The black hole behind the cut

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30/05/2023

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Introduction

Introduction

Black hole physics poses theoretical problems: singularity, loss of unitarity...

It is believed that these problems should be resolved by quantum corrections. This is of course hard to check.

Holography comes to help:

$$\text{gravity in } AdS_{d+1} = \text{CFT on the } d\text{-dimensional boundary}$$

$$G_N^{-1} \sim c \Rightarrow G_N \to 0 \sim c \to \infty \,,$$

where c counts the number of degrees of freedom in the CFT.

Sharp reformulation in terms of CFT correlators.

Introduction

A different philosophy (fuzzball proposal): what if there is a classical solution?

That is, how inconsistencies are solved if black hole are regular geometries (classical microstates), and the horizon emerges as an IR description?

We will present a 3d solution of the EOMs that looks like a black hole up to the horizon scale and ends with a smooth cap.

Focusing on boundary 2d correlators, we will show that black hole behavior indeed arises as an effective description of our smooth geometry.

Plan of the talk

Plane of the talk

- Holography and boundary correlators
- Heavy states and black hole microstates
- An exact solution to wave equations
- An exact solution for boundary correlators

Holography and boundary correlators

Holography and boundary correlators

The boundary (2d) CFT is covariant under

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{m+n}\,.$$

where c is the central charge. On (primary) fields

$$[L_n, \mathcal{O}(z)] = \left(z^{n+1}\partial_z + \Delta(n+1)z^n\right)\mathcal{O}(z)\,.$$

A special role is played by

$$L_0 \sim H\,, \quad \text{and } L_0\mathcal{O}(0)|0\rangle = \Delta\mathcal{O}(0)|0\rangle\,.$$

$\mathcal{O}(0)$ on the vacuum creates a state of energy (scaling dimension) Δ.

Holography and boundary correlators

We are interested in the large c limit of the theory. An operator is *light* if $\Delta_L = \mathcal{O}(c^0)$ and *heavy* if $\Delta_H = \mathcal{O}(c^1)$. Since

$$\text{gravity in } AdS_3 = \text{CFT on the 2d boundary}\,,$$

$$c \sim G_N^{-1}\,,$$

states in large c CFT $\sim$ states in classical gravity (fields configurations).

$$|0\rangle \sim \text{empty } AdS_3$$

$$\mathcal{O}_L|0\rangle \sim \text{(scalar) field in } AdS_3$$

$$\mathcal{O}_H|0\rangle \sim \,?$$

Since $\Delta_H \sim G_N^{-1}$, we can't neglect backreaction on the AdS_3 metric: heavy states generates nontrivial backgrounds $|H\rangle$.

Holography and boundary correlators

Playing with heavy state one can construct bulk geometries that differ from black holes at the horizon state: a BH microstate.

A convenient way to understand its properties is to probe it with perturbations.

response to perturbations $\sim$ 2 pt functions

So we are led to study the HHLL correlator

$$\langle H|\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})|H\rangle = \langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle$$

Holography and boundary correlators

Some generalities on CFT correlators:

The dependence of the position of insertions is fixed by conformal invariance in 2 and 3 pt functions, e.g.

$$\langle \mathcal{O}_1(1)\mathcal{O}_2(z,\bar{z})\rangle = \frac{1}{|1-z|^{2\Delta_2}}\delta_{\Delta_1\Delta_2}\,.$$

4 point functions are more complicated. A useful tool to study them is the OPE:

$$\mathcal{O}_1(z)\mathcal{O}_2(0) = \sum_k C_{12k}\sum_n c_n z^{\Delta_k-\Delta_1-\Delta_2+n}L_{-n}\mathcal{O}_k(0)\,.$$

Holography and boundary correlators

At fixed k

$$\mathcal{O}_1(z)\mathcal{O}_2(0) \supset \sum_n c_n z^{\Delta_k - \Delta_1 - \Delta_2 + n} L_{-n}\mathcal{O}_k(0)\,.$$

The operator $\mathcal{O}_k(0)$ (+ descendants) contributes with a series in z: a conformal block $\mathcal{V}_k(z)$. Then

$$\sum_k C_{12k}\mathcal{V}_k(z)\,.$$

$k \in$ spectrum with weights C_{12k}.

$\mathcal{V}_k(z)$ is universal, but the spectrum and the C_{12k}'s are not.

Back to HHLL:

Holography and boundary correlators

We compute HHLL from the 3d bulk. The response to perturbations is captured by the wave equation

$$\Box_H \phi(\tau,\rho,\sigma) = \Delta_L(\Delta_L - 2)\phi(\tau,\rho,\sigma)\,,$$

where $\Delta_L(\Delta_L - 2)$ is the squared mass of the perturbation.

