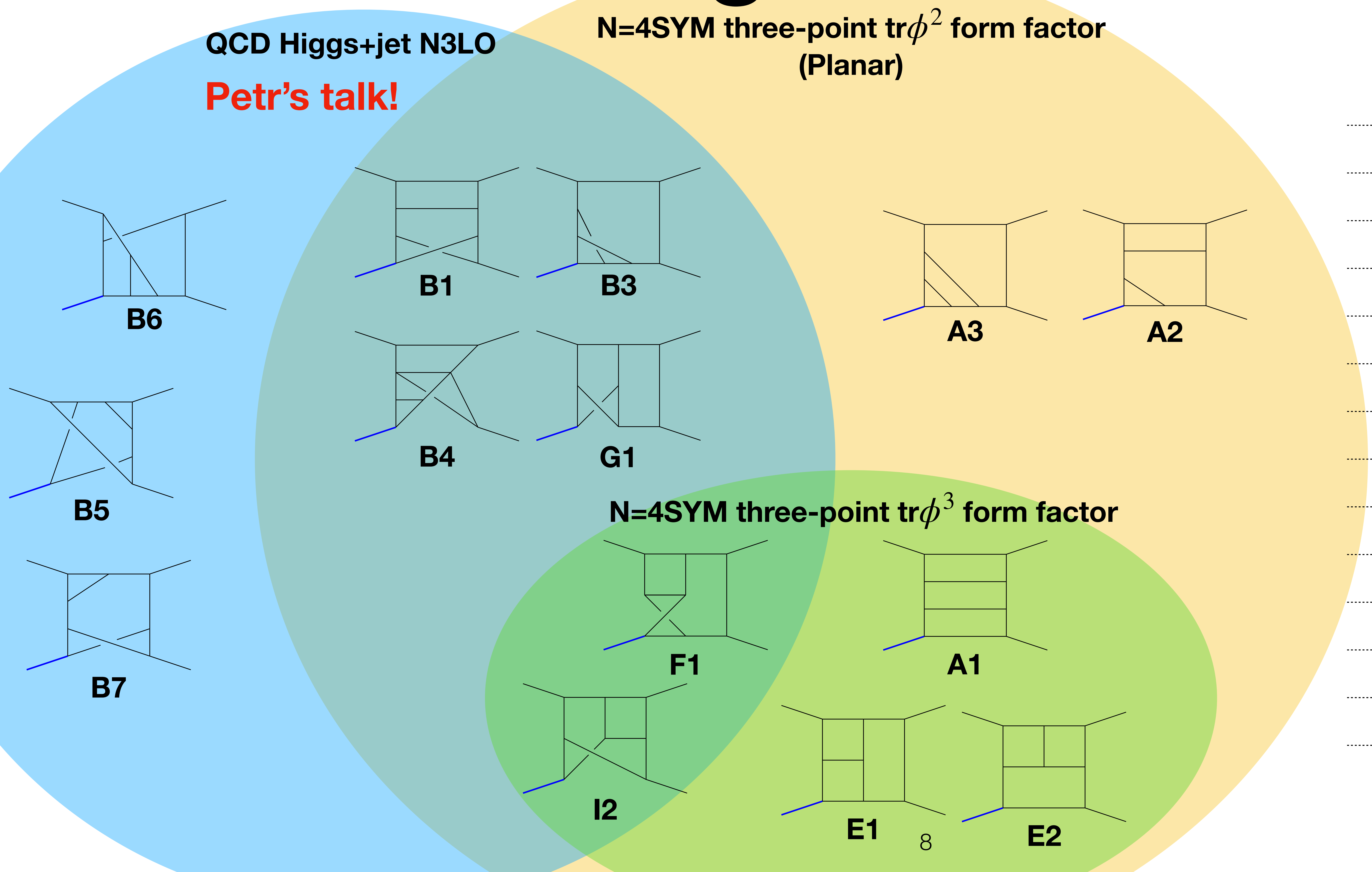


[Gehrmann, Henn, Jakubčík, JL, Mella, Syrrakos, Tancredi, Torres Bobadilla (to appear)]

