MAX-PLANCK-INSTITUT
FÜR PHYSIK

# Towards a fully general subtraction scheme: nested soft-collinear 2.0


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*HP2 2024*

In collaboration with: Federica Devoto, Kirill Melnikov, Raoul Röntsch, Davide Maria Tagliabue

Based on: JHEP02(2024)016

# Disclaimers:

- ❖ This talk is **not** about a **new subtraction method***

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Only talks containing the word "subtraction" in the title!
(see the references therein and the rest of the timetable)

**Antenna Subtraction beyond NNLO**

Subtraction, slicing ... | Subtraction, slicing an...

Sep 11, 2024, 11:00 AM
25m
Room C1

Speaker

Matteo Marcoli (University of Durham)

**A Numerical Implementation of the LASS Subtraction Scheme**

Subtraction, slicing ... | Subtraction, slicing an...

Sep 11, 2024, 11:25 AM
25m
Room C1

Speaker

Adam Kardos (University of Debrecen)

**Antenna subtraction for processes with identified particles at hadron colliders**

Subtraction, slicing ... | Subtraction, slicing an...

Sep 12, 2024, 11:00 AM
25m
Room C1

Speaker

Leonardo Bonino (University of Zurich)

**NNLO corrections with local subtractions**

Plenary

Sep 13, 2024, 9:00 AM
30m
Room B3

Speaker

Francesco Tramontano (Universita` di Napoli e INFN)

(*) many schemes are available at NNLO, and some of them have already been applied to outstanding phenomenological studies.

Antenna *[Gehrmann-De Ridder et al. '05]*, CoLoRFul *[Del Duca et al. '16]*, STRIPPER *[Czakon '10]*, Projection to Born *[Cacciari et al. '15]*,

Local analytic sector *[Magnea, CSS et al. '18]*, …

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❖ This talk is **not** about a **new subtraction method***

❖ On the contrary, it is a **“fresh look” at a pre-existent method** (**nested soft-collinear subtraction** *[Caola, Melnikov, Röntsch ‘17]*) that we believe features many desirable properties that are non-trivial to enforce, especially at NNLO.

❖ The aim is three folds:

- Improve our understanding on the **interplay occurring among different terms** that arise in intermediate steps → leading to simpler and more **compact final results**, minimising the brute-force evaluation of the counterterms, and exposing crucial cancellations.
- Prove that the original framework is robust enough to **tackle processes with arbitrary multiplicity**.
- Provide **final formulas** that can be adapted to **any QCD process** (i.e. treating the number of partons as a free parameter, not using any process-specific simplification) and implemented straightforwardly in **any numerical framework**.

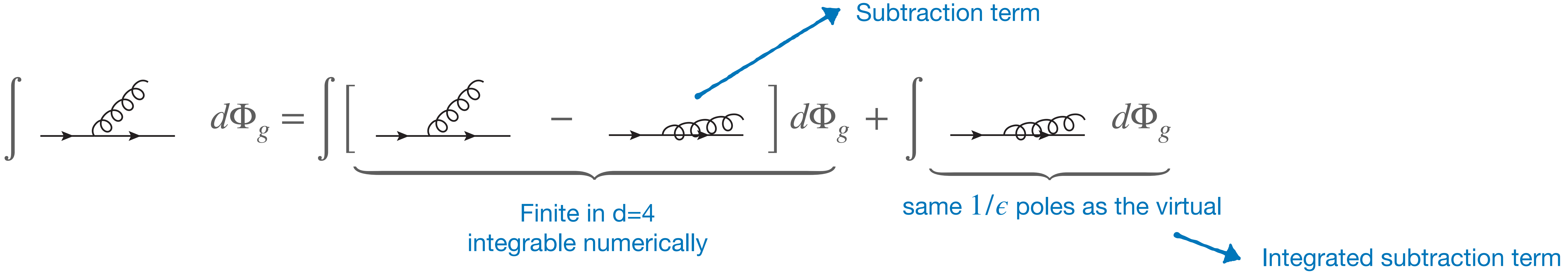
(*) many schemes are available at NNLO, and some of them have already been applied to outstanding phenomenological studies.

Antenna *[Gehrmann-De Ridder et al. ‘05]*, CoLoRFul *[Del Duca et al. ’16]*, STRIPPER *[Czakon ’10],* Projection to Born *[Cacciari et al. ’15]*,

Local analytic sector *[Magnea, CSS et al. ’18],* …

# Notation and generalities:

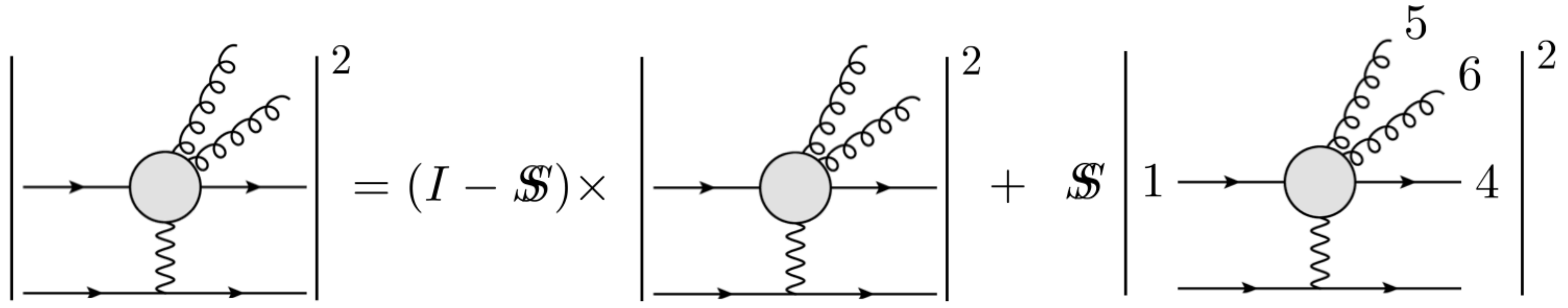
❖ **General principle** and terminology:




❖ **Skeleton** of the subtraction procedure for DIS *[Asteriadis, Calola, Melnikov Röntsch '19]*:

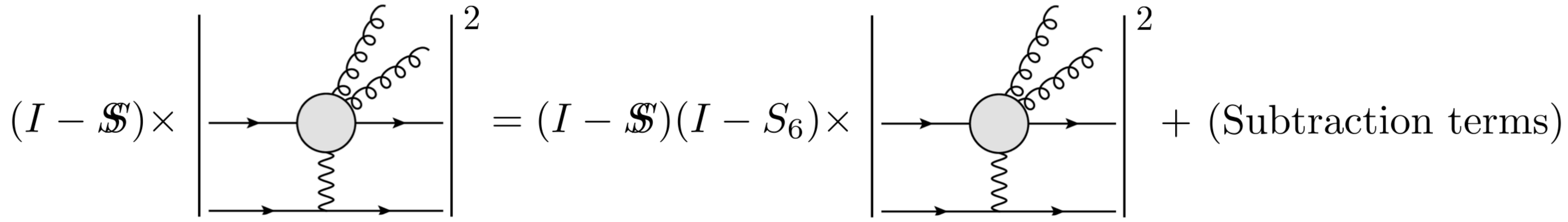
- Extract double soft singularities first (*global*) ($E_5 \sim E_6 \to 0$)

$$I = (I - \mathbb{S}) + \mathbb{S}$$

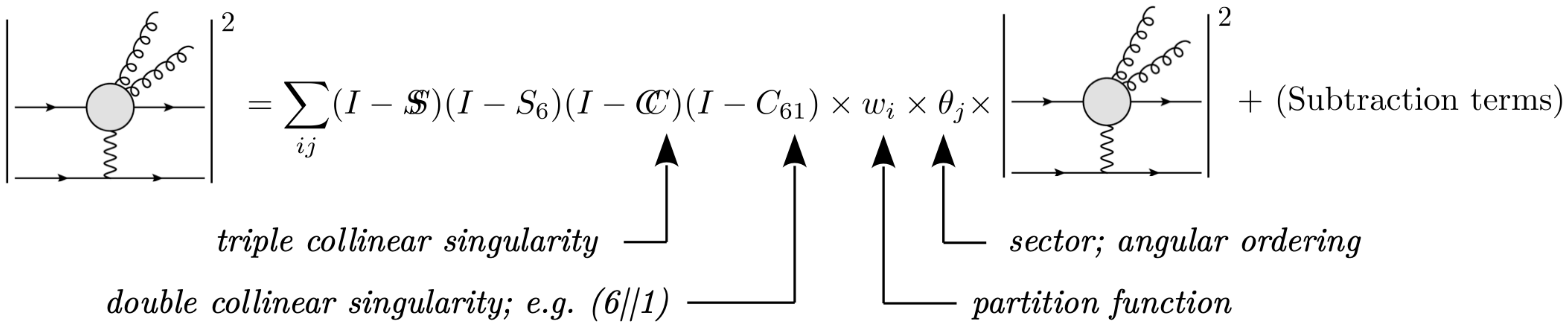



- Then single soft (energy ordering of the emissions to reduce the number of singularities)

$$I = (I - S_6) + S_6$$




- Collinear singularities (*local*): partition function + sectoring [separate overlapping singularities]



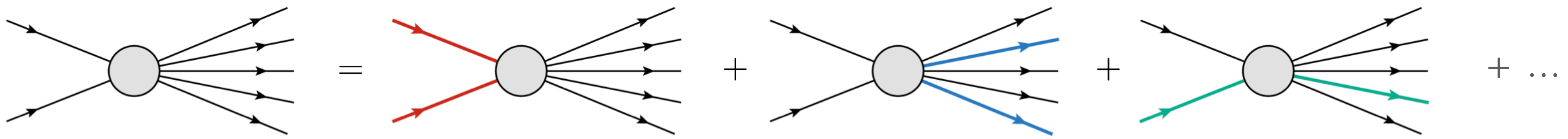

*[figures curtsy of K. Asteriadis]*

# Ingredients, tests and goals

❖ Singular **kernels** for initial- and final-state emission are **known**.

❖ **Integration** of the most complicated **double-unresolved limits** already **performed for arbitrary kinematics.**
For nested-soft collinear using *reverse unitarity** *[Anastasiou, Melnikov '02]* to map phase space integrals onto loop integrals *[Caola, Delto, Frellesvig, Melnikov '18, '19]*

❖ **Bottom-up approach**: building $pp \to N$ onto simpler processes. **Application to low-multiplicity** processes worked out straightforwardly.

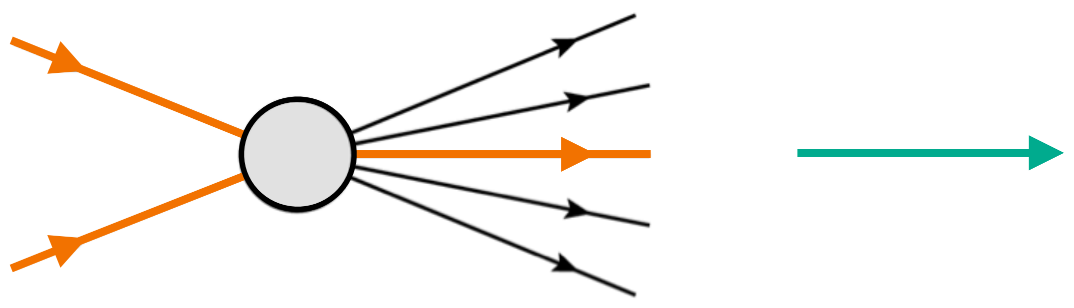


**DY** *[Caola, Melnikov, Röntsch '19]*

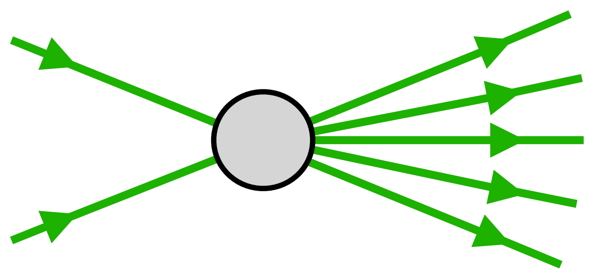
**Higgs decay** *[Caola, Melnikov, Röntsch '19]*

**DIS** *[Asteriadis, Calola, Melnikov Röntsch '19]*

❖ Natural generalisation: **V+j @ NNLO QCD**



Prototype for

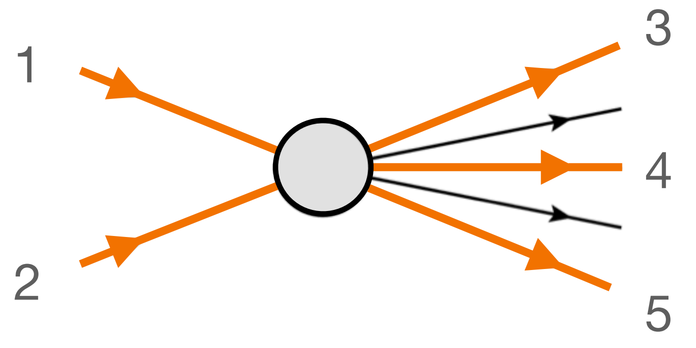


(*) different approaches also feasible with standard techniques *[Magnea, Pellicciolli, CSS, Torrielli, Uccirati '20]*

# Application to V+j

❖ In the case of gluon final state the formal expression for the **regularised double real** correction is quite simple.

❖ This can be done because we know how to deal with **multiple radiators** [**partitioning**, **energy ordering**]




$$\frac{1}{3!}\langle F_{\rm LM}(1_q,2_{\bar q};3_g,4_g,5_g)\rangle = \langle S_{45}\,\Delta^{(45)}F_{\rm LM}^{4>5}\rangle + \langle (I-S_4)S_5\,\Delta^{(45)}F_{\rm LM}^{4>5}\rangle$$
$$+\Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\Big[\Theta^{(a)}C_{45,i}(I-C_{5i})+\Theta^{(b)}C_{45,i}(I-C_{45})$$
$$+\Theta^{(c)}C_{45,i}(I-C_{4i})+\Theta^{(d)}C_{45,i}(I-C_{45})\Big]\omega_{4i5i}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$
$$-\Big\langle (I-S_{45})(I-S_5)\sum_{(ij)\in{\rm DC}}C_{4i}C_{5j}\,\omega_{4i5j}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$
$$+\Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\big[\Theta^{(a)}C_{5i}+\Theta^{(b)}C_{45}+\Theta^{(c)}C_{4i}+\Theta^{(d)}C_{45}\big]\,\omega_{4i5i}$$
$$+\sum_{(ij)\in{\rm DC}}\big[C_{4i}+C_{5j}\big]\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$

Subtraction terms

Fully regulated term

$$+\Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\Big[\Theta^{(a)}(I-C_{45,i})(I-C_{5i})+\Theta^{(b)}(I-C_{45,i})(I-C_{45})$$
$$+\Theta^{(c)}(I-C_{45,i})(I-C_{4i})+\Theta^{(d)}(I-C_{45,i})(I-C_{45})\Big]\omega_{4i5i}$$
$$+\sum_{(ij)\in{\rm DC}}(I-C_{4i})(I-C_{5j})\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$

$$(ij)\in{\rm DC}\longrightarrow (ij)\in\{(12),(13),(21),(23),(31),(32)\}$$
$$i\in{\rm TC}\longrightarrow i\in\{1,2,3\}\,.$$

# Application to V+j

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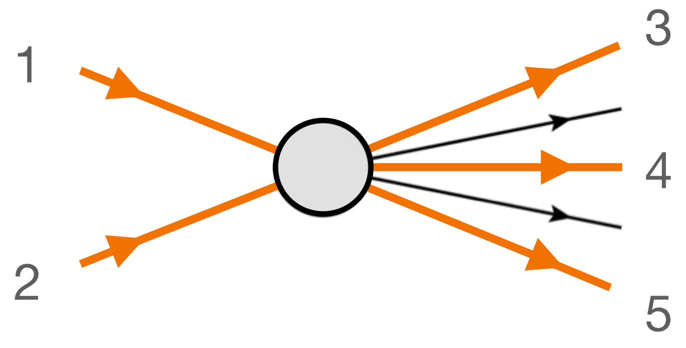


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$$+\sum_{(ij)\in{\rm DC}}\big[C_{4i}+C_{5j}\big]\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$
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$$+\sum_{(ij)\in{\rm DC}}(I-C_{4i})(I-C_{5j})\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$

$$(ij)\in{\rm DC}\longrightarrow(ij)\in\{(12),(13),(21),(23),(31),(32)\}$$
$$i\in{\rm TC}\longrightarrow i\in\{1,2,3\}\,.$$

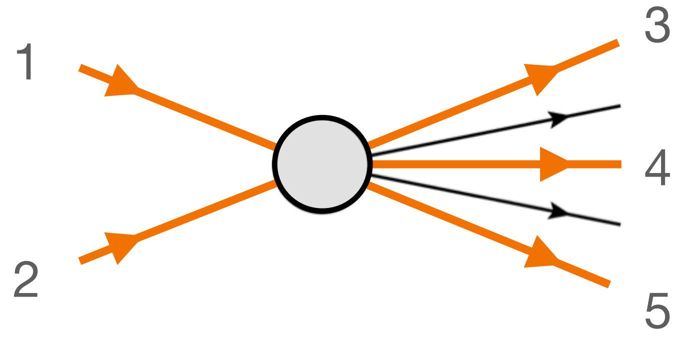
- Such expression can easily accommodates for higher multiplicity

HOWEVER

- In the first implementation of the scheme all **subtraction terms were calculated separately**, and pole cancellation verified after putting everything together.
- This approach becomes immediately **cumbersome as the number of final state partons increases** → large number of subtraction terms.