We will discuss separable backgrounds, such that after Fourier transforming

$$\phi(\tau,\rho,\sigma) = \frac{1}{(2\pi)^2}\sum_{\ell\in\mathbb{Z}}\int d\omega\, e^{i\omega\tau + i\ell\sigma}\psi(\rho)$$

the wave equation reduces to an ODE for $\psi(\rho)$.

Holography and boundary correlators

If our geometry is regular we want our perturbation to be regular as well, so we impose

$$\psi(\rho) = \rho^{|\ell|}(1 + \mathcal{O}(\rho)) \to 0\,, \quad \text{as}\, \rho \to 0\,.$$

Close to the AdS_3 boundary ($\rho \to \infty$) we have

$$\psi(\rho) = \mathcal{A}(\omega, \ell)\rho^{\Delta_L - 2}(1 + \mathcal{O}(\rho^{-2})) + \mathcal{B}(\omega, \ell)\rho^{-\Delta_L}(1 + \mathcal{O}(\rho^{-2}))\,.$$

The solution is a superposition of a non normalizable mode (source) and a normalizable one (response). Normalizing by $\mathcal{A}$:

$$\begin{aligned} &1 \times \rho^{\Delta_L - 2} \sim \delta(\sigma)\delta(\tau)\rho^{\Delta_L - 2} \quad \text{perturbation} \\ &\mathcal{B}/\mathcal{A}\, \rho^{-\Delta_L} \sim \text{response} \end{aligned}$$

Holography and boundary correlators

Then the holographic information is all encoded in $\mathcal{A}$ and $\mathcal{B}$:

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle = \mathcal{N}\sum_{\ell}\int d\omega\, e^{i\omega\tau + i\ell\sigma}G(\omega,\ell)$$

where

$$z = e^{i(\tau+\sigma)} = e^{iv}\,, \quad \bar{z} = e^{i(\tau-\sigma)} = e^{iu}\,, \quad G(\omega,\ell) = \frac{\mathcal{B}(\omega,\ell)}{\mathcal{A}(\omega,\ell)}$$

where u, v are the lightcone coordinates.

In other words, $\mathcal{B}/\mathcal{A}$ is the correlator in momentum space.

Holography and boundary correlators

The easiest example: $\mathcal{O}_H = 1$, that is, empty AdS_3.

The wave equation is just the hypergeometric equation. The regular solution at $\rho = 0$ is $(x = \rho^2/(1+\rho^2))$

$$\psi(\rho) = (1-x)^{\frac{\Delta}{2}} x^{\frac{1+|\ell|}{2}} {}_2F_1\left(\frac{1}{2}(|\ell| - \omega + \Delta), \frac{1}{2}(|\ell| + \omega + \Delta), 1 + |\ell|, x\right)$$

${}_2F_1(\ldots, x)$ is given as a convergent series centered around $x = 0$ with radius of convergence 1.

In order to compute the correlator a nontrivial analytic continuation is needed.

Holography and boundary correlators

In this easy case, the analytic continuation around $x \sim 1$ is well known:

$$\psi(\rho) = \frac{\Gamma(2-\Delta)}{\Gamma\left(\frac{|\ell|-\omega+\Delta}{2}\right)\Gamma\left(\frac{|\ell|+\omega+\Delta}{2}\right)}(1-x)^{\frac{\Delta}{2}}(1+\dots)+ \\ + \frac{\Gamma(\Delta-2)}{\Gamma\left(\frac{|\ell|-\omega+2-\Delta}{2}\right)\Gamma\left(\frac{|\ell|+\omega+2-\Delta}{2}\right)}(1-x)^{\frac{1-\Delta}{2}}(1+\dots),$$

and

$$G(\omega,\ell) = \frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\frac{\Gamma\left(\frac{|\ell|+\omega+\Delta}{2}\right)\Gamma\left(\frac{|\ell|-\omega+\Delta}{2}\right)}{\Gamma\left(\frac{|\ell|+\omega+2-\Delta}{2}\right)\Gamma\left(\frac{|\ell|-\omega+2-\Delta}{2}\right)}.$$

Holography and boundary correlators

$$\langle \mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\rangle \propto \sum_{\ell}\int d\omega\, e^{i\omega\tau+i\ell\sigma}\frac{\Gamma\left(2-\Delta\right)}{\Gamma\left(\Delta-2\right)}\prod_{\pm}\frac{\Gamma\left(\frac{|\ell|\pm\omega+\Delta}{2}\right)}{\Gamma\left(\frac{|\ell|\pm\omega+2-\Delta}{2}\right)}\,.$$

The Γ's have poles on the real ω axis: we need to choose a $i\epsilon$ prescription to integrate.