Integral Families

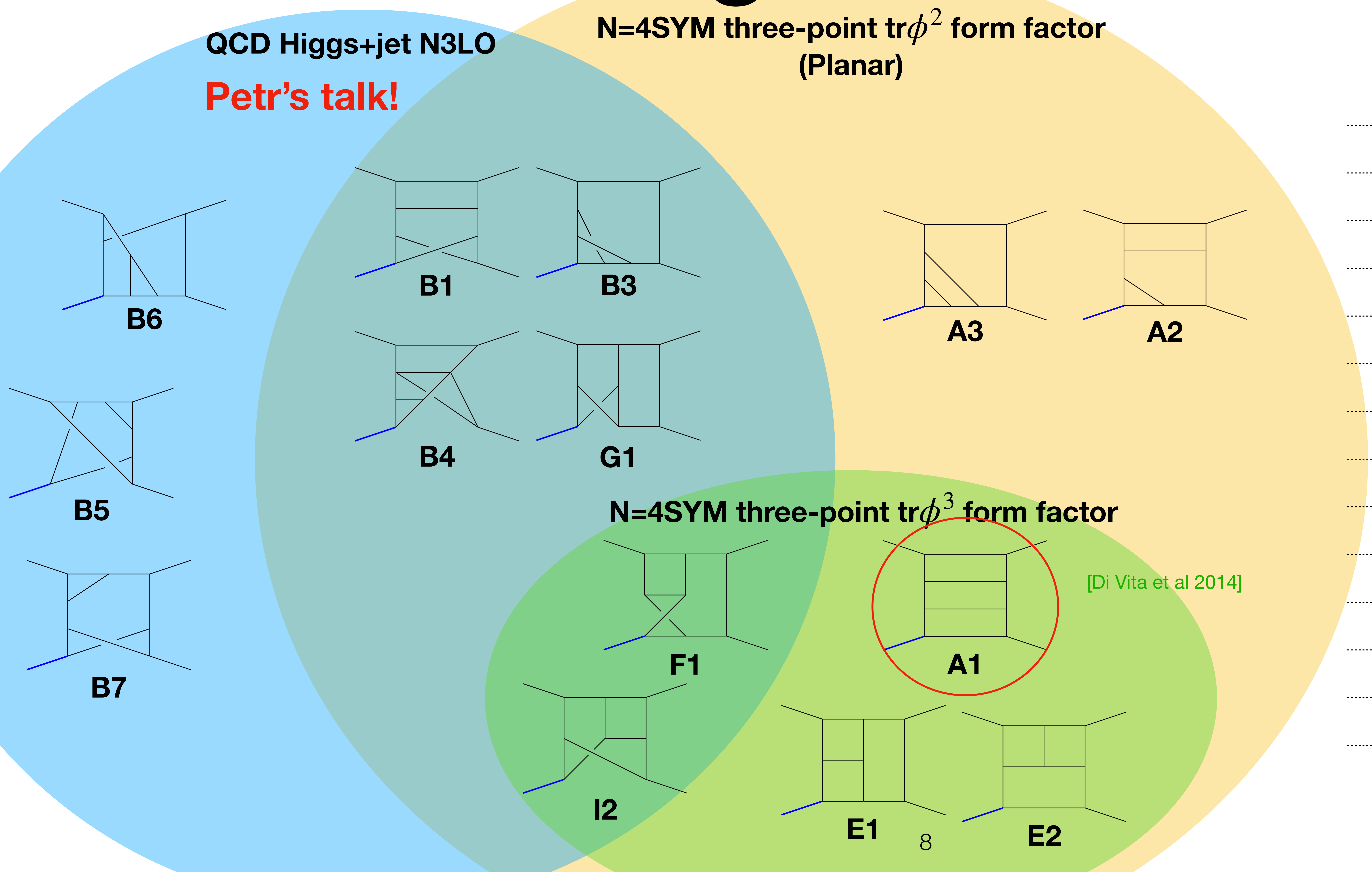


	# MI
A1	83
A2	100
A3	80
B1	150
B3	90
B4	143
B5	70
B6	150
B7	89
E1	166
E2	117
F1	214
G1	254
I2	305

(Family : reducible top sector)

[Gehrmann, Henn, Jakubčík, JL, Mella, Syrrakos, Tancredi, Torres Bobadilla (to appear)]