# Application to V+j: the lesson

❖ The expression of **double-real** poles and finite remainder reflects such growth and results in **large, non-transparent bunch of terms** (what if we now want to change the flavour of final-state partons? Do we have to start from the beginning?)

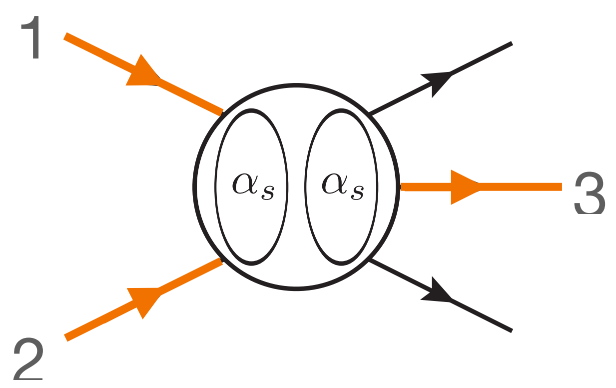



```
Series[RRpoles, {ϵ, 0, -1}]

Out[•]=
1/(4 ϵ^4) (3 asontwopi² CA² FLM[p1_q, p2_q, p3_g] + 10 asontwopi² CA CF FLM[p1_q, p2_q, p3_g] + 2 asontwopi² CF² FLM[p1_q, p2_q, p3_g] +
   5 asontwopi² CF² delta[1 - z] × FLM[p1_q, z p2_q, p3_g, z] + 5 asontwopi² CF² delta[1 - z] × FLM[z p1_q, p2_q, p3_g, z] -
   4 asontwopi² CF² delta[1 - z1] × delta[1 - z2] × FLM[z1 p1_q, z2 p2_q, p3_g, z1, z2]) +
(77 asontwopi² CA² FLM[p1_q, p2_q, p3_g] + 110 asontwopi² CA CF FLM[p1_q, p2_q, p3_g] + 72 asontwopi² CA² L3 FLM[p1_q, p2_q, p3_g] + 36 asontwopi² CA CF FLM[p1_q, z p2_q, p3_g, z] + ⋯100⋯) / (24 ϵ^3) +
(1381/72 asontwopi² CA² FLM[p1_q, p2_q, p3_g] + 67/4 asontwopi² CA CF FLM[p1_q, p2_q, p3_g] + ⋯509⋯ + 1/2 asontwopi² CA CF delta[1 - z] × FLM[z p1_q, p2_q, p3_g, z] PolyLog[2, 1 - eta[2, 3]]) / ϵ^2 +
(3151/27 asontwopi² CA² FLM[p1_q, p2_q, p3_g] + 2813/36 asontwopi² CA CF FLM[p1_q, p2_q, p3_g] + 265/18 asontwopi² CA² L3 FLM[p1_q, p2_q, p3_g] - 11/2 ⋯3⋯ FLM[p1_q, p2_q, p3_g] + ⋯2016⋯) / ϵ + O[ϵ]^0

large output   show less   show more   show all   set size limit...
```

❖ Interestingly, the **IR structure of the double-virtual** corrections does not reflect the same complexity *[Catani '98]*




$$
\begin{aligned}
\langle \mathcal{M}|\mathcal{M}\rangle_{\alpha_s^2} = & \left\langle \mathcal{M}_0 \left| \frac{1}{2} I_1^2(\epsilon) + \frac{1}{2}\left(I_1^\dagger(\epsilon)\right)^2 + I_1^\dagger(\epsilon) I_1(\epsilon) + \left(\mathcal{H}_2 + \mathcal{H}_2^\dagger\right) \right| \mathcal{M}_0 \right\rangle \\
& + \left\langle \mathcal{M}_0 \left| -\frac{\beta_0}{\epsilon}\left(I_1(\epsilon) + I_1^\dagger(\epsilon)\right) + c_\epsilon \left(\frac{\beta_0}{\epsilon} + K\right)\left(I_1(2\epsilon) + I_1^\dagger(2\epsilon)\right) \right| \mathcal{M}_0 \right\rangle \\
& + 2\mathrm{Re}\left[\langle \mathcal{M}_0 \,|\, I_1(\epsilon) + I_1^\dagger(\epsilon) \,|\, \mathcal{M}_1^{\mathrm{fin}} \rangle\right] + 2\mathrm{Re}\left[\langle \mathcal{M}_0 | \mathcal{M}_2^{\mathrm{fin}} \rangle\right] + \langle \mathcal{M}_1^{\mathrm{fin}} | \mathcal{M}_1^{\mathrm{fin}} \rangle \,.
\end{aligned}
$$

```
Series[VVpoles, {ϵ, 0, -1}]
asontwopi² (CA + 2 CF)² FLM[p1_q, p2_q, p3_g] / (2 ϵ^4) +
asontwopi² (CA + 2 CF) FLM[p1_q, p2_q, p3_g] (12 CF + 7 β0 + 4 (CA - 2 CF) Log[s12/mu2] - 4 CA (Log[s13/mu2] + Log[s23/mu2])) / (4 ϵ^3) + 1/ϵ^2 asontwopi²
 ((-CA - 2 CF) FLVfin[p1_q, p2_q, p3_g] + FLM[p1_q, p2_q, p3_g] (1/72 (-2 CA CF (67 + 9 π²) + CA² (-67 + 33 π²) + 20 CA Nf TR + 4 CF (CF (81 - 42 π²) + 10 Nf TR)) +
   9 CF β0/2 + β0² + 1/4 (8 CF (-CA + 2 CF) Log[s12/mu2]² + 4 CA (CA + CF) Log[s13/mu2]² + Log[s13/mu2] (-3 CA (CA + 6 CF) - 4 (2 CA + CF) β0 + 4 CA² Log[s23/mu2]) + Log[s23/mu2]
     (-3 CA (CA + 6 CF) - 4 (2 CA + CF) β0 + 4 CA (CA + CF) Log[s23/mu2]) - 2 (CA - 2 CF) Log[s12/mu2] (-3 (CA + 4 CF + β0) + 2 CA (Log[s13/mu2] + Log[s23/mu2]))))) +
1/ϵ asontwopi² (FLVfin[p1_q, p2_q, p3_g] (-3 CF - β0 - (CA - 2 CF) Log[s12/mu2] + CA (Log[s13/mu2] + Log[s23/mu2])) +
  1/864 FLM[p1_q, p2_q, p3_g] (3 CA² (60 + 227 π²) + 2 CA (CF (-1922 + 333 π²) - 2 Nf (232 + 3 π²) TR + 3 (-268 + 51 π²) β0) +
   4 (-27 CF² (3 + 52 π²) + 2 CF Nf (184 + 9 π²) TR - 315 CF π² β0 + 40 Nf TR (2 Nf TR + 3 β0)) -
   12 (24 (CA - 4 CF) (CA - 2 CF) Log[s12/mu2]³ + 24 CA (2 CA + CF) Log[s13/mu2]³ + 9 Log[s13/mu2]² (-9 CA (CA + 2 CF) - 2 (5 CA + 2 CF) β0 + 4 CA² Log[s23/mu2]) -
     18 (CA - 2 CF) Log[s12/mu2]² (3 CA - 2 (12 CF + β0) + 2 CA (Log[s13/mu2] + Log[s23/mu2])) + 2 (CA - 2 CF) Log[s12/mu2]
      (67 CA + 3 (CA + 28 CF) π² - 2 (81 CF + 10 Nf TR + 27 β0) + 9 (9 CA + 2 β0) Log[s13/mu2] - 18 CA Log[s13/mu2]² + 9 Log[s23/mu2] (9 CA + 2 β0 - 2 CA Log[s23/mu2])) +
     2 Log[s13/mu2] (CA (-67 CA + 81 CF + 3 (5 CA - 16 CF) π² + 20 Nf TR) + 27 (CA + 2 CF) β0 + 18 β0² + 18 CA Log[s23/mu2] (-3 CA - 2 β0 + CA Log[s23/mu2])) +
     Log[s23/mu2] (2 CA (-67 CA + 81 CF + 3 (5 CA - 16 CF) π² + 20 Nf TR) + 54 (CA + 2 CF) β0 + 36 β0² +
      3 Log[s23/mu2] (-27 CA (CA + 2 CF) - 6 (5 CA + 2 CF) β0 + 8 CA (2 CA + CF) Log[s23/mu2]))) + 6 (CA² - 62 CA CF + 88 CF²) Zeta[3]))) + O[ϵ]^0
```

# Application to V+j: the lesson

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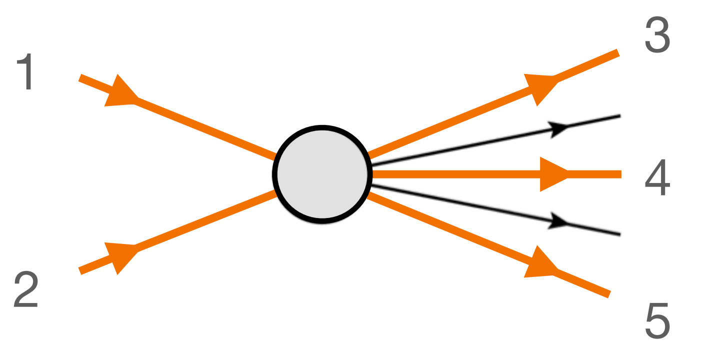
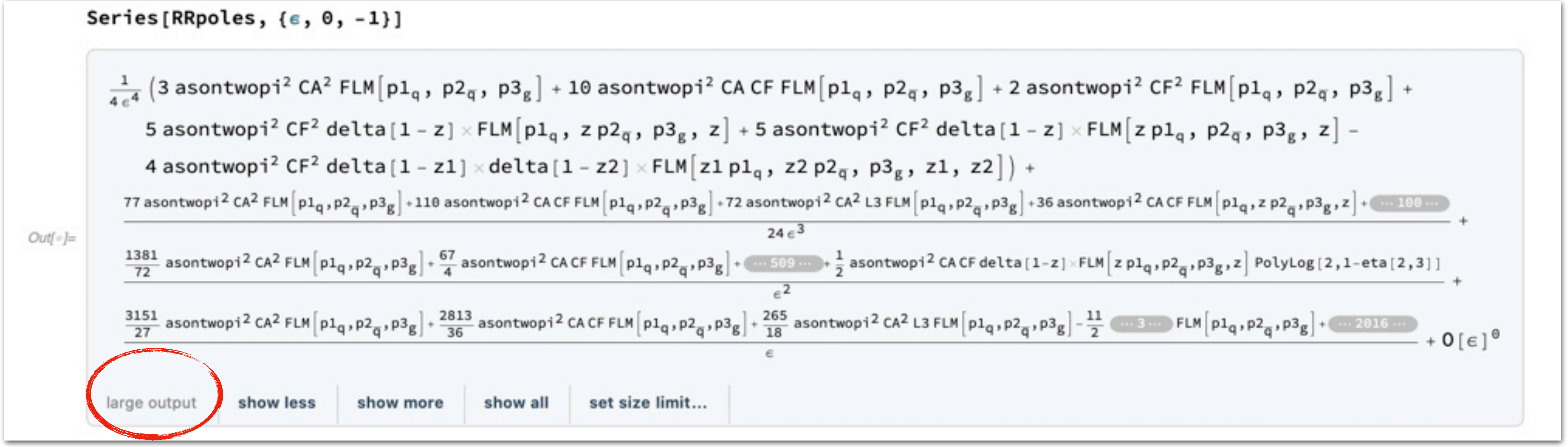
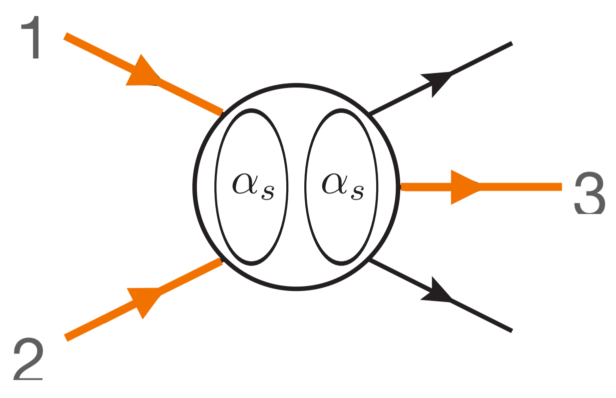
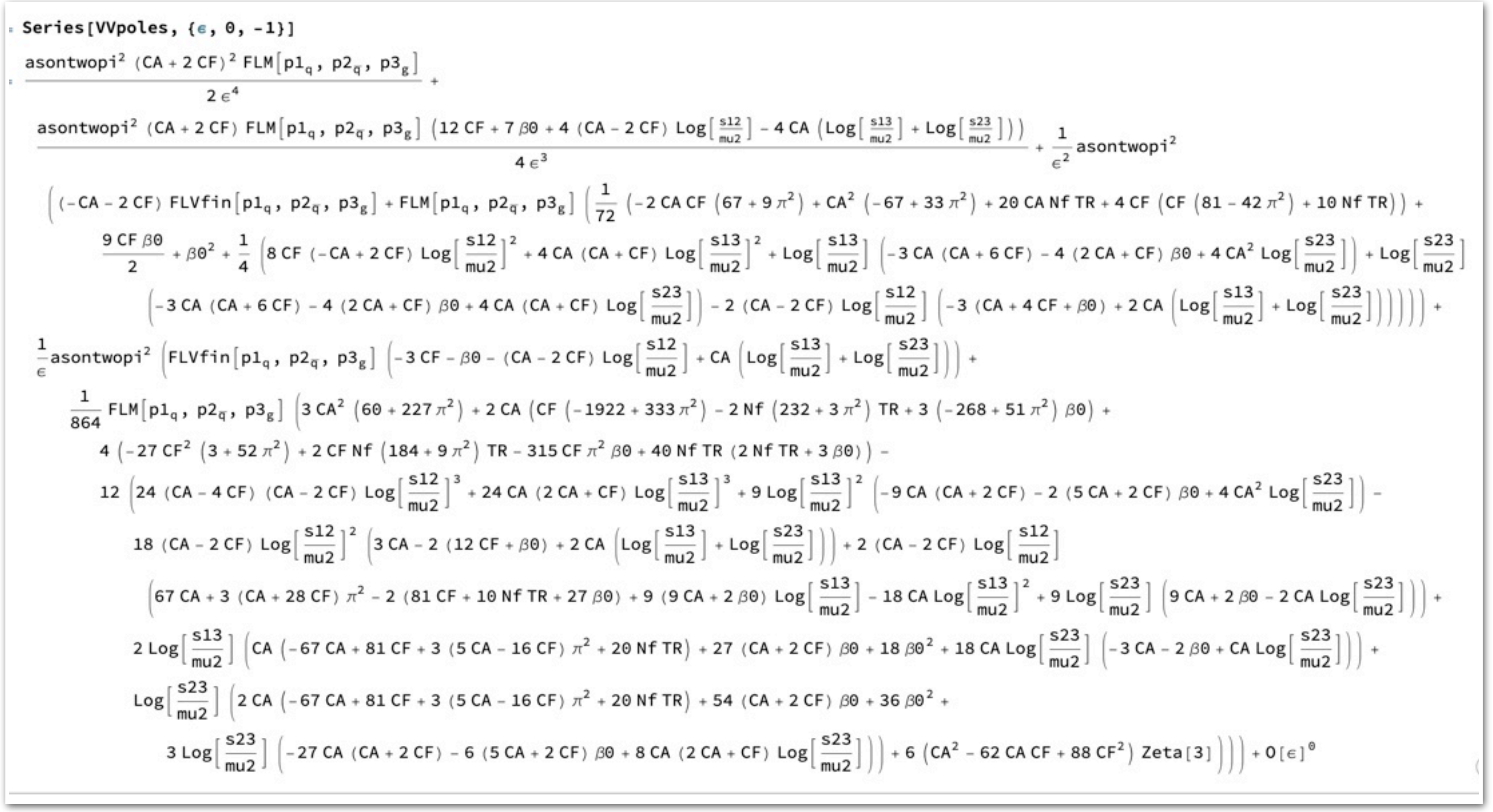