Different choices define different correlators (retarded, Feynman...). To make things explicit it's useful to consider a dispersive representation. From the Cauchy theorem (subtractions are needed: $G(\omega,\ell)\sim\omega^{2\Delta-2}$)

$$\frac{G(\omega,\ell)}{\omega^{\Delta}} = \frac{1}{2\pi i}\oint_{\gamma_\omega}\frac{1}{\omega^{\Delta}}\frac{G(\omega',\ell)}{\omega'-\omega}$$

Holography and boundary correlators

Blowing up the contour at infinity we get

$$G(\omega,\ell) = \sum_n \left(\frac{\omega}{\omega_n}\right)^\Delta \frac{\mathrm{Res}(G,\omega_n)}{\omega-\omega_n}\,.$$

Now we can define

$$G_F(\omega,\ell) = \sum_{\omega_n>0} \left(\frac{\omega}{\omega_n}\right)^\Delta \frac{\mathrm{Res}(G,\omega_n)}{\omega-\omega_n+i\epsilon} + \sum_{\omega_n<0} \left(\frac{\omega}{\omega_n}\right)^\Delta \frac{\mathrm{Res}(G,\omega_n)}{\omega-\omega_n-i\epsilon}\,.$$

$$G_R(\omega,\ell) = \sum_n \left(\frac{\omega}{\omega_n}\right)^\Delta \frac{\mathrm{Res}(G,\omega_n)}{\omega-\omega_n+i\epsilon}\,.$$

Holography and boundary correlators

Here $\omega_n = \pm(|\ell| + 2n + \Delta_L)$: these are the scaling dimensions of the descendants of $\mathcal{O}_L$, that is the dimensions of the operators appearing in

$$\mathcal{O}_L(z,\bar{z})1(0) = \sum c_n z^{n+\ell} \bar{z}^n (\partial\bar{\partial})^n \partial^\ell \mathcal{O}(0) + \text{c.c.}\,.$$

Accordingly,

$$\sum_\ell \int d\omega\, e^{i\omega\tau + i\ell\sigma} G_F(\omega,\ell) = \sum_\ell \sum_n e^{i\omega_n\tau + i\ell\sigma} \text{Res}(G,\omega_n) = \frac{1}{|1-z|^{2\Delta_L}}\,.$$

All in all

$$\omega_n \sim \text{operators in the } \mathcal{O} \times 1 \text{ OPE}\,.$$

What is the bulk interpretation of these poles?

Holography and boundary correlators

$G(\omega, \ell)$ has a pole every time $\mathcal{A}$ in

$$\psi(\rho) = \mathcal{A}(\omega, \ell)\rho^{\Delta_L - 2}(1 + \mathcal{O}(\rho^{-2})) + \mathcal{B}(\omega, \ell)\rho^{-\Delta_L}(1 + \mathcal{O}(\rho^{-2}))$$

vanishes.

When this happens $\psi(\rho)$ becomes a normalizable wavefunction: these poles corresponds to bound states in the gravitational background.

$$\omega_n \sim \text{excitations around the background}\,.$$

In the geodesics approximation $\Delta_L \gg 1$, the ω_n's are the energies of orbits around the heavy object.

Holography and boundary correlators

Before moving on note that

$$\operatorname{Im} G_R(\omega, \ell) = \sum_n \pi \operatorname{Res}(G, \omega_n) \delta(\omega - \omega_n)\,.$$

This captures the density of *on-shell* states exchanged in the OPE.

This is on the same ground of what happens in ordinary QFT:

$$G \sim \frac{1}{p^2 - m^2 + i\epsilon} \Rightarrow \operatorname{Im} G \sim \delta(p^2 - m^2)\,.$$

Holography and boundary correlators

Let us now consider a different example: a BTZ black hole.

A black hole is a finite-temperature object so the relevant boundary state has to be a thermal ensemble.

The holographic presciption to compute the thermal two point function is different as well. Instead of imposing regularity at $\rho = 0$,

$$\psi(\rho) = \psi_{in}(\rho)\,, \quad \text{as } \rho \sim \rho_+\,,$$

and now

$$G_R(\omega, \ell) = \frac{\mathcal{B}(\omega, \ell)}{\mathcal{A}(\omega, \ell)}\,.$$

Holography and boundary correlators

Poles of G_R are now the QNMs of the black holes!