Integral Families



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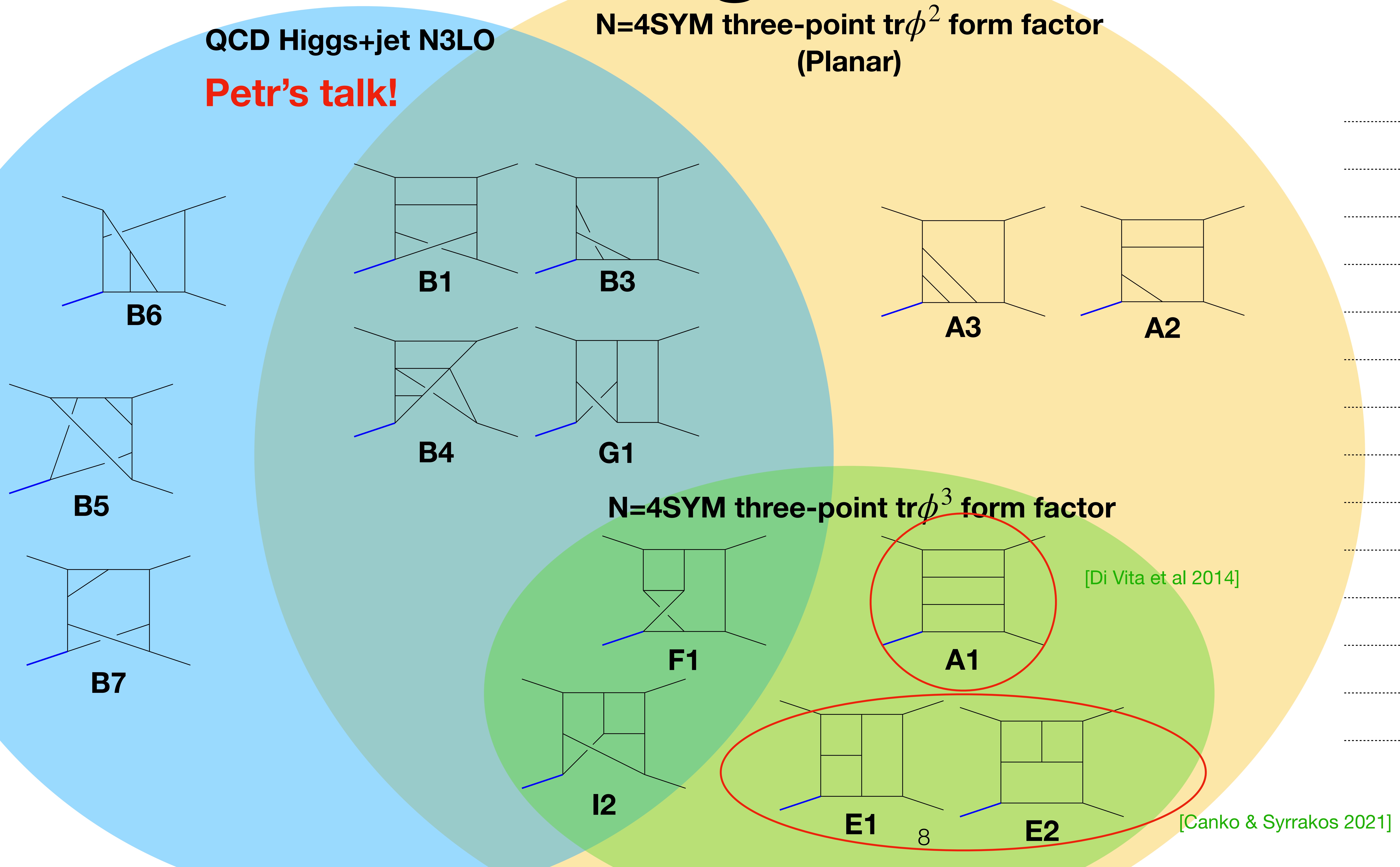

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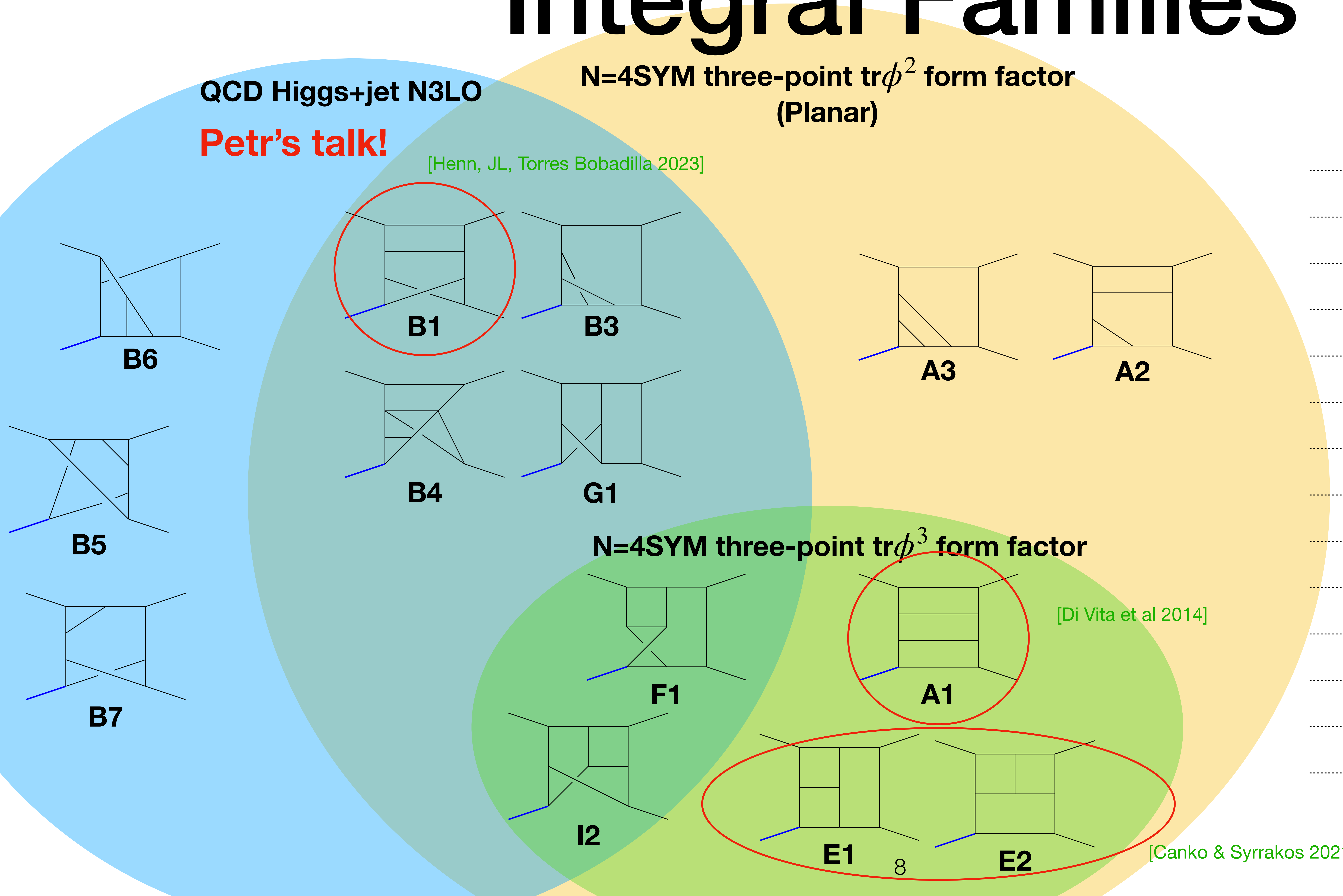