❖ Interestingly, the **IR structure of the double-virtual** corrections does not reflect the same complexity *[Catani '98]*




$$\langle\mathcal{M}|\mathcal{M}\rangle_{\alpha_s^2} = \left\langle \mathcal{M}_0 \left| \frac{1}{2} I_1^2(\epsilon) + \frac{1}{2}\left(I_1^\dagger(\epsilon)\right)^2 + I_1^\dagger(\epsilon) I_1(\epsilon) + \left(\mathcal{H}_2 + \mathcal{H}_2^\dagger\right) \right| \mathcal{M}_0 \right\rangle$$
$$+ \left\langle \mathcal{M}_0 \left| -\frac{\beta_0}{\epsilon}\left(I_1(\epsilon) + I_1^\dagger(\epsilon)\right) + c_\epsilon \left(\frac{\beta_0}{\epsilon} + K\right)\left(I_1(2\epsilon) + I_1^\dagger(2\epsilon)\right) \right| \mathcal{M}_0 \right\rangle$$
$$+ 2\mathrm{Re}\left[\langle \mathcal{M}_0 \,|\, I_1(\epsilon) + I_1^\dagger(\epsilon) \,|\, \mathcal{M}_1^{\mathrm{fin}}\rangle\right] + 2\mathrm{Re}\left[\langle \mathcal{M}_0 | \mathcal{M}_2^{\mathrm{fin}}\rangle\right] + \langle \mathcal{M}_1^{\mathrm{fin}} | \mathcal{M}_1^{\mathrm{fin}}\rangle \, .$$




**"Asymmetry": VV very simple pole structure, RR structure obscured by energy ordering, partitioning…**

# Outlook of the talk:

Can we identify structures **early on** in the calculations so that cancellation of divergences can be seen "by eye", even for a **generic process**?

**Main idea: look at the pole structure of the virtual corrections to infer similar structures for the subtraction terms**

→ reverses the standard logic guiding the construction of most established infrared subtraction schemes*

→ by product: **get rid of color correlations and reduce the rest to a sum over external-leg contributions.**

Case of study: $q\bar{q} \to X + Ng$

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**A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to $N$-gluon final states in $q\bar{q}$ annihilation**

Federica Devoto,[a] Kirill Melnikov,[b] Raoul Röntsch,[c] Chiara Signorile-Signorile[b,d,e] and Davide Maria Tagliabue[c]

Work in progress: $gq \to X + (N-1)g + q$

**NLO and NNLO QCD contributions to the channel $gq \to X + (N-1)g + q$**

Federica Devoto,[a] Kirill Melnikov,[b] Raoul Röntsch,[c] Chiara Signorile-Signorile,[d] Davide Maria Tagliabue[c]

(*) similar idea also explored in the context of *local analytic sector subtraction* [Magnea, Milloy, CSS, Torrielli '24] and *antenna subtraction* [Gehrmann, Glover, Marcoli '23]

# Warm up @NLO: $q\bar{q} \to X + Ng$

$$2s\,\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}}$$

1) Virtual corrections:
   color-correlations, elastic terms

$$I_{\mathrm{V}}(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^{\dagger}(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2}\sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\mathrm{sing}}(\epsilon)}{\boldsymbol{T}_i^2}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\mathrm{sing}}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}.$$

2) Real corrections: soft: color-correlations, elastic terms

$$I_{\mathrm{S}}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{(ij)}^{N_p}\eta_{ij}^{-\epsilon}K_{ij}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)$$

$$I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon)$$

- Highest pole trivially cancels
- Color correlations cancel
- Remnant elastic single pole

# Warm up @NLO: $q\bar{q} \to X + Ng$

$$2s\, \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}}$$

1) Virtual corrections:
   color-correlations, elastic terms

$$I_{\mathrm{V}}(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^{\dagger}(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2}\sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\mathrm{sing}}(\epsilon)}{\boldsymbol{T}_i^2}(\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\left(\frac{\mu^2}{2p_i \cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\mathrm{sing}}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$$

2) Real corrections:

soft: color-correlations, elastic terms

$$I_{\mathrm{S}}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)$$

$$I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon)$$

- Highest pole trivially cancels
- Color correlations cancel
- Remnant elastic single pole

hard-collinear: no color-correlations, boosts + elastic terms

$$\mathcal{P}_{aa}^{\mathrm{gen}} \otimes F_{\mathrm{LM}}$$

$$I_{\mathrm{C}}(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}$$

"generalised anomalous dimensions"

$$\Gamma_{i,f_i} = \gamma_i + 2\boldsymbol{T}_i^2 L_i + \mathcal{O}(\epsilon)$$

$$\Longrightarrow \quad I_{\mathrm{T}}(\epsilon) = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon) \quad \text{FINITE}$$

# Warm up @NLO: $q\bar{q} \to X + Ng$

$$2s\, \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}}$$

1) Virtual corrections: color-correlations, elastic terms

$$I_{\mathrm{V}}(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^{\dagger}(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2} \sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\mathrm{sing}}(\epsilon)}{\boldsymbol{T}_i^2} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \left(\frac{\mu^2}{2p_i \cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\mathrm{sing}}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$$

2) Real corrections: soft: color-correlations, elastic terms

$$I_{\mathrm{S}}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)$$

$$I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon)$$

- Highest pole trivially cancels
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hard-collinear: no color-correlations, boosts + elastic terms

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$$\Longrightarrow \quad I_{\mathrm{T}}(\epsilon) = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon) \quad \text{FINITE}$$

3) PDFs renormalisation: no color-correlations, boosts $\hat{P}_{aa}^{(0)} \otimes F_{\mathrm{LM}}$

4) Sum:

$$2s\, \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \frac{\alpha_s(\mu)}{2\pi} \langle I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LM}} \rangle + \frac{\alpha_s(\mu)}{2\pi} \left[ \langle \mathcal{P}_{aa}^{\mathrm{NLO}} \otimes F_{\mathrm{LM}} \rangle + \langle F_{\mathrm{LM}} \otimes \mathcal{P}_{bb}^{\mathrm{NLO}} \rangle \right] + \langle F_{\mathrm{LV}}^{\mathrm{fin}} \rangle + \langle \mathcal{O}_{\mathrm{NLO}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m}) \rangle$$

# Lesson from NLO

**Simple interplay between** $\underbrace{\left[V + S_i R + (I - S_i) C_{ij} R\right]_{\text{elastic}}}_{I_{\mathrm{T}}(\epsilon) = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon)}$ **and** $\underbrace{\left[(1 - S_i) C_{ij} R\right]_{\text{boost}} + \text{PDFs}}_{\langle \mathcal{P}_{aa}^{\mathrm{NLO}} \otimes F_{\mathrm{LM}} \rangle + \langle F_{\mathrm{LM}} \otimes \mathcal{P}_{bb}^{\mathrm{NLO}} \rangle}$

# Moving forward to NNLO

Starting from **IR poles of double-virtual** *[Catani '98]* we want to find **subtraction terms** that can "**complete**" it: identify structures similar to those encountered at NLO → we want to push the idea of writing NNLO $\sim$ NLO$^2$ as much as possible

❖ Rewrite the VV in term of NLO operators:

$$\begin{aligned}\langle F_{\mathrm{VV}} \rangle = [\alpha_s]^2 \Big\langle \Big[ \frac{1}{2} I_{\mathrm{V}}^2(\epsilon) - \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_{\mathrm{E}}}} \Big( \frac{\beta_0}{\epsilon} I_{\mathrm{V}}(\epsilon) - \Big( \frac{\beta_0}{\epsilon} + K \Big) I_{\mathrm{V}}(2\epsilon) \Big) \Big] \cdot F_{\mathrm{LM}} \Big\rangle \\ + [\alpha_s]^2 \Big\langle \Big[ -\frac{1}{2} \big[ \bar{I}_1(\epsilon), \bar{I}_1^{\dagger}(\epsilon) \big] + \mathcal{H}_{2,\mathrm{tc}} + \mathcal{H}_{2,\mathrm{tc}}^{\dagger} + \mathcal{H}_{2,\mathrm{cd}} + \mathcal{H}_{2,\mathrm{cd}}^{\dagger} \Big] \cdot F_{\mathrm{LM}} \Big\rangle \\ + [\alpha_s] \langle I_{\mathrm{V}}(\epsilon) \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \rangle + \langle F_{\mathrm{LV}^2}^{\mathrm{fin}} \rangle + \langle F_{\mathrm{VV}}^{\mathrm{fin}} \rangle \,.\end{aligned}$$

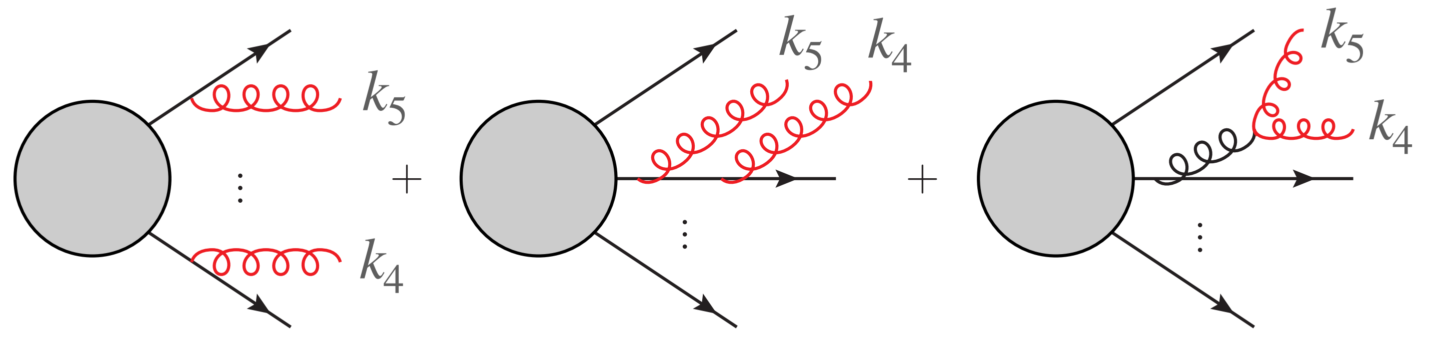
❖ Identify features that can guide you

- different **powers/arguments/prefactors**
- different type of **color-correlations**

$$\left\{ \begin{array}{l} T_i \cdot T_j \\ T_i \cdot T_j \cdot T_k \\ (T_i \cdot T_j) \cdot (T_k \cdot T_l) \end{array} \right.$$

→ **specific pattern of cancellation.**

# Follow the (colored) crumbs




- ❖ Search for similar features among the various subtraction terms.
  For instance double and quartic color correlations:

**Double soft**

*[Catani, Grazzini '99]*

Factorised term $\quad (T_i \cdot T_j)\cdot(T_k \cdot T_l) \quad \Longrightarrow \quad \langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^4} = [\alpha_s]^2 \frac{1}{2}\langle I_{\rm S}^2(\epsilon)\cdot F_{\rm LM}\rangle$

- ❖ Combine terms such that manifest cancellations occur without any process-aware manipulation

$I_{\rm S}^2(\epsilon) + I_{\rm V}^2(\epsilon)$ free of quartic color-correlated poles

Iterations of NLO*

- ❖ Some bits require some further massage

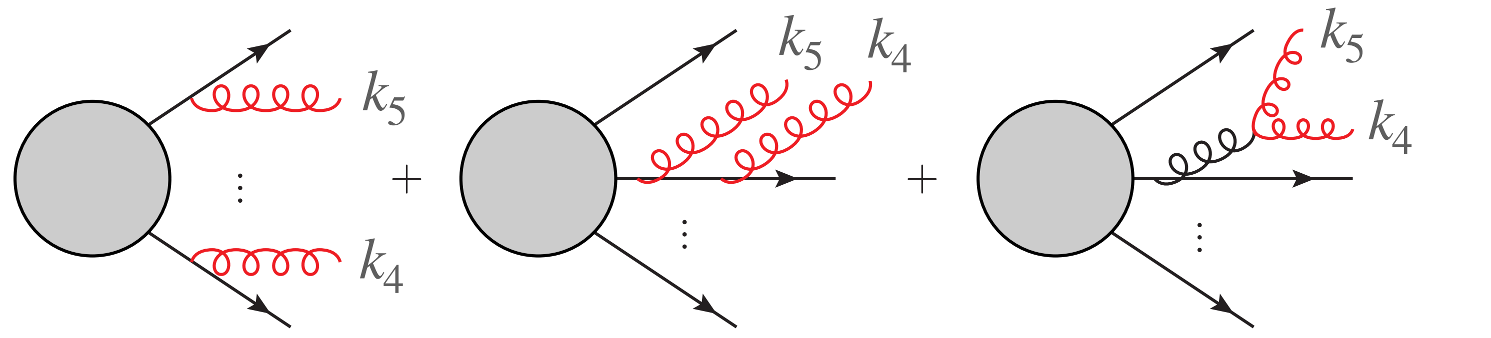
Non-factorised term $\quad T_i \cdot T_j \quad \Longrightarrow \quad \langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}$

*[Caola, Delto, Frellesvig, Melnikov '18]*

$$
\begin{aligned}
\mathbb{S}_{ij}^{(gg)} = (2E_{\max})^{-4\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2 \Bigg\{ & \frac{1}{2\epsilon^4} + \frac{1}{\epsilon^3}\left[\frac{11}{12} - \ln(s^2)\right] \\
& + \frac{1}{\epsilon^2}\left[2\mathrm{Li}_2(c^2) + \ln^2(s^2) - \frac{11}{6}\ln(s^2) + \frac{11}{3}\ln 2 - \frac{\pi^2}{4} - \frac{16}{9}\right] \\
& + \frac{1}{\epsilon}\Bigg[6\mathrm{Li}_3(s^2) + 2\mathrm{Li}_3(c^2) + \left(2\ln(s^2) + \frac{11}{3}\right)\mathrm{Li}_2(c^2) - \frac{2}{3}\ln^3(s^2) \\
& \quad + \left(3\ln(c^2) + \frac{11}{6}\right)\ln^2(s^2) - \left(\frac{22}{3}\ln 2 + \frac{\pi^2}{2} - \frac{32}{9}\right)\ln(s^2) \\
& \quad - \frac{45}{4}\zeta_3 - \frac{11}{3}\ln^2 2 - \frac{11}{36}\pi^2 - \frac{137}{18}\ln 2 + \frac{217}{54}\Bigg] \\
& + 4\mathrm{G}_{-1,0,0,1}(s^2) - 7\mathrm{G}_{0,1,0,1}(s^2) + \frac{22}{3}\mathrm{Ci}_3(2\delta) + \frac{1}{3\tan(\delta)}\mathrm{Si}_2(2\delta) \\
& + 2\mathrm{Li}_4(c^2) - 14\mathrm{Li}_4(s^2) + 4\mathrm{Li}_4\left(\frac{1}{1+s^2}\right) - 2\mathrm{Li}_4\left(\frac{1-s^2}{1+s^2}\right) \\
& + 2\mathrm{Li}_4\left(\frac{s^2-1}{1+s^2}\right) + \mathrm{Li}_4(1-s^4) + \Bigg[10\ln(s^2) - 4\ln\left(1+s^2\right) \\
& + \frac{11}{3}\Bigg]\mathrm{Li}_3(c^2) + \left[14\ln(c^2) + 2\ln(s^2) + 4\ln\left(1+s^2\right) + \frac{22}{3}\right]\mathrm{Li}_3(s^2) \\
& + 4\ln(c^2)\mathrm{Li}_3(-s^2) + \frac{9}{2}\mathrm{Li}_2^2(c^2) - 4\mathrm{Li}_2(c^2)\mathrm{Li}_2(-s^2) + \Bigg[7\ln(c^2)\ln(s^2) \\
& - \ln^2(s^2) - \frac{5}{2}\pi^2 + \frac{22}{3}\ln 2 - \frac{131}{18}\Bigg]\mathrm{Li}_2(c^2) + \left[\frac{2}{3}\pi^2 - 4\ln(c^2)\ln(s^2)\right]\times \\
& \mathrm{Li}_2(-s^2) + \frac{\ln^4(s^2)}{3} + \frac{\ln^4\left(1+s^2\right)}{6} - \ln^3(s^2)\left[\frac{4}{3}\ln(c^2) + \frac{11}{9}\right] \\
& + \ln^2(s^2)\left[7\ln^2(c^2) + \frac{11}{3}\ln(c^2) + \frac{\pi^2}{3} + \frac{22}{3}\ln 2 - \frac{32}{9}\right] - \frac{\pi^2}{6}\ln^2\left(1+s^2\right) \\
& + \zeta_3\left[\frac{17}{2}\ln(s^2) - 11\ln(c^2) + \frac{7}{2}\ln\left(1+s^2\right) - \frac{21}{2}\ln 2 - \frac{99}{4}\right] + \ln(s^2)\times \\
& \left[-\frac{7\pi^2}{2}\ln(c^2) + \frac{22}{3}\ln^2 2 - \frac{11}{18}\pi^2 + \frac{137}{9}\ln 2 - \frac{208}{27}\right] - 12\mathrm{Li}_4\left(\frac{1}{2}\right) \\
& + \frac{143}{720}\pi^4 - \frac{\ln^4 2}{2} + \frac{\pi^2}{2}\ln^2 2 - \frac{11}{6}\pi^2\ln 2 + \frac{125}{216}\pi^2 + \frac{22}{9}\ln^3 2 \\
& + \frac{137}{18}\ln^2 2 + \frac{434}{27}\ln 2 - \frac{649}{81} + \mathcal{O}(\epsilon)\Bigg\},
\end{aligned}
$$

$\delta = \frac{\delta_{12}}{2}$, $s = \sin\frac{\delta_{12}}{2}$, $c = \cos\frac{\delta_{12}}{2}$ $\quad \mathrm{Ci}_n(z) = \frac{\mathrm{Li}_n(e^{iz}) + \mathrm{Li}_n(e^{-iz})}{2}$, $\mathrm{Si}_n(z) = \frac{\mathrm{Li}_n(e^{iz}) - \mathrm{Li}_n(e^{-iz})}{2i}$

# Follow the (colored) crumbs

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For instance double and quartic color correlations:

**Double soft**

[Catani, Grazzini '99]

Factorised term $(T_i \cdot T_j) \cdot (T_k \cdot T_l) \implies \langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m}, \mathfrak{n}) \rangle_{T^4} = [\alpha_s]^2 \frac{1}{2} \langle I_{\mathrm{S}}^2(\epsilon) \cdot F_{\mathrm{LM}} \rangle$

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Iterations of NLO*

❖ Some bits require some further massage

Non-factorised term $T_i \cdot T_j \implies \langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m}, \mathfrak{n}) \rangle_{T^2}$

$$= [\alpha_s]^2 \left[ \frac{C_A}{\epsilon^2} c_1(\epsilon) + \frac{\beta_0}{\epsilon} c_2(\epsilon) + \beta_0 c_3(\epsilon) \right] \langle \widetilde{I}_{\mathrm{S}}(2\epsilon) \cdot F_{\mathrm{LM}} \rangle + \langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m}, \mathfrak{n}) \rangle_{T^2}^{\mathrm{fin}}$$

New structure, but pole content reducible to "variants" of NLO

$$\widetilde{I}_{\mathrm{S}}(2\epsilon) = I_{\mathrm{S}}(2\epsilon) + \mathcal{O}(\epsilon)$$

$$c_1(\epsilon) = 1 + \left(\frac{\pi^2}{6} - \frac{32}{9}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

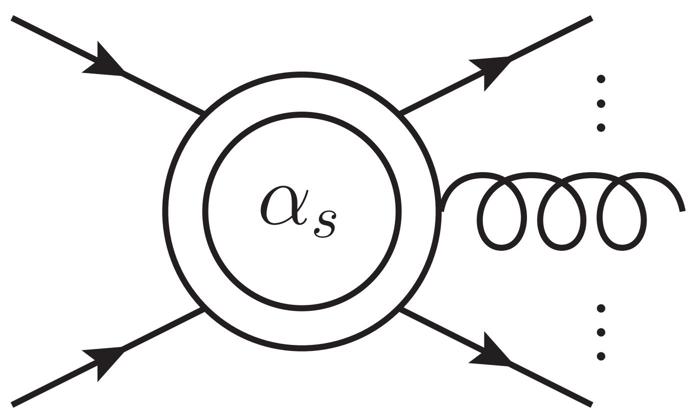
$$c_2(\epsilon) = 1 + \frac{\pi^2}{3}\epsilon^2$$

$$c_3(\epsilon) = 4 \log 2 + 8\epsilon \log^2 2$$

(*) almost $(I_{\mathrm{S}} + I_{\mathrm{V}})^2 \to$ to obtain a NLO$^2$ object we need the product

# Follow the (colored) crumbs

❖ The "missing" products we look at the limits of the real-virtual contribution. For instance




**Soft real-virtual**

*[Catani, Grazzini '00]*

$$
\begin{aligned}
& S_{\mathfrak{m}}\, F_{\mathrm{RV}}(\mathfrak{m}) \\
& = -\, g_{s,b}^2 \sum_{(ij)}^{N_p} \Big\{ 2\, S_{ij}(p_{\mathfrak{m}})\ (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LV}} - \frac{\alpha_s(\mu)}{2\pi} \frac{\beta_0}{\epsilon}\, 2\, S_{ij}(p_{\mathfrak{m}})\ (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LM}} \\
& \quad -\, 2\, \frac{[\alpha_s]}{\epsilon^2}\, C_A\, A_K(\epsilon) \Big( S_{ij}(p_{\mathfrak{m}}) \Big)^{1+\epsilon} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LM}} \\
& \quad -\, [\alpha_s]\, \frac{4\pi\, \Gamma(1+\epsilon) \Gamma^3(1-\epsilon)}{\epsilon\, \Gamma(1-2\epsilon)} \sum_{\substack{k=1 \\ k \neq i,j}}^{N_p} \kappa_{ij}\, S_{ki}(p_{\mathfrak{m}}) \Big( S_{ij}(p_{\mathfrak{m}}) \Big)^{\epsilon} f_{abc}\, T_k^a\, T_i^b\, T_j^c\, F_{\mathrm{LM}} \Big\}
\end{aligned}
$$

$$
S_{ij}(p_{\mathfrak{m}}) = \frac{p_i \cdot p_j}{2 (p_i \cdot p_{\mathfrak{m}}) (p_j \cdot p_{\mathfrak{m}})}
$$

$$
A_K = \frac{\Gamma^3(1+\epsilon)\, \Gamma^5(1-\epsilon)}{\epsilon^2\, \Gamma(1+2\epsilon)\, \Gamma^2(1-2\epsilon)}
$$

Triple-color correlations:

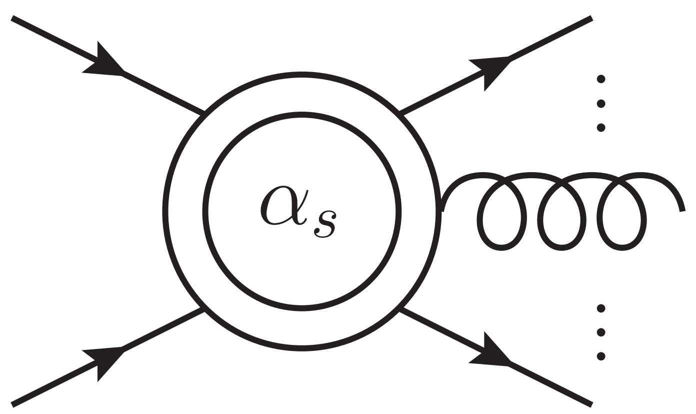
- Vanish for $N_p \geq 4$
- Non-trivial phase space integral
- Finite after integration for FSR

❖ The integrated subtraction term can be almost fully written in terms of NLO-like operators

$$
\begin{aligned}
\big\langle S_{\mathfrak{m}}\, F_{\mathrm{RV}}(\mathfrak{m}) \big\rangle = {} & [\alpha_s]^2 \Big\langle \frac{1}{2} \Big[ I_{\mathrm{S}}(\epsilon) \cdot I_{\mathrm{V}}(\epsilon) + I_{\mathrm{V}}(\epsilon) \cdot I_{\mathrm{S}}(\epsilon) \Big] \cdot F_{\mathrm{LM}} \Big\rangle \\
& + [\alpha_s] \Big\langle I_{\mathrm{S}}(\epsilon) \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \Big\rangle - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_E}} \frac{\beta_0}{\epsilon} \Big\langle I_{\mathrm{S}}(\epsilon) F_{\mathrm{LM}} \Big\rangle \\
& - \frac{[\alpha_s]^2}{\epsilon^2}\, C_A\, A_K(\epsilon) \Big\langle \widetilde{I}_{\mathrm{S}}(2\epsilon) \cdot F_{\mathrm{LM}} \Big\rangle \\
& + [\alpha_s]^2 \Big\langle \Big( \frac{1}{2} \Big[ I_{\mathrm{S}}(\epsilon)\, ,\, \overline{I}_1(\epsilon) - \overline{I}_1^{\dagger}(\epsilon) \Big] + I_{\mathrm{tri}}^{\mathrm{RV}}(\epsilon) \Big) \cdot F_{\mathrm{LM}} \Big\rangle
\end{aligned}
$$

# Follow the (colored) crumbs

❖ The "missing" products we look at the limits of the real-virtual contribution. For instance




**Soft real-virtual**

[Catani, Grazzini '00]

$$
\begin{aligned}
&S_{\mathfrak{m}}\, F_{\rm RV}(\mathfrak{m}) \\
&= - g_{s,b}^2 \sum_{(ij)}^{N_p} \Big\{ 2\, S_{ij}(p_{\mathfrak{m}})\ (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\rm LV} - \frac{\alpha_s(\mu)}{2\pi} \frac{\beta_0}{\epsilon}\, 2\, S_{ij}(p_{\mathfrak{m}})\ (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\rm LM} \\
&\quad - 2\, \frac{[\alpha_s]}{\epsilon^2}\, C_A\, A_K(\epsilon) \left( S_{ij}(p_{\mathfrak{m}}) \right)^{1+\epsilon} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\rm LM} \\
&\quad - [\alpha_s] \frac{4\pi\, \Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\, \Gamma(1-2\epsilon)} \sum_{\substack{k=1 \\ k\neq i,j}}^{N_p} \kappa_{ij}\, S_{ki}(p_{\mathfrak{m}}) \left( S_{ij}(p_{\mathfrak{m}}) \right)^{\epsilon} f_{abc}\, T_k^a\, T_i^b\, T_j^c\, F_{\rm LM} \Big\}
\end{aligned}
$$

$$
S_{ij}(p_{\mathfrak{m}}) = \frac{p_i \cdot p_j}{2 (p_i \cdot p_{\mathfrak{m}})(p_j \cdot p_{\mathfrak{m}})}
$$

$$
A_K = \frac{\Gamma^3(1+\epsilon)\,\Gamma^5(1-\epsilon)}{\epsilon^2\,\Gamma(1+2\epsilon)\,\Gamma^2(1-2\epsilon)}
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- Vanish for $N_p \geq 4$
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❖ The integrated subtraction term can be almost fully written in terms of NLO-like operators

$$
\begin{aligned}
\langle S_{\mathfrak{m}}\, F_{\rm RV}(\mathfrak{m}) \rangle &= [\alpha_s]^2 \left\langle \frac{1}{2} \left[ I_{\rm S}(\epsilon) \cdot I_{\rm V}(\epsilon) + I_{\rm V}(\epsilon) \cdot I_{\rm S}(\epsilon) \right] \cdot F_{\rm LM} \right\rangle \\
&+ [\alpha_s] \left\langle I_{\rm S}(\epsilon) \cdot F_{\rm LV}^{\rm fin} \right\rangle - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \frac{\beta_0}{\epsilon} \left\langle I_{\rm S}(\epsilon) F_{\rm LM} \right\rangle \\
&- \frac{[\alpha_s]^2}{\epsilon^2} C_A\, A_K(\epsilon) \left\langle \widetilde{I}_{\rm S}(2\epsilon) \cdot F_{\rm LM} \right\rangle \\
&+ [\alpha_s]^2 \left\langle \left( \frac{1}{2} \left[ I_{\rm S}(\epsilon)\, ,\, \overline{I}_1(\epsilon) - \overline{I}_1^{\dagger}(\epsilon) \right] + I_{\rm tri}^{\rm RV}(\epsilon) \right) \cdot F_{\rm LM} \right\rangle
\end{aligned}
$$

**Structures** and **color coefficients** already encountered in **double-virtual** and **double-soft**.

**A pattern begins to arise…**

# The pie so far

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f$$

$$c_1(\epsilon) = 1 + \left(\frac{\pi^2}{6} - \frac{32}{9}\right)\epsilon^2 + \cdots$$

$$A_K(\epsilon) = 1 - \frac{\pi^2}{3}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

| | | | | |
|---|---|---|---|---|
| $\langle F_{\mathrm{LVV}} \rangle$ | $\frac{1}{2} I_{\mathrm{V}}^2(\epsilon)$ | $\frac{\beta_0}{\epsilon} I_{\mathrm{V}}(\epsilon)$ | $K\, I_{\mathrm{V}}(2\epsilon)$ | $\frac{\beta_0}{\epsilon} I_{\mathrm{V}}(2\epsilon)$ |
| $\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle$ | $\frac{1}{2} I_{1,R}^2(\epsilon)$ | | $\frac{C_A}{\epsilon^2} c_1(\epsilon)\, \widetilde{I}_{\mathrm{S}}(2\epsilon)$ | $\frac{\beta_0}{\epsilon}\, \widetilde{I}_{\mathrm{S}}(2\epsilon)$ |
| $\langle S_{\mathfrak{m}}\, F_{\mathrm{RV}}(\mathfrak{m}) \rangle$ | $\frac{1}{2}\left[I_{\mathrm{S}}(\epsilon)\cdot I_{\mathrm{V}}(\epsilon) + I_{\mathrm{V}}(\epsilon)\cdot I_{\mathrm{S}}(\epsilon)\right]$ | $\frac{\beta_0}{\epsilon} I_{\mathrm{S}}(\epsilon)$ | $-\frac{C_A}{\epsilon^2} A_K(\epsilon)\, \widetilde{I}_{\mathrm{S}}(2\epsilon)$ | |

Almost reconstruct $I_{\mathrm{T}}^2(\epsilon)$ → look at collinear

Almost reconstruct $I_{\mathrm{T}}(\epsilon)$ but with extra $1/\epsilon$ → look at collinear

Clear interplay → $C_A, 2\epsilon$

non-transparent cancellation

Almost reconstruct $I_{\mathrm{T}}(2\epsilon)$ but with extra $1/\epsilon$ → look at collinear

$$I_{\mathrm{T}}(\epsilon) = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon)$$ finite

# Cancellation of quartic and double color correlations (cc)

❖ To obtain complete iterations of NLO we need to add also collinear contributions.

❖ They feature diverse kinematics dependences and are non-trivial to manipulate [partition functions, sectoring, definitions of collinear limits action on matrix elements and phase space]. However, only few of them can lead to color correlations, namely those arising from real-virtual corrections.