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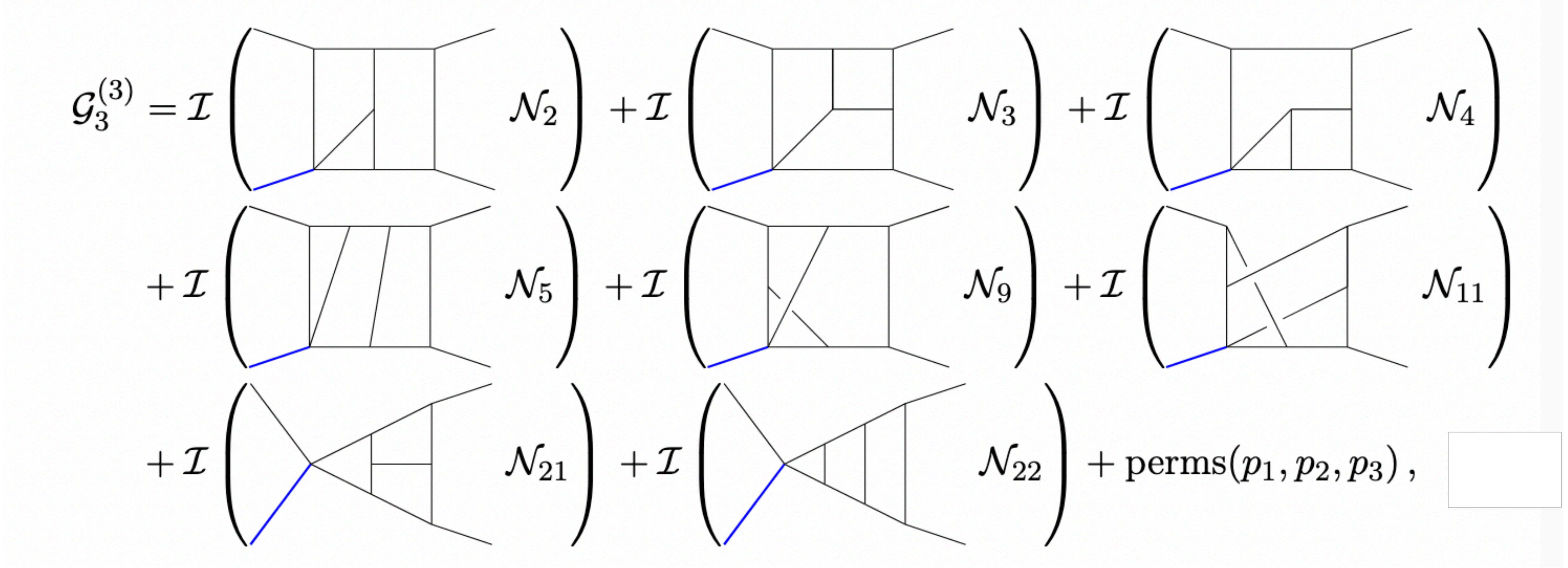
Computing trϕ^3 form factor

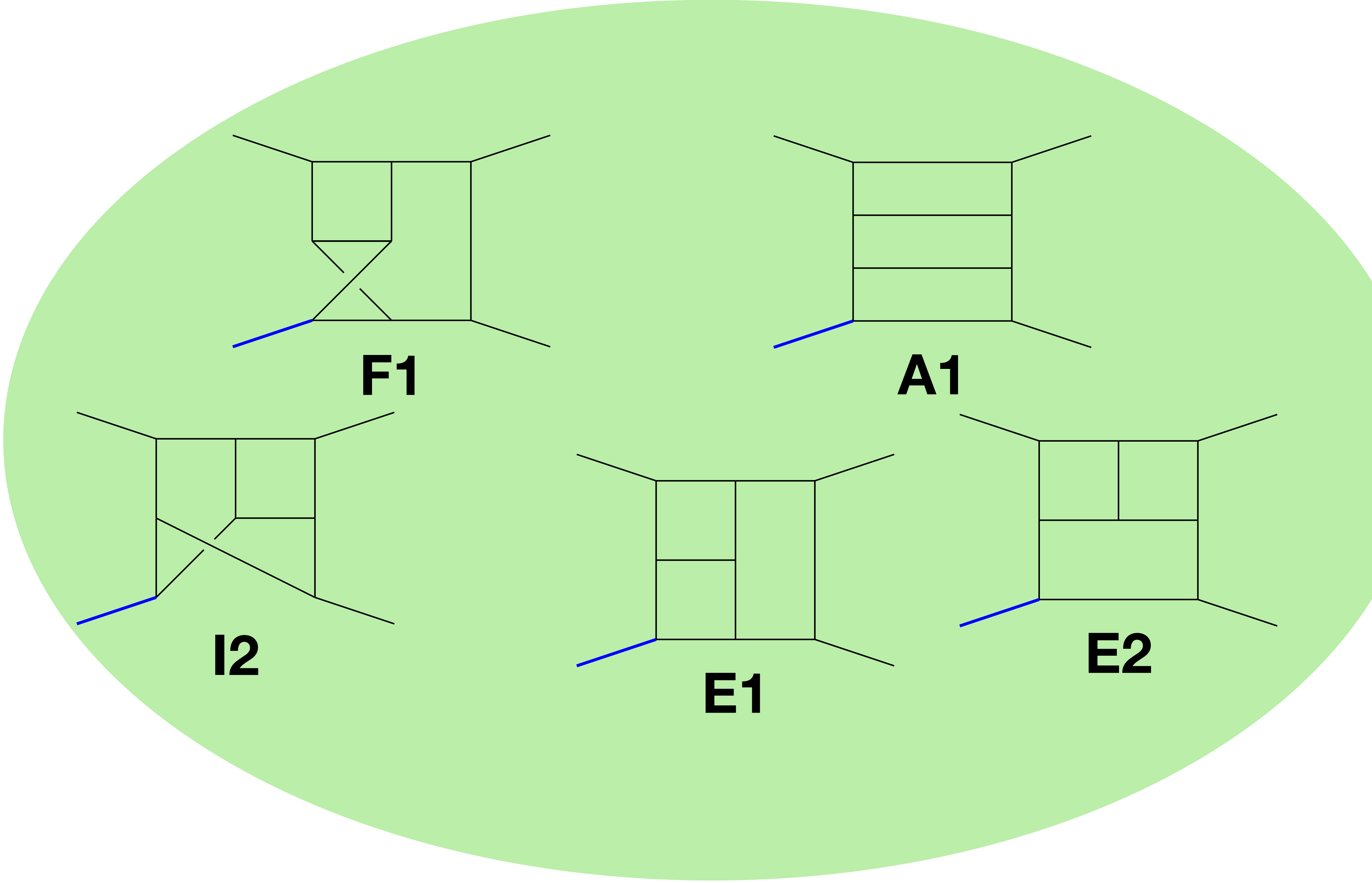
[Henn, JL, Torres Bobadilla (to appear)]



[Lin, Yang, Zhang 2021]

IBP reduction

[Peraro, 2019]