❖ Here we focus on contributions that contain at least one virtual or one soft operator and feature elastic, LO-like kinematics:

$$
\begin{aligned}
\Sigma_N^{(\mathrm{V+S}),\mathrm{el}} = {} & [\alpha_s]^2 \frac{1}{2} \Big\langle \big[ I_{\mathrm{V}}^2 + I_{\mathrm{V}} I_{\mathrm{S}} + I_{\mathrm{S}} I_{\mathrm{V}} + I_{\mathrm{S}}^2 + 2 I_{\mathrm{C}} I_{\mathrm{V}} + 2 I_{\mathrm{C}} I_{\mathrm{S}} \big] \cdot F_{\mathrm{LM}} \Big\rangle \\
& + [\alpha_s]^2 \frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_{\mathrm{E}}}} \Big\langle \Big[ -\big[ I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon) \big] + I_{\mathrm{V}}(2\epsilon) + \tilde{c}(\epsilon)\, \widetilde{I}_{\mathrm{S}}(2\epsilon) \Big] \cdot F_{\mathrm{LM}} \Big\rangle \\
& + [\alpha_s]^2 \Big\langle \Big[ K \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_{\mathrm{E}}}}\, I_{\mathrm{V}}(2\epsilon) + C_A \Big( \frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon} \delta_g^{C_A}(\epsilon) \Big) \widetilde{I}_{\mathrm{S}}(2\epsilon) \Big] \cdot F_{\mathrm{LM}} \Big\rangle \\
& + [\alpha_s] \Big\langle \big[ I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) \big] \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \Big\rangle \,,
\end{aligned}
$$

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\begin{aligned}
\Sigma_N^{(\mathrm{V+S}),\mathrm{el}} &= [\alpha_s]^2 \frac{1}{2} \Big\langle \overbrace{\left[I_\mathrm{V}^2 + I_\mathrm{V} I_\mathrm{S} + I_\mathrm{S} I_\mathrm{V} + I_\mathrm{S}^2 + 2 I_\mathrm{C} I_\mathrm{V} + 2 I_\mathrm{C} I_\mathrm{S}\right]}^{= I_\mathrm{T}^2 - I_\mathrm{C}^2 \quad \text{no singular c.c.}} \cdot F_\mathrm{LM} \Big\rangle \\
&+ [\alpha_s]^2 \frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_\mathrm{E}}} \Big\langle \Big[ -\left[I_\mathrm{S}(\epsilon) + I_\mathrm{V}(\epsilon)\right] + I_\mathrm{V}(2\epsilon) + \tilde{c}(\epsilon)\, \widetilde{I}_\mathrm{S}(2\epsilon) \Big] \cdot F_\mathrm{LM} \Big\rangle \\
&+ [\alpha_s]^2 \Big\langle \Big[ K \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_\mathrm{E}}}\, I_\mathrm{V}(2\epsilon) + C_A \Big( \frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon} \delta_g^{C_A}(\epsilon) \Big) \widetilde{I}_\mathrm{S}(2\epsilon) \Big] \cdot F_\mathrm{LM} \Big\rangle \\
&+ [\alpha_s] \underbrace{\Big\langle \left[ I_\mathrm{V}(\epsilon) + I_\mathrm{S}(\epsilon) \right]}_{= I_\mathrm{T} - I_\mathrm{C} \quad \text{no singular c.c.}} \cdot F_\mathrm{LV}^\mathrm{fin} \Big\rangle \,,
\end{aligned}
$$

# Cancellation of quartic and double color correlations (cc)

❖ To obtain complete iterations of NLO we need to add also collinear contributions.

❖ They feature diverse kinematics dependences and are non-trivial to manipulate [partition functions, sectoring, definitions of collinear limits action on matrix elements and phase space]. However, only few of them can lead to color correlations, namely those arising from real-virtual corrections.

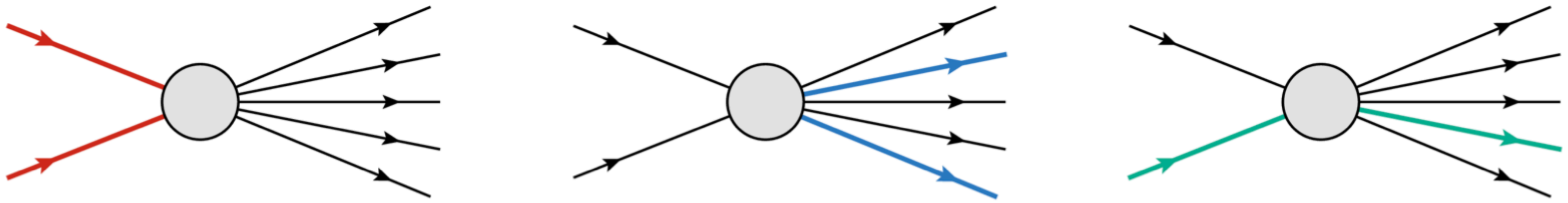
❖ Here we focus on contributions that contain at least one virtual or one soft operator and feature elastic, LO-like kinematics:

$$
\begin{aligned}
\Sigma_N^{(\mathrm{V+S}),\mathrm{el}} = {} & [\alpha_s]^2 \frac{1}{2} \Big\langle \overbrace{\big[ I_\mathrm{V}^2 + I_\mathrm{V} I_\mathrm{S} + I_\mathrm{S} I_\mathrm{V} + I_\mathrm{S}^2 + 2 I_\mathrm{C} I_\mathrm{V} + 2 I_\mathrm{C} I_\mathrm{S} \big]}^{= I_\mathrm{T}^2 - I_\mathrm{C}^2 \ \text{no singular c.c.}} \cdot F_\mathrm{LM} \Big\rangle \\
& + [\alpha_s]^2 \frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_\mathrm{E}}} \Big\langle \Big[ \overbrace{- \big[ I_\mathrm{S}(\epsilon) + I_\mathrm{V}(\epsilon) \big] + I_\mathrm{V}(2\epsilon) + \tilde{c}(\epsilon)\, \tilde{I}_\mathrm{S}(2\epsilon)}^{\sim \tilde{I}_\mathrm{S}(2\epsilon)} \Big] \cdot F_\mathrm{LM} \Big\rangle \\
& + [\alpha_s]^2 \Big\langle \Big[ K \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_\mathrm{E}}} I_\mathrm{V}(2\epsilon) + C_A \Big( \frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon} \delta_g^{C_A}(\epsilon) \Big) \tilde{I}_\mathrm{S}(2\epsilon) \Big] \cdot F_\mathrm{LM} \Big\rangle \\
& + [\alpha_s] \underbrace{\Big\langle \big[ I_\mathrm{V}(\epsilon) + I_\mathrm{S}(\epsilon) \big]}_{= I_\mathrm{T} - I_\mathrm{C} \ \text{no singular c.c.}} \cdot F_\mathrm{LV}^\mathrm{fin} \Big\rangle \,,
\end{aligned}
$$

$$
\sim - \underbrace{I_\mathrm{V+S}(\epsilon) + I_\mathrm{V+S}(2\epsilon)}_{\mathcal{O}(\epsilon)} + \big( \underbrace{\tilde{c}(\epsilon) - 1}_{\mathcal{O}(\epsilon^2)} \big) \tilde{I}_\mathrm{S}(2\epsilon) + \underbrace{\tilde{I}_\mathrm{S}(2\epsilon) - I_\mathrm{S}(2\epsilon)}_{\mathcal{O}(\epsilon)}
$$

# Pictorial conclusions

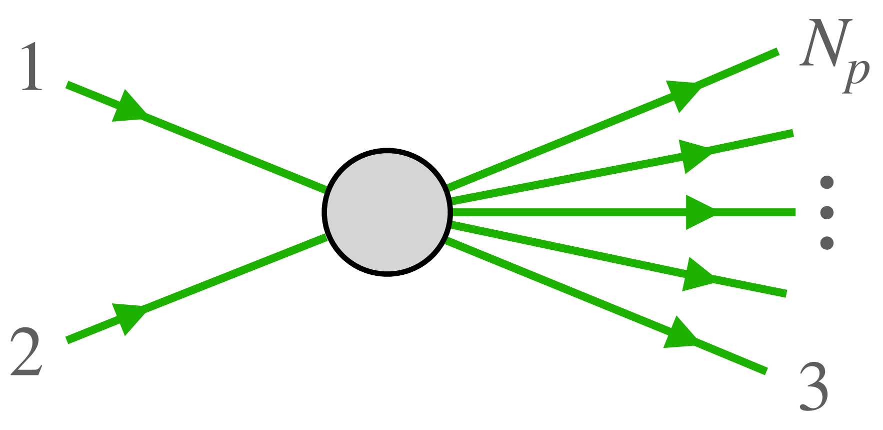
## Previous studies:



## Can we generalise? yes!

[Devoto, Melnikov, Röntsch, CSS, Tagliabue '24]

❖ Introduce **NLO-like universal operators** that describe **virtual, soft and collinear singularities,** and combine into **finite quantities**




Virtual component — Soft component — Collinear component

$$I_{\mathrm{T}}(\epsilon) = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon)$$

**Free of poles**
**Fully general in the number of partons**

❖ Reduce **NNLO** corrections to **iterations of these operators** → demonstrate **cancellations prior to explicit evaluation**

$$d\hat{\sigma}^{\mathrm{NNLO}} = d\hat{\sigma}^{\mathrm{VV}} + d\hat{\sigma}^{\mathrm{RV}} + d\hat{\sigma}^{\mathrm{RR}} + d\hat{\sigma}^{\mathrm{pdf}} = \frac{[\alpha_s]^2}{2}\left\langle M_0 \middle| \left[I_{\mathrm{V}} + I_{\mathrm{S}} + I_{\mathrm{C}}\right]^2 \middle| M_0 \right\rangle + \ldots \equiv \langle M_0 | I_{\mathrm{T}}^2 | M_0 \rangle + \ldots$$

Double-Virtual, Real-Virtual, Double-Real, PDFs Renor.

# Standard conclusions

1. Subtraction schemes are necessary ingredients to obtain precise theoretical predictions.

2. **Nested-soft collinear subtraction** provides an efficient method to deal with n-parton processes:

   I. combine different subtraction terms **to get rid of color-correlations** (and boosted contributions),

   II. reduce the subtraction terms to **few, recurring structures**.

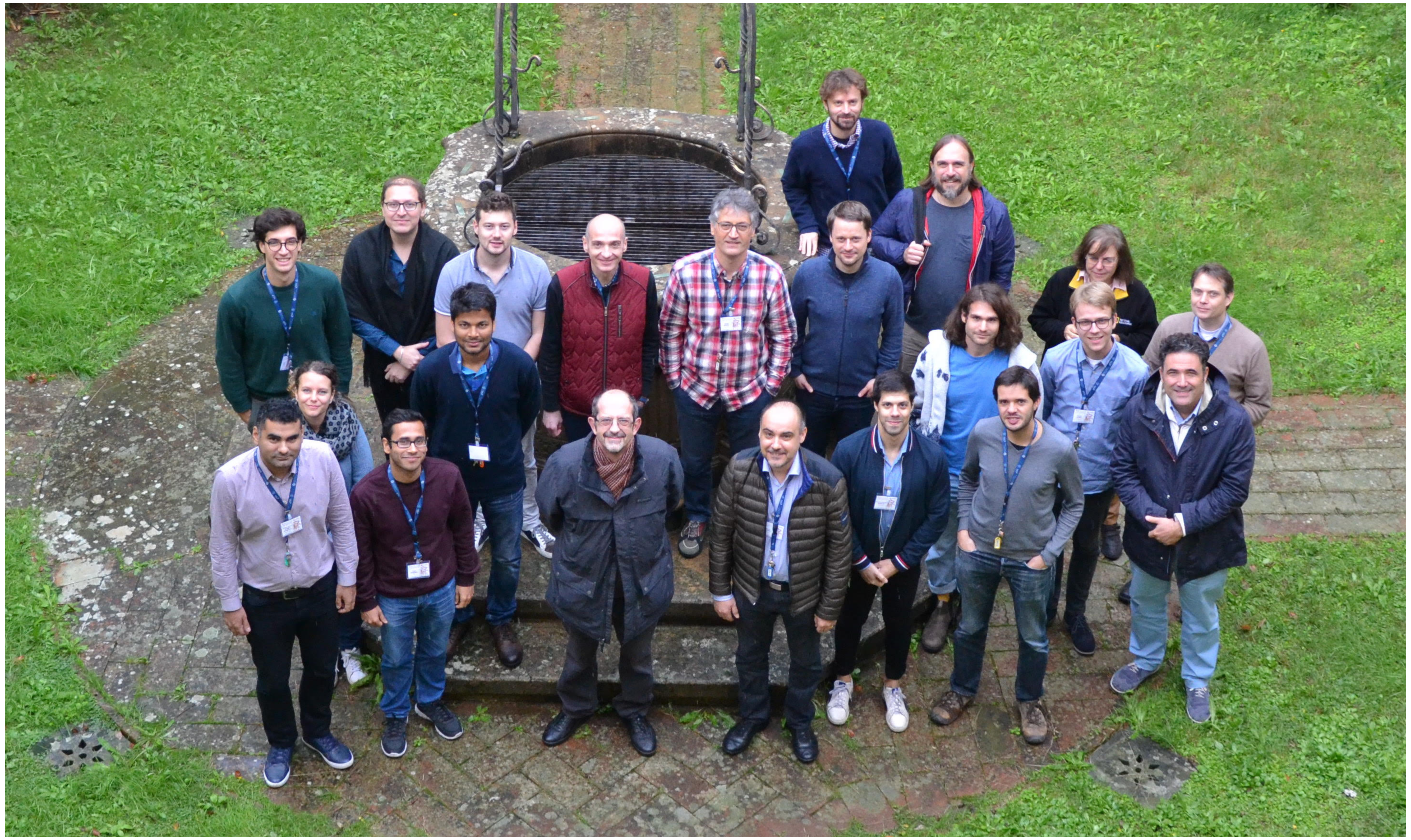
3. Pole cancellation proven analytically for the case-study $q\bar{q} \to X + Ng$.

→ **Finite remainders in agreement with the standard approach for** $q\bar{q} \to X + g\,@NNLO$

# Work in progress

Generalisation to arbitrary final- and initial-state partons.

Implementation



Novembre 2019
GGI, Florence

# Thank you!

# Backup

# Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\left\langle\left[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\right]F_{\rm LM}\right\rangle$$

$$\frac{\beta_0}{\epsilon}[\alpha_s]I_{1,T}(\epsilon)$$

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+[\alpha_s]^2\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\left\langle\left[-[\alpha_s]^2C_A\,A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\right]F_{\rm LM}\right\rangle$$

$$\frac{\alpha_s}{2\pi}[\alpha_s]\frac{\beta_0}{\epsilon}\left\langle I_{1,T}(2\epsilon)\,F_{\rm LM}\right\rangle-\frac{\alpha_s}{2\pi}[\alpha_s]\frac{\beta_0}{\epsilon}\left\langle I_C(2\epsilon)\,F_{\rm LM}\right\rangle+\Sigma^{(1)}_{T_i\cdot T_j,\,{\rm fin}}$$

No singular, color-correlated contributions

# Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\left\langle\left[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\right]F_{\rm LM}\right\rangle$$

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+[\alpha_s]^2\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\left\langle\left[-[\alpha_s]^2C_A\,A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\right]F_{\rm LM}\right\rangle$$

$$\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_{1,T}(2\epsilon)\,F_{\rm LM}\right\rangle-\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle-\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_C(2\epsilon)\,F_{\rm LM}\right\rangle$$

finite

Singular and color-correlated

color-uncorrelated

# Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\left\langle\left[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\right]F_{\rm LM}\right\rangle$$

$\widetilde{I}_{1,R}(2\epsilon) \longrightarrow I_{1,R}(2\epsilon)$

finite

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+[\alpha_s]^2\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\left\langle\left[-[\alpha_s]^2C_AA_K\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\right]F_{\rm LM}\right\rangle$$

$$\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_{1,T}(2\epsilon)\,F_{\rm LM}\right\rangle-\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle-\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_C(2\epsilon)\,F_{\rm LM}\right\rangle$$

finite — Singular and color-correlated — color-uncorrelated

$-C_AA_K+\frac{C_A}{\epsilon^2}c_1$ finite

$\widetilde{I}_{1,R}(2\epsilon) \longrightarrow I_{1,R}(2\epsilon)$

# Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\Big\langle\Big[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\Big]F_{\rm LM}\Big\rangle$$

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}\,c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}\,c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+[\alpha_s]^2\,\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\Big\langle\Big[-[\alpha_s]^2\,C_A\,A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2 c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\Big]F_{\rm LM}\Big\rangle$$

$1/\epsilon$ color-uncorrelated

$$[\alpha_s]\,\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\Big\langle\Big(I_{1,T}(2\epsilon)-I_{1,T}(\epsilon)\Big)F_{\rm LM}\Big\rangle$$

$$\propto\frac{C_A(C_A+2C_F)}{\epsilon^2}\left(-\frac{131}{72}+\frac{\pi^2}{6}+\frac{11}{6}\log 2\right)+\frac{1}{\epsilon}[\text{color}-\text{correlations}]$$

$$+\Big\langle\Big[[\alpha_s]^2\Big(-C_A\,A_K+\frac{C_A}{\epsilon^2}c_1(\epsilon)+\beta_0\,c_3(\epsilon)\Big)-\frac{\alpha_s}{2\pi}[\alpha_s]\,c_\epsilon\,K\Big]I_{1,R}(2\epsilon)\,F_{\rm LM}\Big\rangle$$

$$-\frac{\alpha_s}{2\pi}[\alpha_s]\Big\langle\Big(c_\epsilon\,K+\frac{\beta_0}{\epsilon}\Big)\,I_C(2\epsilon)\,F_{\rm LM}\Big\rangle$$