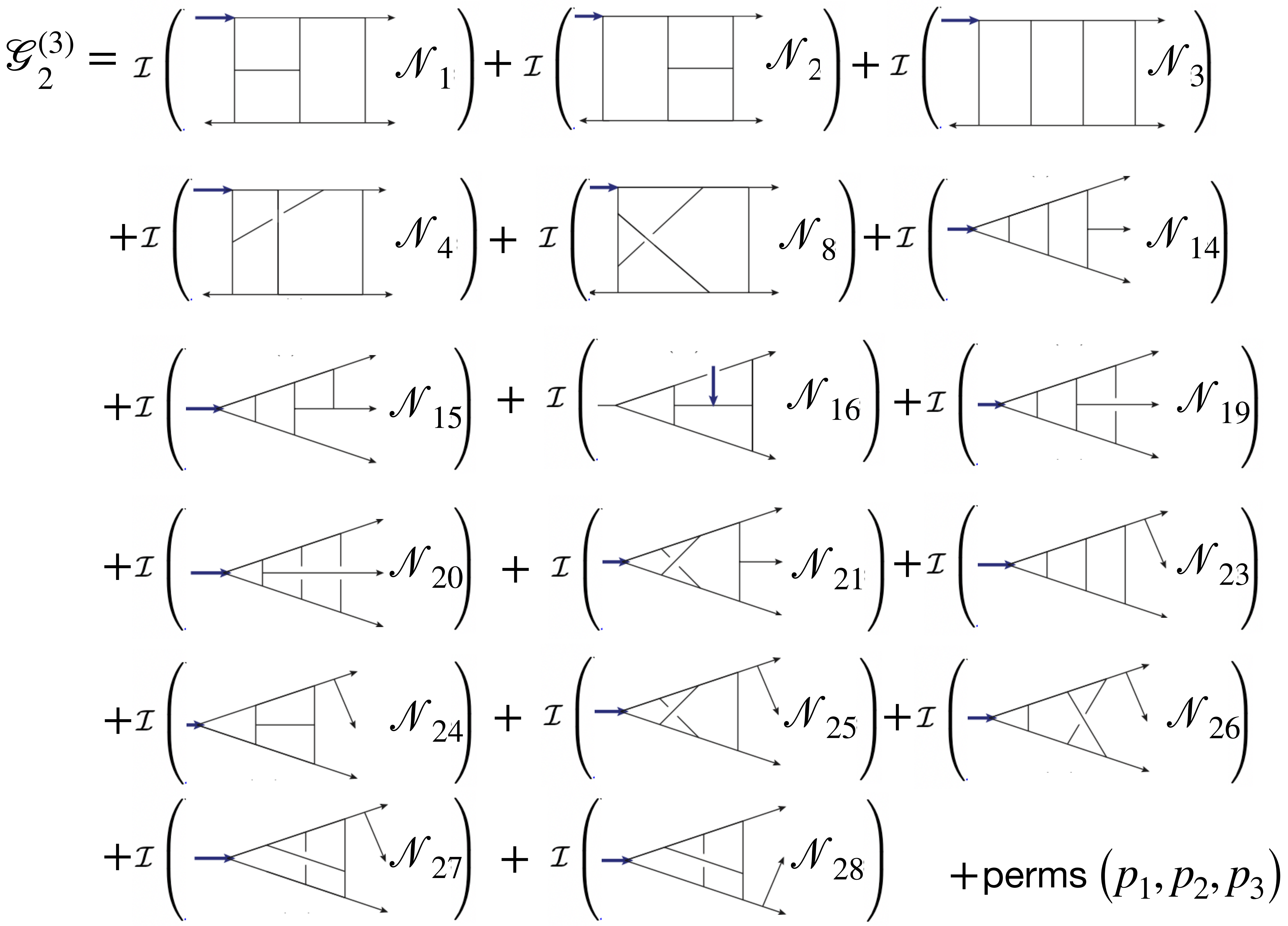
$$\mathcal{G}_3^{(3)} = c_1 ut[Ax123,1] + c_2 ut[Ax123,2] + \ldots$$

$c_1, c_2, \ldots$ are rational numbers!

	Ordering						
Family	123	132	213	231	312	321	Total
A	78	73	46	71	44	40	352
E1	12	10	11	10	11	10	64
E2	23	20	18	19	18	14	112
F1	32	26	28	22	23	19	150
I2	2	0	0	0	0	0	2

Computing trϕ^2 form factor

[Gehrmann, Henn, Jakubčík, JL, Mella, Syrrakos, Tancredi, Torres Bobadilla (to appear)]

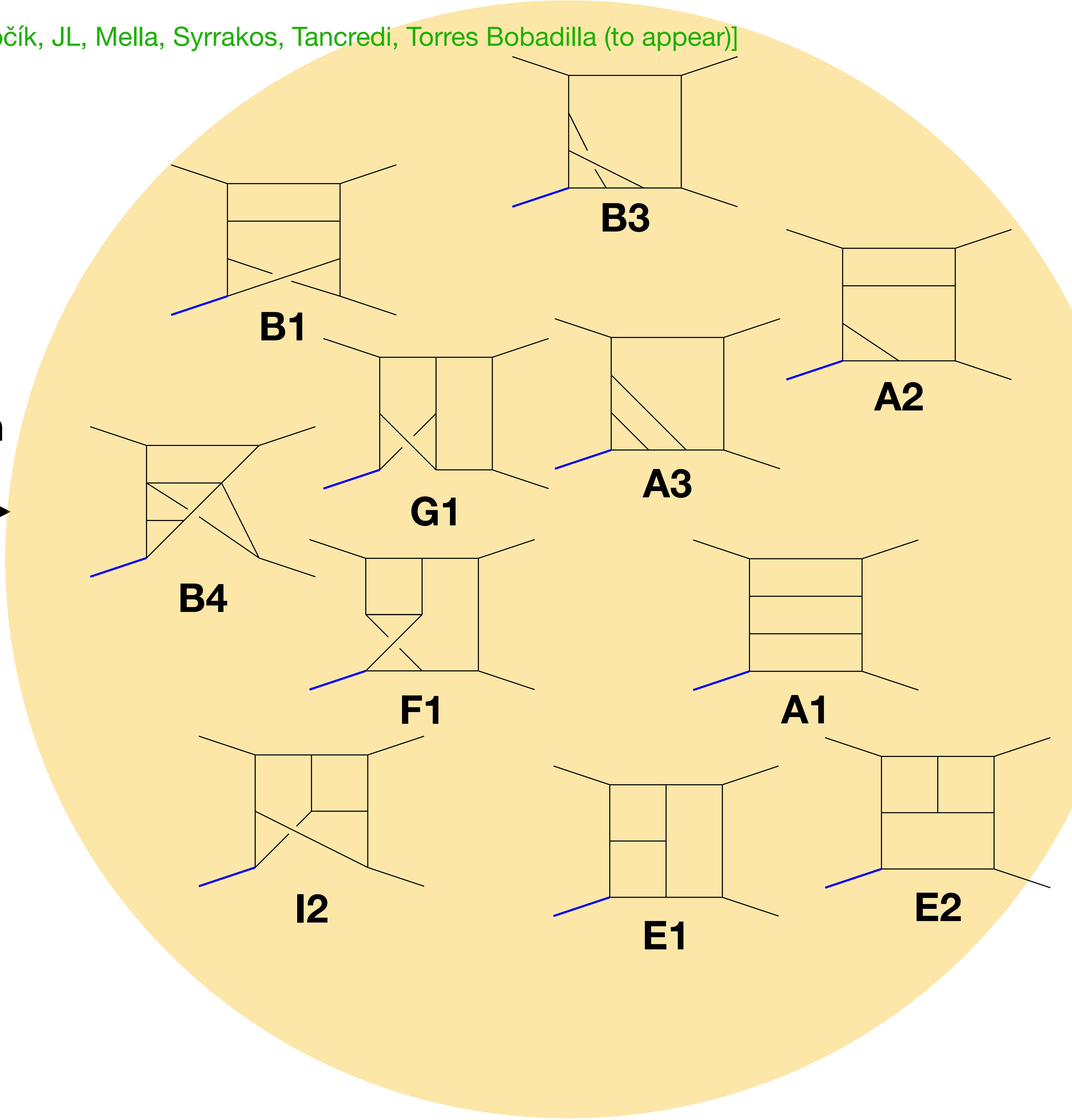


[Lin, Yang, Zhang 2021]

IBP reduction

[Peraro, 2019]

→



$$\mathcal{G}_2^{(3)} = c_1 ut[Ax123,1] + c_2 ut[Ax123,2] + \ldots$$

$c_1, c_2, \ldots$ are rational numbers!

Different normalisation of finite remainders

There's still freedom in choice of finite part of 1-loop function.

BDS finite remainder [Bern, Dixon, Smirnov 2005]

$$\mathcal{G}_{\mathcal{O}} = \mathcal{G}_{\mathcal{O}}^{\text{BDS}} \exp\left(\mathcal{R}\right)$$

$$\mathcal{R} = \sum_{L=1}^{\infty} g^{2L} \mathcal{R}^{(L)}$$

$$\mathcal{R}^{(2)} = \mathcal{G}_{\mathcal{O}}^{(2)} - \frac{1}{2}\left(\mathcal{G}_{\mathcal{O}}^{(1)}\right)^2 - f^{(2)}(\epsilon)\,\mathcal{G}_{\mathcal{O}}^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

$$\mathcal{R}^{(3)} = \mathcal{G}_{\mathcal{O}}^{(3)} + \frac{1}{3}\left(\mathcal{G}_{\mathcal{O}}^{(1)}\right)^3 - \mathcal{G}_{\mathcal{O}}^{(2)}(\epsilon)\,\mathcal{G}_{\mathcal{O}}^{(1)}(\epsilon) - f^{(3)}(\epsilon)\,\mathcal{G}_{\mathcal{O}}^{(1)}(3\epsilon) - C^{(3)} + \mathcal{O}(\epsilon)$$