$1/\epsilon^2$ color-uncorrelated

Peculiar dependence in the color-correlations, that fits perfectly a contribution from triple-collinear sectors $\Theta^{(b)}$

$$\Big\langle\sum_{i\in\rm TC}(I-S_{45})\,C_{45}\Theta^{(b)}(F_{\rm LM}-2S_5F_{\rm LM}^{4>5})\,\omega_{4i5i}\,\Delta^{(45)}\Big\rangle$$

$$\rightarrow\;-4[\alpha_s]^2C_A2^{-2\epsilon}\delta_g(\epsilon)\Big\langle I_{1,R}(2\epsilon)F_{\rm LM}\Big\rangle+\Sigma^{(2)}_{T_i\cdot T_j,\,\rm fin}\;\;\propto-\frac{C_A(C_A+2C_F)}{\epsilon^2}\left(-\frac{131}{72}+\frac{\pi^2}{6}+\frac{11}{6}\log 2\right)+\mathcal{O}(\epsilon^{-1})$$

# Useful relations:

$$I_{1,R}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{i\neq j}^{n} \eta_{ij}^{-\epsilon} K_{ij} \mathbf{T}_i \cdot \mathbf{T}_j \, ,$$

$$\begin{aligned} K_{ij} &= \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij}^{1+\epsilon} \, {}_2F_1(1,1,1-\epsilon,1-\eta_{ij}) \\ &= \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \, {}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-\eta_{ij}) \end{aligned}$$

$$\widetilde{I}_{1,R}(2\epsilon) = -\frac{(2E_{\max}/\mu)^{-4\epsilon}}{(2\epsilon)^2} \sum_{i\neq j}^{n} \eta_{ij}^{-2\epsilon} \widetilde{K}_{ij} \mathbf{T}_i \cdot \mathbf{T}_j$$

$$\begin{aligned} \widetilde{K}_{ij} &= \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)} \eta_{ij}^{1+3\epsilon} {}_2F_1(1+\epsilon,1+\epsilon,1-\epsilon,1-\eta_{ij}) \\ &= \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)} \, {}_2F_1(-2\epsilon,-2\epsilon;1-\epsilon,1-\eta_{ij}) \, . \end{aligned}$$

$$\widetilde{K}_{ij}(\epsilon) = K_{ij}(2\epsilon) \left[ \frac{{}_2F_1(-2\epsilon,-2\epsilon;1-\epsilon,1-\eta_{ij})}{{}_2F_1(-2\epsilon,-2\epsilon,1-2\epsilon,1-\eta_{ij})} \right] = K_{ij}(2\epsilon) \left[ 1 + \mathcal{O}(\epsilon^3) \right]$$

$$\tilde{I}_{1,R}(2\epsilon) = I_{1,R}(2\epsilon) + \mathcal{O}(\epsilon)$$

# Useful definitions:

$$\hat{\Gamma}_q = \frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{2E_1}{\mu}\right)^{-2\epsilon}\left[\gamma_q + \frac{C_F}{\epsilon}(1-e^{-2\epsilon L_1})\right]F_{\rm LM}(1\dots N) \sim \frac{1}{\epsilon}(\gamma_q + 2C_F\, L_1) + \mathcal{O}(\epsilon^0)$$

$$\hat{\Gamma}_g = \frac{1}{\epsilon}C_A\left(\frac{2E_n}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\gamma^{22}_{z,g\to gg} + \frac{1}{\epsilon}(1-e^{-2\epsilon L_n})\right] \qquad \gamma^{22}_{z,g\to gg} \sim \frac{11}{6} + \frac{1}{9}\left(67-6\pi^2\right)\epsilon + \dots$$

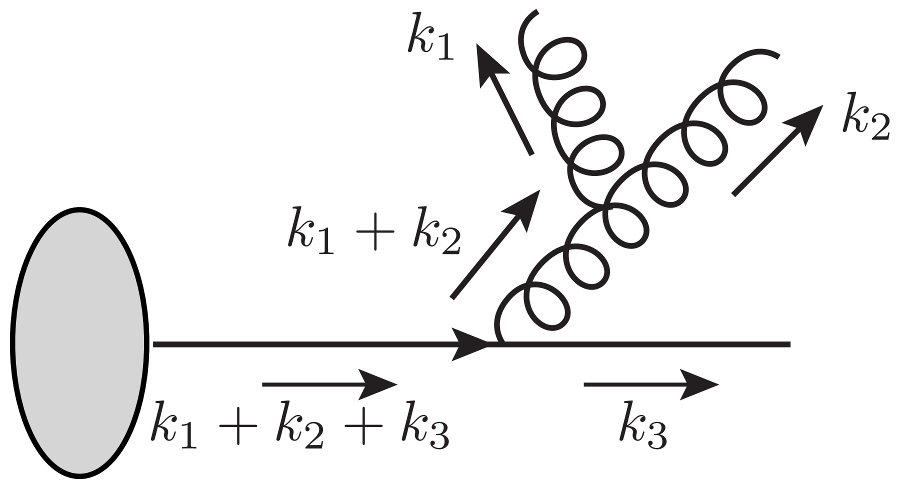
$$\hat{\Gamma}_g(2\epsilon) = \frac{1}{2\epsilon}C_A\left(\frac{2E_n}{\mu}\right)^{-4\epsilon}\frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\left[\gamma^{44}_{z,g\to gg} + \frac{1}{2\epsilon}(1-e^{-4\epsilon L_n})\right]$$

$$P^{\rm gen}_{qq}(z) = -\frac{1}{\epsilon}\hat{P}^{\rm AP,0}_{qq}(z) + P'_{\rm fin,qq}(z)$$

$$G^{(1)}(z)\,F^{(1)}_{\rm LM} = \frac{1}{2}[\alpha_s]^2\left[-P^{\rm gen}_{qq}\otimes\Gamma^{(1)}_q(z)F^{(1)}_{\rm LM}(1_q,2_{\bar{q}};3_g|z) + \Gamma^{(1)}_q P^{\rm gen}_{qq}\otimes F^{(1)}_{\rm LM}(1_q,2_{\bar{q}};3_g|z)\right]$$

$$G^{(3)}(L_3) = \frac{1}{2}\frac{[\alpha_s]^2}{\epsilon^2}C_A^2\left(\frac{2E_3}{\mu}\right)^{-4\epsilon}\left(\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right)^2\left(\gamma^{22}_{z,g\to gg} + \frac{1}{\epsilon}\right)\left(\gamma^{42}_{z,g\to gg} - \gamma^{22}_{z,g\to gg}\right)$$

## 1. Clear understanding of which singular configurations do actually contribute



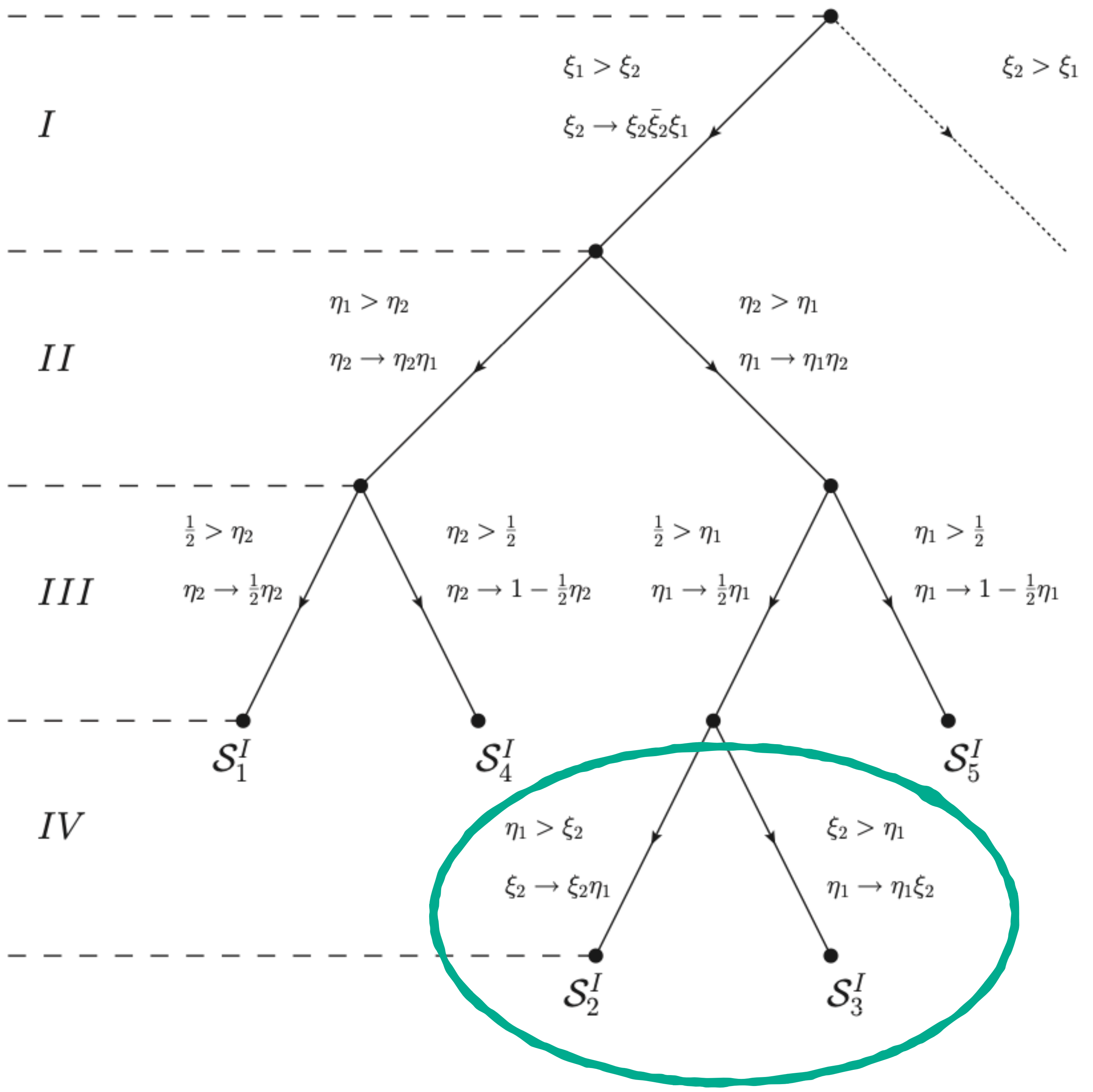

$$\sim \frac{1}{(k_1+k_2)^2}\frac{1}{(k_1+k_2+k_3)^2} = \frac{1}{2k_1\cdot k_2}\frac{1}{2k_1\cdot k_2 + 2k_1\cdot k_3 + 2k_2\cdot k_3} \Longleftrightarrow k_1 \to 0 \text{ and } k_2 \parallel k_3$$

Entangled soft-collinear limits of diagrams can not be treated in a process-independent way.
**Do non-commutative limits actually contribute?**

STRIPPER was implemented taking into account all the possible choices of soft and collinear limits order -> redundant configurations were included

**Gauge invariant amplitudes are free of entangled singularities** thanks to color coherence: soft parton does not resolve angles of the collinear partons

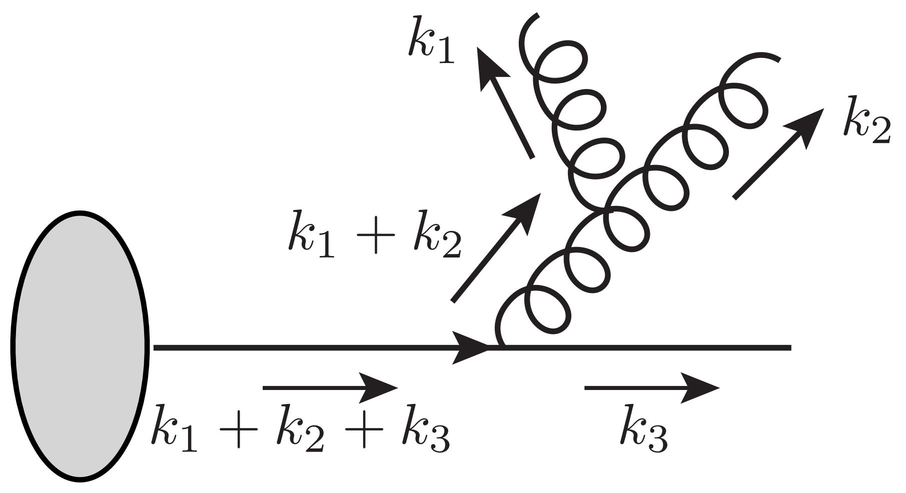
Soft-collinear limits can be described by taking the known soft and collinear limits sequentially




*[Czakon 1005.0274]*

## 2. Get to the point where the problem is well defined

a) Identify the overlapping singularities
b) Regulate them



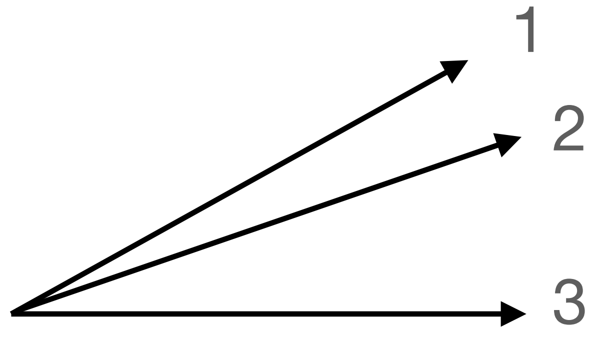

$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

**Soft origin**: $E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$

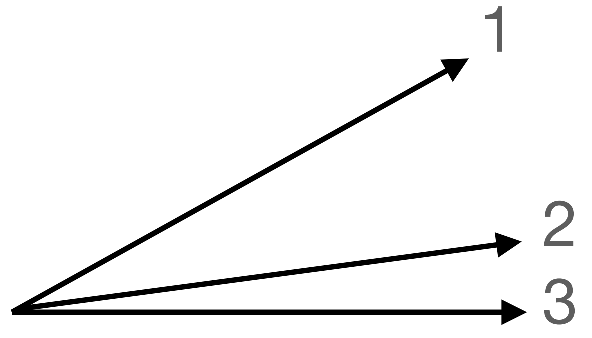
**Collinear origin**: $\vec{n}_1 \parallel \vec{n}_2 \quad \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$

$E_1 \ll E_2 , \quad E_2 \ll E_1$

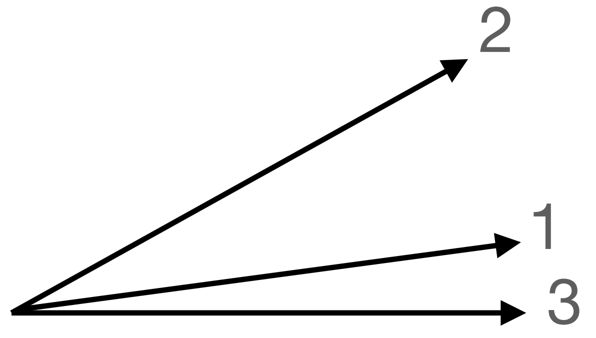
Includes **strongly ordered** configurations




$\vec{n}_1 \cdot \vec{n}_2 < \vec{n}_1 \cdot \vec{n}_3$




$\vec{n}_2 \cdot \vec{n}_3 < \vec{n}_1 \cdot \vec{n}_3$

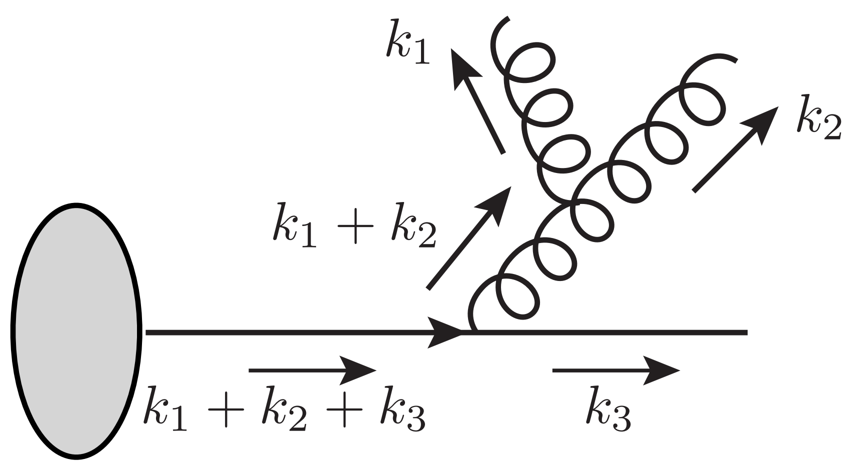



$\vec{n}_1 \cdot \vec{n}_3 < \vec{n}_2 \cdot \vec{n}_3$

Soft and collinear modes do not intertwine: soft subtraction can be done globally. Collinear singularities have still to be regulated.
Strongly ordered configurations have to be properly taken into account.