$$f^{(2)}(\epsilon) = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2$$

$$C^{(2)} = 4\zeta_4$$

$$f^{(3)} = 4\left(\frac{11}{2}\zeta_4 + \left(6\zeta_5 + 5\zeta_2\zeta_3\right)\epsilon + \left(\frac{1909}{48}\zeta_6 + 31\zeta_3^2\right)\epsilon^2\right)$$

$$C^{(3)} = 16\zeta_3^2 - \frac{181}{3}\zeta_6$$

BDS-like finite remainder [Dixon, McLeod, Wilhelm 2020]

$$\mathcal{G}_2 = \mathcal{G}_2^{\text{BDS-like}} \times \mathcal{E}$$

$$\mathcal{E} = \exp\left[\frac{1}{4}\Gamma_{\textbf{cusp}}\mathcal{E}^{(1)} + \mathcal{R}\right]$$

with $\Gamma_{\text{cusp}} = 4g^2 - 8\zeta_2 g^4 + 88\zeta_4 g^6 - 4\left[219\zeta_6 + 8\zeta_3^2\right] g^8 + \ldots,$

$$\mathcal{E} = \sum_{L=1}^{\infty} g^{2L}\mathcal{E}^{(L)}$$

$$\mathcal{E}^{(2)} = \mathcal{R}^{(2)} + \frac{1}{2}\left(\mathcal{E}^{(1)}\right)^2 - 2\zeta_2\mathcal{E}^{(1)}$$

$$\mathcal{E}^{(3)} = \mathcal{R}^{(3)} + \mathcal{E}^{(1)}\mathcal{E}^{(2)} - \frac{1}{3}\left(\mathcal{E}^{(1)}\right)^3 + 22\zeta_4\mathcal{E}^{(1)}$$

$$\mathcal{E}^{(1)}_{\text{tr}\phi^2} = 2\text{Li}_2\left(1 - \frac{1}{u}\right) + 2\text{Li}_2\left(1 - \frac{1}{v}\right) + 2\text{Li}_2\left(1 - \frac{1}{w}\right)$$

$$\mathcal{E}^{(1)}_{\text{tr}\phi^3} = -\frac{1}{2}\left[\log^2\left(\frac{u}{v}\right) + \log^2\left(\frac{v}{w}\right) + \log^2\left(\frac{w}{u}\right)\right] - 3\zeta_2$$

Three-loop trϕ^2, trϕ^3 form factor

- **Function space**

$$\vec{\alpha} = \{u, v, w, 1-u, 1-v, 1-w\} \quad \text{, with } u = \frac{-s}{-p_4^2}, v = \frac{-t}{-p_4^2}, w = 1-u-v.$$

Quadratic letters and square root letters are cancelled out!

- **Adjacency conditions**

$$a = \frac{u}{vw}, b = \frac{v}{wu}, c = \frac{w}{uv}, d = \frac{1-u}{u}, e = \frac{1-v}{v}, f = \frac{1-w}{w}.$$

1. ~~$\ldots d \otimes e \ldots$~~, ~~$\ldots e \otimes f \ldots$~~, ~~$\ldots f \otimes d \ldots$~~ + swap
2. ~~$\ldots a \otimes d \ldots$~~, ~~$\ldots b \otimes e \ldots$~~, ~~$\ldots c \otimes f \ldots$~~ + swap
3. ~~$\ldots a \otimes abc \otimes b \ldots$~~ + dihedral

		$\mathscr{R}$	$\mathscr{E}$
trϕ^2	2-loop	-	1,2,3
	3-loop	-	1,2,3
trϕ^3	2-loop	1,2,3	1,2,3
	3-loop	1	1,2,3

- **Check**

 - Perfect agreement with the analytic expressions obtained from bootstrap approach → trϕ^2 computed up to 8 loop [Dixon, McLeod, Wilhelm 2020]; → trϕ^3 computed up to 6 loop [Basso, Dixon, Tumanov (to appear)]

 - Perfect agreement with numerical result. [Guan, Lin, Liu, Ma, Yang 2023]

 + GPL expression !

Summary & Outlook

- We compute three-loop trϕ^2 and trϕ^3 three-point form factor analytically with first-principle method computing Feynman integrals.

- When noramlised appropriately, the trϕ^3 form factor satisfies the same adjacency relations as in the trϕ^2 case. [Dixon's talk at Simons center for geometry and physics, 2024]

- We find the perfect agreement with the analytic expressions from bootstrap and numerical result.
  [Dixon, McLeod, Wilhelm 2020]
  [Basso, Dixon, Tumanov (to appear)]
  [Guan, Lin, Liu, Ma, Yang 2023]

- Can we compute 3loop contribution of Higgs+jets amplitudes N3LO? And will they have nice properties of form factor? **Petr's talk!**

Thank You