# Nested soft-collinear subtraction at NNLO: generalities *[Caola, Melnikov, Röntsch 1702.01352]*

Extension of **FKS subtraction** *[Frixione, Kunst, Signer 9512328]* **to NNLO and inspired by STRIPPER** *[Czakon 1005.0274]*



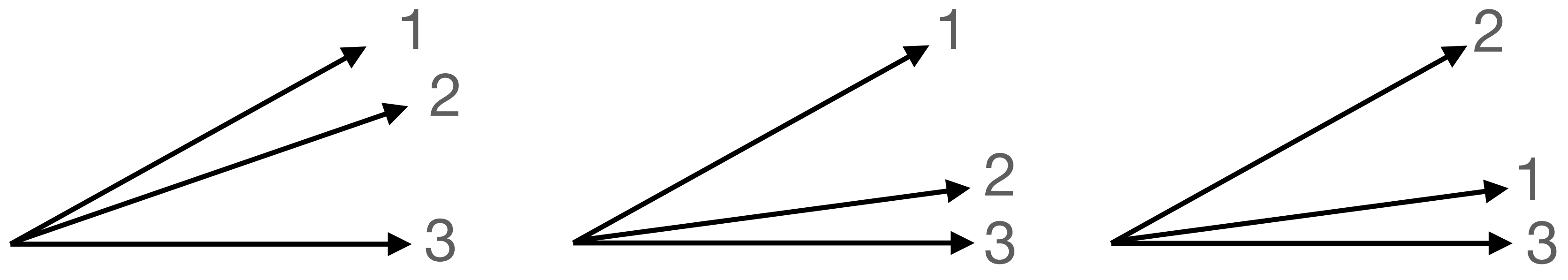

$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

$E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$

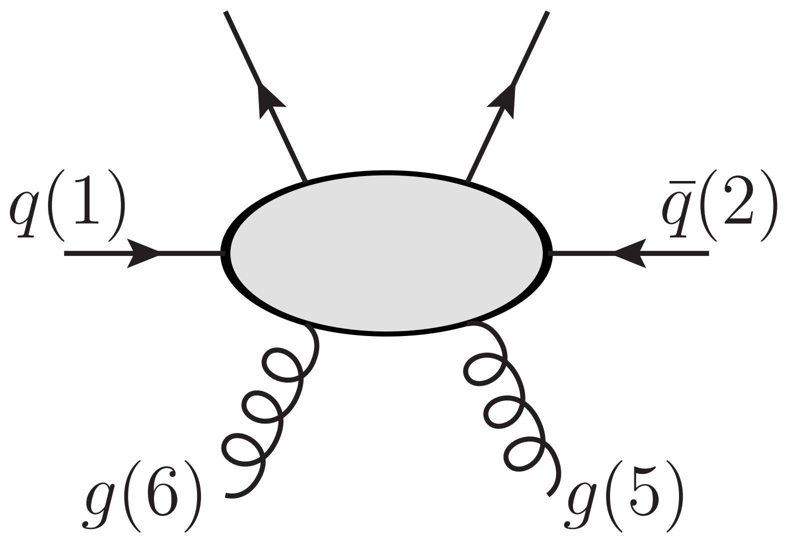
$\vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$

$\vec{n}_1 \parallel \vec{n}_2$

**Strongly-ordered** configurations have also to be included: $E_1 \ll E_2\,, \quad E_2 \ll E_1$



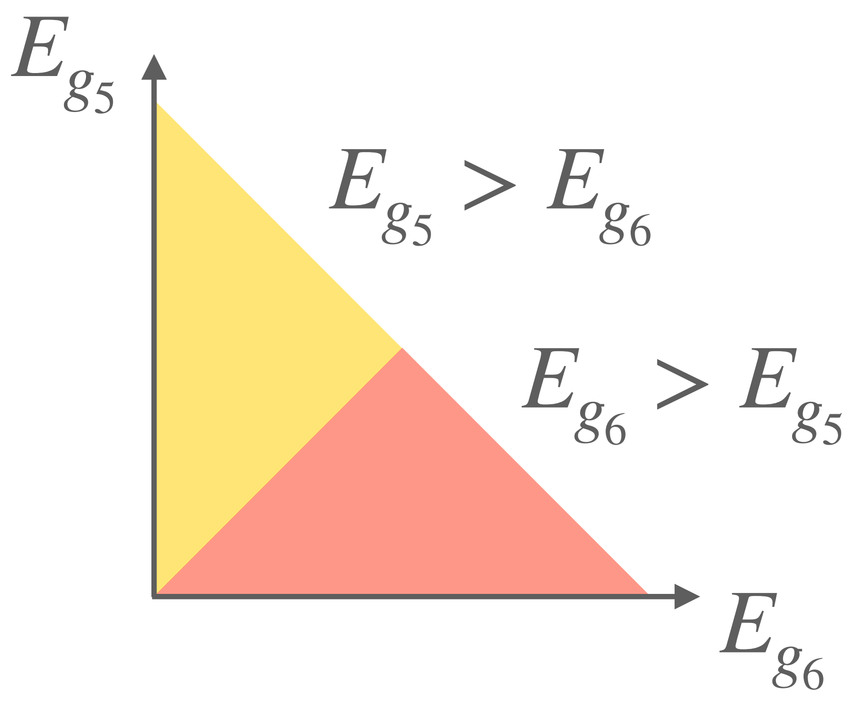

$\vec{n}_1 \cdot \vec{n}_2 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_2 \cdot \vec{n}_3 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_1 \cdot \vec{n}_3 < \vec{n}_2 \cdot \vec{n}_3$




**Soft limits:**

- Non-trivial structure of double-soft eikonal
- Strongly-ordered limits to disentangle

$$1 = \theta\left(E_{g_5} - E_{g_6}\right) + \theta\left(E_{g_6} - E_{g_5}\right)$$



# Nested soft-collinear subtraction at NNLO: generalities *[Caola, Melnikov, Röntsch 1702.01352]*

Extension of **FKS subtraction** *[Frixione, Kunst, Signer 9512328]* **to NNLO and inspired by STRIPPER** *[Czakon 1005.0274]*

- Exploit colour-coherence to discard interplay between soft and collinear

  → subtract soft limits first, then collinear

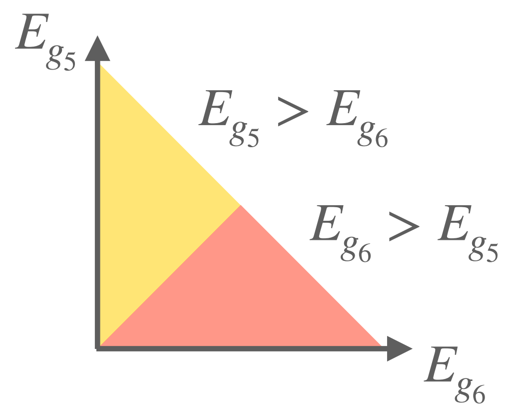
**"nested approach"**

- Define subtraction terms in 3 steps:

  - Globally remove double soft singularities

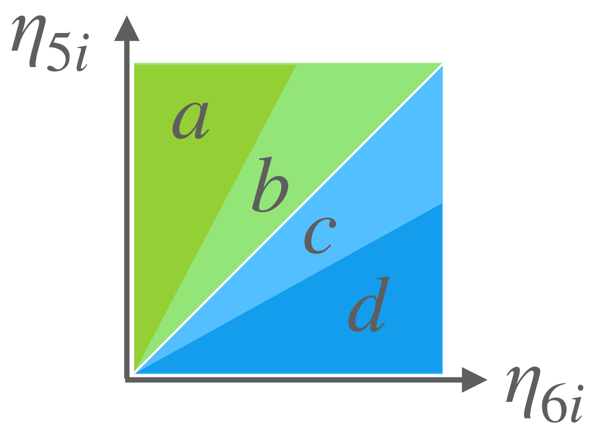
  - Globally remove single soft singularities [using energy ordering]

  - FKS partition and sectoring to treat the minimum number of collinear singularities at a time




$$1 = \sum_{i,j} \omega^{i5,j6}$$

$$\omega^{5i,6i} = \omega^{5i,6i} \left(\theta_a + \theta_b + \theta_c + \theta_d\right)$$




- Integrate subtraction terms analytically using Reverse Unitarity *[Anastasiou, Melnikov '02]*:
  **map phase space integrals onto loop integrals** *[Caola, Delto, Frellesvig, Melnikov '18, '19]*

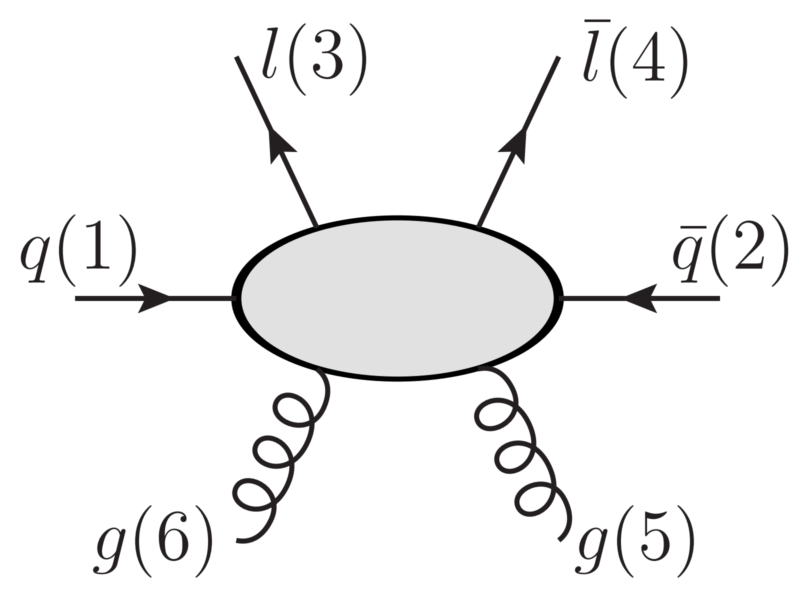
# Phase space partitions

Efficient way to simplify the problem: introduce **partition functions** (following FKS philosophy):

- **Unitary partition**
- Select a **minimum number of singularities** in each sector
- Do **not affect** the **analytic integration** of the counterterms

Definition of partition functions benefits from remarkable degree of **freedom**: different approaches can be implemented

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ [Caola, Melnikov, Röntsch 1702.01352]




$$1 = \omega^{51,61} + \omega^{52,62} + \omega^{51,62} + \omega^{52,61}$$

$$\omega^{51,61} = \frac{\rho_{25}\,\rho_{26}}{d_5\,d_6}\left(1 + \frac{\rho_{15}}{d_{5621}} + \frac{\rho_{16}}{d_{5612}}\right) \qquad \omega^{51,62} = \frac{\rho_{25}\,\rho_{16}\,\rho_{56}}{d_5\,d_6\,d_{5612}}$$

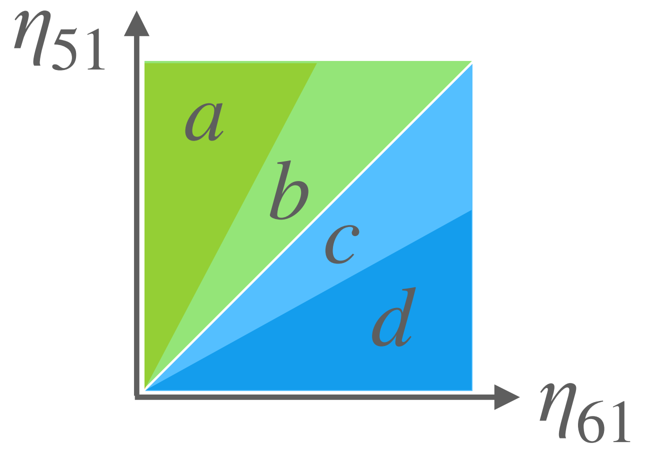
$$\omega^{52,62} = \frac{\rho_{15}\,\rho_{16}}{d_5\,d_6}\left(1 + \frac{\rho_{25}}{d_{5621}} + \frac{\rho_{26}}{d_{5612}}\right) \qquad \omega^{52,61} = \frac{\rho_{15}\,\rho_{26}\,\rho_{56}}{d_5\,d_6\,d_{5621}}$$

$$\rho_{ab} = 1 - \cos\vartheta_{ab}\,,\ \eta_{ab} = \rho_{ab}/2$$

$$d_{i=5,6} = \rho_{1i} + \rho_{2i} = 2$$

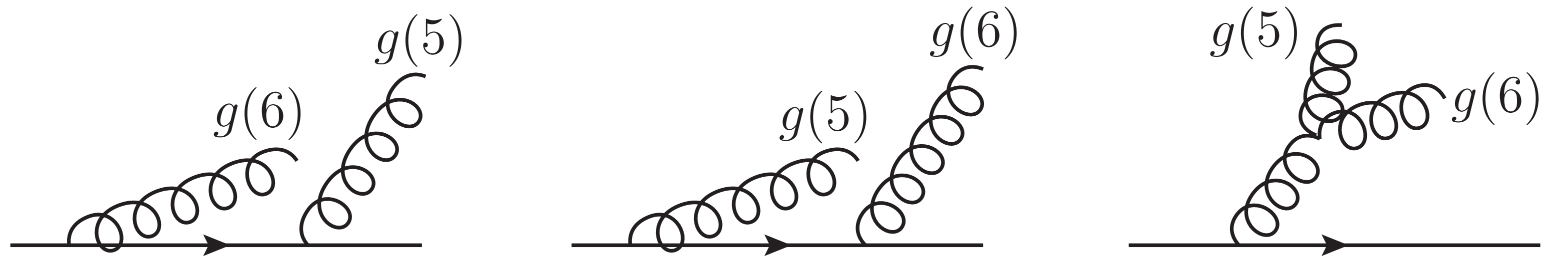
$$d_{5621} = \rho_{56} + \rho_{52} + \rho_{61}$$

$$d_{5612} = \rho_{56} + \rho_{51} + \rho_{62}$$




$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$

$$= \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}$$



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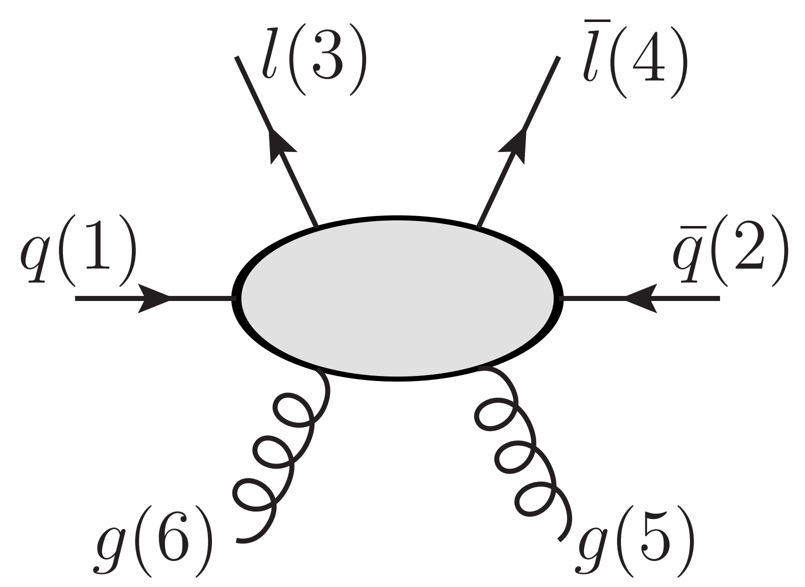



Advantages:

1. Simple definition
2. Structure of collinear singularities fully defined
3. Same strategy holds for NNLO mixed QCDxEW processes
4. **Minimum number of sector**

Disadvantages:

1. Partition based on angular ordering -> Lorentz invariance not preserved
   -> angles defined in a given reference frame
2. Theta function

# 3. Solve the PS integrals

The problem is now well defined:

A. **Singular kernels** and their nested limits have to be **subtracted from the double real correction** to get integrable object

$$\int d\Phi_{n+2}\, RR_{n+2} = \int d\Phi_{n+2} \left[ RR_{n+2} - K_{n+2} \right] + \int d\Phi_{n+2}\, K_{n+2}$$

$$K_{n+2} \supset C_{ij},\, C_{kl},\, S_i,\, S_{ij},\, C_{ijk}$$

B. **Counterterms** have to be **integrated over the unresolved phase space**

$$I = \int \mathrm{PS}_{\mathrm{unres.}} \otimes \mathrm{Limit} \otimes \mathrm{Constraints}$$

The ‘Limit’ component is universal and known. The phase space is well defined. Constraints may vary depending on the scheme.

Several kinematic structures have to be integrated **analytically** over a 6-dim PS.

**Different approximations and techniques** can be applied: the results assume different form depending on the adopted strategy

Two main structure are the most complicated ones and affect most of the physical processes:

- **Double soft**
- **Triple collinear**

# Kernels integration

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ *[Caola, Delto, Frellesvig, Melnikov 1807.05835, Delto, Melnikov 1901.05213]*

Two soft parton (5,6) and two hard massless radiator (1,2): arbitrary relative angle between the three-momenta of the radiators

$$I_{12}^{(gg)(56)} = \frac{(1-\epsilon)(s_{51}s_{62} + s_{52}s_{61}) - 2s_{56}s_{12}}{s_{56}^2(s_{51}+s_{61})(s_{52}+s_{62})} + s_{12}\frac{s_{51}s_{62}+s_{52}s_{61}-s_{56}s_{12}}{s_{56}s_{51}s_{62}s_{52}s_{61}}\left[1-\frac{1}{2}\frac{s_{51}s_{62}+s_{52}s_{61}}{(s_{51}+s_{61})(s_{52}+s_{62})}\right]$$

$$I_{S_{56}}^{(gg)} = \int [\mathrm{d}k_5]\,[\mathrm{d}k_6]\,\theta(E_{\max}-E_5)\,\theta(E_5-E_6)\,I_{12}^{(gg)(56)}\left(k_1,k_2,k_5,k_6\right) \qquad [\,\mathrm{d}f_i\,] = \frac{\mathrm{d}^d k_i}{(2\pi)^d}(2\pi)\,\delta_+(k_i^2)$$

$$E_5 = E_{\max}\,\xi \qquad E_6 = E_{\max}\,\xi\, z \qquad 0<\xi<1\,,\,0<z<1$$

Reverse unitarity: **map phase space integrals onto loop integrals** *[Anastasiou, Melnikov 0207004]*
after defining integral families, integration-by-part identities. Differential equations w.r.t. the ratio of energies of emitted gluons at fixed angle.
Boundary conditions for z=0, and arbitrary angle

# Double virtual contribution

Universal structure, regulated by Catani's operator, valid for any number of external coloured partons *[Catani '98]* . Features a single structure with color-correlations

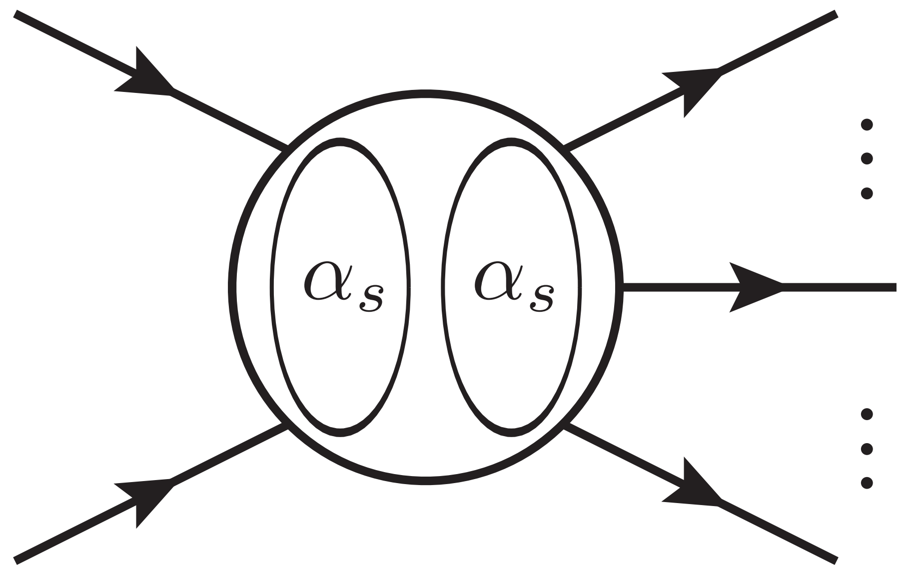
$$\langle F_{\rm LVV}\rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \Big\langle \frac{1}{2}\Big[2\Re(\mathcal{I}_1(\epsilon))\Big]^2 F_{\rm LM} - \frac{\beta_0}{\epsilon} 2\Re(\mathcal{I}_1(\epsilon)) F_{\rm LM}$$
$$+ \frac{e^{-\epsilon\gamma_{\rm E}}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \frac{\beta_0}{\epsilon} 2\Re(\mathcal{I}_1(2\epsilon)) F_{\rm LM} + \frac{e^{-\epsilon\gamma_{\rm E}}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} K\, 2\Re(\mathcal{I}_1(2\epsilon)) F_{\rm LM}$$
$$+ 2\,\underbrace{\frac{e^{\epsilon\gamma_{\rm E}}}{4\epsilon\,\Gamma(1-\epsilon)}\mathcal{H}_2(\epsilon)}_{\text{Process-dependent}}\, F_{\rm LM} + 2\Re(\mathcal{I}_1(\epsilon)) \underbrace{F_{\rm LV}^{\rm fin} + F_{\rm LVV}^{\rm fin} + F_{\rm LV^2}^{\rm fin}}_{\text{Finite remainders from 2-loop and (1-loop)}^2\text{ amplitudes}} \Big\rangle ,$$

Process-dependent

Finite remainders from 2-loop and (1-loop)$^2$ amplitudes

Color-correlations inside $\mathcal{I}_1(\epsilon)$ (already encountered at NLO)

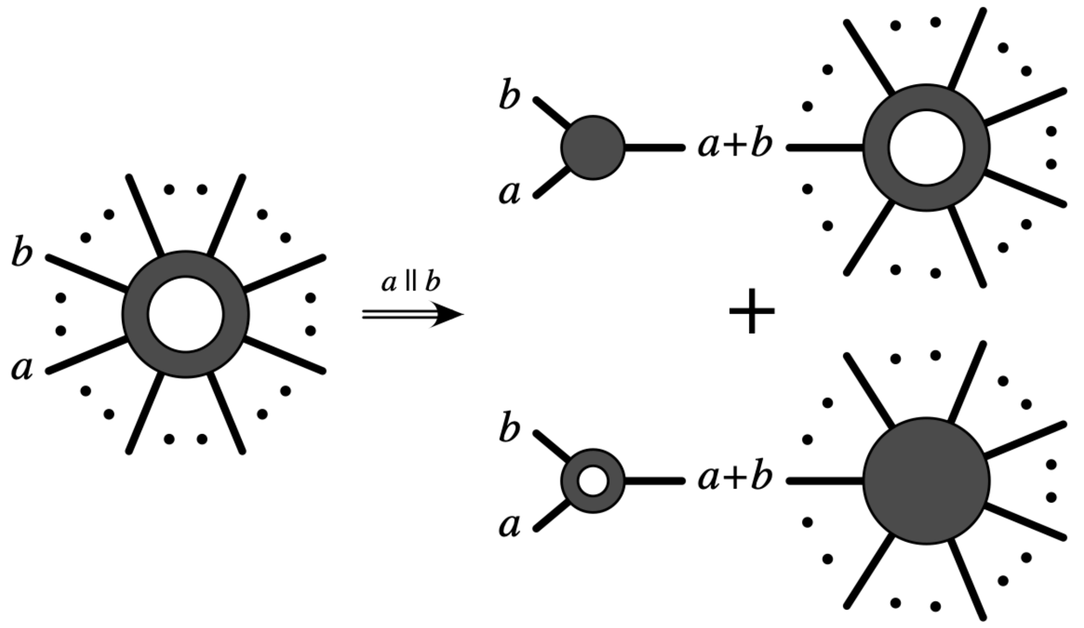
$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f \,.$$



# Hard-collinear real-virtual and single soft RR

Also in this case the IR structure is know in full generality *[Kosower '99, Bern, Del Duca et al. '99]*.

For $q\bar{q} \to V + ggg$ the integrated contribution reads




$$\sum_{i=1}^{3} \left\langle (I - S_4) C_{4i}\, \Delta^{(4)}\, F_{\mathrm{LV}}(4) \right\rangle = [\alpha_s]^2 \left\langle I_C(\epsilon)\, 2\Re\big(\bar{\mathcal{I}}_1(\epsilon)\big) F_{\mathrm{LM}} \right\rangle + \frac{\alpha_s}{2\pi}[\alpha_s] \left\langle I_C(\epsilon) F_{\mathrm{LV}}^{\mathrm{fin}} \right\rangle$$

$$- [\alpha_s]\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon} \left\langle I_C(\epsilon) F_{\mathrm{LM}} + \sum_{k=1}^{2} P_{qq}^{\mathrm{gen}}(z) \otimes F_{\mathrm{LM}}^{(k)}(z) \right\rangle$$

$$+ [\alpha_s]^2 \left\langle \Gamma_g^{\mathrm{1loop}}\, F_{\mathrm{LM}} \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \sum_{i=k}^{2} \left\langle P_{qq}^{\mathrm{1loop}} \otimes F_{\mathrm{LM}}^{(k)}(z) \right\rangle$$

$$+ [\alpha_s]^2 \sum_{k=1}^{2} \left\langle P_{qq}^{\mathrm{gen}}(z) \otimes 2\mathrm{Re}\big(\bar{\mathcal{I}}_1(z,\epsilon)\big) F_{\mathrm{LM}}^{(k)}(z) \right\rangle + [\alpha_s]\frac{\alpha_s}{2\pi} \sum_{k=1}^{2} \left\langle P_{qq}^{\mathrm{gen}}(z) \otimes F_{\mathrm{LV}}^{\mathrm{fin},\,(k)}(z) \right\rangle$$

Same structure as NLO

One-loop splitting functions,
known analytically

**Single soft**: different subtraction terms combined → careful with the limits order

$$\sum_{i=1}^{3} \Big\langle (I - S_4) C_{4i} \left[ \left\langle S_5\, \Delta^{(45)}\, F_{\mathrm{LM}}^{4>5}(4,5) \right\rangle \right] + S_5\, (I - S_4)\, C_{4i}\, \Delta^{(45)}\, F_{\mathrm{LM}}^{5>4}(4,5) \Big\rangle =$$

$$+ [\alpha_s]^2 \sum_{k=1}^{2} \left\langle I_{1R}(\epsilon) P_{qq}^{\mathrm{gen}}(z) \otimes F_{\mathrm{LM}}^{(k)}(z) \right\rangle + [\alpha_s]^2 \left\langle I_{1R}(\epsilon)\, I_C(\epsilon)\, F_{\mathrm{LM}} \right\rangle$$

$$+ \frac{[\alpha_s]^2}{\epsilon^2} N_s\, C_A \left[ \sum_{k=1}^{2} \left\langle \left(\frac{2E_k}{\mu}\right)^{-2\epsilon} \tilde{P}_{qq}^{\mathrm{gen},}(z) \otimes F_{\mathrm{LM}}^{(k)}(z) \right\rangle + \sum_{k=1}^{3} \left\langle \left(\frac{2E_k}{\mu}\right)^{-2\epsilon} \hat{\Gamma}^{(k)\,\mathrm{e.o.}}\, F_{\mathrm{LM}} \right\rangle \right]$$

# Status so far

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f \, .$$

| | | | | | |
|---|---|---|---|---|---|
| $\langle F_{\mathrm{LVV}} \rangle$ | $\frac{1}{2}\left[2\Re(\mathcal{I}_1(\epsilon))\right]^2$ | $\frac{\beta_0}{\epsilon}\, 2\Re(\mathcal{I}_1(\epsilon))$ | $K\, 2\Re(\mathcal{I}_1(2\epsilon))$ | $\frac{\beta_0}{\epsilon}\, 2\Re(\mathcal{I}_1(2\epsilon))$ | |
| $\langle S_{45} F_{\mathrm{LM}}^{4>5}(4,5) \rangle$ | $\frac{1}{2} I_{1,R}^2(\epsilon)$ | | $\frac{C_A}{\epsilon^2} \widetilde{I}_{1,R}(2\epsilon)$ | $\frac{\beta_0}{\epsilon} \widetilde{I}_{1,R}(2\epsilon)$ | $\beta_0 \widetilde{I}_{1,R}(2\epsilon)$ |
| $\langle S_4\, F_{\mathrm{LRV}}(4) \rangle$ | $I_{1,R}(\epsilon)\, 2\Re(\mathcal{I}_1(\epsilon))$ | $\frac{\beta_0}{\epsilon} I_{1,R}(\epsilon)$ | $C_A\, A_K\, \widetilde{I}_{1,R}(2\epsilon)$ | | |
| $\langle (I - S_4) C_{4i}\, \Delta^{(4)}\, F_{\mathrm{LV}}(4) \rangle$ | $I_C(\epsilon)\, 2\Re(\bar{\mathcal{I}}_1(\epsilon))$ | $\frac{\beta_0}{\epsilon} I_C(\epsilon)$ | | | |
| $\Big\langle (I - S_4) C_{4i} \left[ \langle S_5\, \Delta^{(45)}\, F_{\mathrm{LM}}^{4>5}(4,5) \rangle \right] + S_5\, (I - S_4)\, C_{4i}\, \Delta^{(45)}\, F_{\mathrm{LM}}^{5>4}(4,5) \Big\rangle$ | $I_{1R}(\epsilon)\, I_C(\epsilon)$ | | | | |

A term $I_C^2(\epsilon)$ needed to reconstruct $(I_1 + I_{1,R} + I_C)^2$

→ look at double-collinear

reconstruct $I_1(\epsilon) + I_{1,R}(\epsilon) + I_C(\epsilon)$ but with extra $1/\epsilon$

Clear interplay → $C_A, 2\epsilon$

non-transparent cancellation

Suggest $I_1(2\epsilon) + I_{1,R}(2\epsilon) + I_C(2\epsilon)$ but with extra $1/\epsilon$

# Hard-collinear real-virtual and single soft RR

Manipulations required to reconstruct recurring structures and match, for instance, PDFs-like corrections

$$\frac{1}{2}\langle \sum_{i,j}(I-S_4)\,(I-S_5)\,C_{4i}C_{5j}\,\Delta^{(45)}F_{\rm LM}(4,5)\rangle = \Big\langle \frac{1}{2}[\alpha_s]^2\Big(I_C(\epsilon)\Big)^2 F_{\rm LM} + \sum_{k=1}^{2} G^{(k)}(z)\,F_{\rm LM}^{(k)}(z) + G^{(3)}\,F_{\rm LM}$$

$$+\frac{1}{2}\,[\alpha_s]^2\sum_{k=1}^{2}\big[P_{qq}^{\rm gen}\otimes P_{qq}^{\rm gen}(z)\big]_{\rm pdf}\,F_{\rm LM}^{(k)}(z) + [\alpha_s]^2\sum_{k=1}^{2}P_{qq}^{\rm gen}\otimes I_C(z,\epsilon)F_{\rm LM}^{(k)}(z)$$

$$+\,[\alpha_s]^2\,P_{qq}^{\rm gen}(z_1)\otimes F_{\rm LM}(z_1,z_2)\otimes P_{qq}^{\rm gen}(z_2)\Big\rangle$$

## Cancellation of the double-color-correlated contributions

$$\frac{1}{2}\Big\langle\Big(\frac{\alpha_s}{2\pi}\,2\Re\big(\mathcal{I}_1(\epsilon)\big)+[\alpha_s]I_{1,R}(\epsilon)+[\alpha_s]I_C(\epsilon)\Big)^2\,F_{\rm LM}\Big\rangle = \frac{1}{2}[\alpha_s]^2\Big\langle I_{1,T}^2(\epsilon)\,F_{\rm LM}\Big\rangle$$

$\longrightarrow$ finite

Same combination encountered at NLO:
finite, and easy to be computed.