They appear at complex ω_n's. For example for extremal BTZ (still from hypergeometrics)

$$G_R = -(i(\ell-\omega))^{\Delta-1}\frac{\Gamma(1-\Delta)}{\Gamma(\Delta-1)}\frac{\Gamma\left(\frac{1}{2}\left(\Delta - i\frac{\ell+\omega}{2}\right)\right)}{\Gamma\left(1-\frac{1}{2}\left(\Delta + i\frac{\ell+\omega}{2}\right)\right)}\,.$$

This has poles at

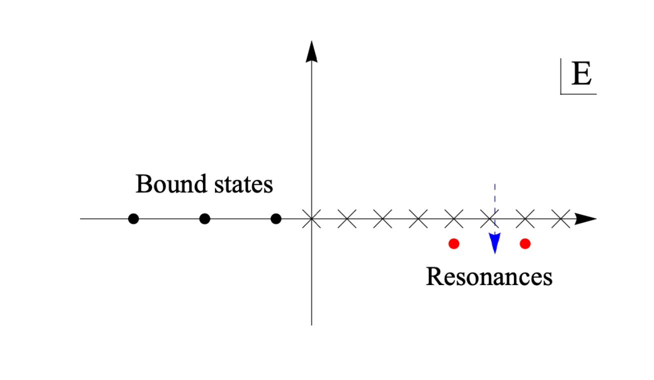
$$\omega_n = -\ell - 2i(2n+\Delta)\,.$$

Moreover $\mathrm{Im}\,G_R$ is smooth on the real axis!

How can we interpret complex ω_n's and smooth $\mathrm{Im}\,G_R$ in the boundary CFT?

Holography and boundary correlators

In QFT continuum of on shell states generate branch cuts on the real axis, and resonances manifest as complex poles of the cut discontinuity[1].



[1] Figure stolen from M. Serone notes on QFT.

Holography and boundary correlators

It is tempting to say that the BTZ $\mathrm{Im}\, G_R$ captures the discontinuity along this cut. However..

- The boundary of AdS_3 is a sphere, and a CFT on a compact set should have discrete spectrum. How did we end up with a cut?

- Real poles: $\langle \mathrm{HHLL} \rangle \sim \sum_n e^{i\omega_n t}$ but complex poles

$$\langle \mathrm{HHLL} \rangle \sim \int d\omega\, e^{i\omega t} \sim \sum_n e^{-\omega_n t}$$

Holography and boundary correlators

Conjecturally this is an effect of the large c (small G_N) limit.

As anticipated we investigate a different viewpoint: black holes are regular geometries that look like black holes at long distances.

Still, one of these geometries is dual to pure $|H\rangle$, and $HHLL$ should only have a discrete set of real poles.

How can this approximate a continuum spectrum with complex resonances?

Heavy states and black hole microstates

Heavy states and black hole microstates

To understand the various steps of the constructions one should dive into the details of the (super-)gravity and its dual (D1D5) CFT. We won't do this, and instead we will just discuss the general ideas.

The simplest way to construct a heavy state is to bind together many light states $\mathcal{O}_H \sim \mathcal{O}_L^{\frac{c}{k}}$ for some $k \in \mathbb{N}$.

The resulting operator creates a coherent state of light excitations.

This is clearly a very atypical microstate. As we will see it still captures some generic features.

Heavy states and black hole microstates

A concrete candidate is [Giusto-Moscato-Russo-...]

$$\mathcal{O}_H = \sum_{p=0}^{c} (1-\eta^2)^{\frac{p}{2}} \eta^{c-p} \left(L_{-1}^n \mathcal{O}_L\right)^p (1)^{c-p} \,.$$

For large c the sum is peaked at $\bar{p} = c(1-\eta^2)$, and

$$\Delta_H \sim c(1-\eta^2) \,.$$

Note that when $\eta = 1$ we get $\mathcal{O}_H = 1$, therefore its dual geometry will be AdS_3. On the other hand as $\eta \to 0$

$$\mathcal{O}_H \to (L_{-1}^n \mathcal{O}_L)^c \,.$$

Heavy states and black hole microstates

One can prove that for $\eta \neq 0$

$$\langle H|\mathcal{O}_L'|H\rangle \neq 0\,,$$

However

$$\lim_{\eta\to 0}\langle H|\mathcal{O}_L'|H\rangle = 0\,,$$

This is what should happen for the thermal ensamble since 1 pt functions are averaged out to 0.

$$\lim_{\eta\to 0}|H\rangle \sim \text{thermal ensemble}\,.$$

Moreover $|H\rangle$ has the same charges of an extremal black hole.

Heavy states and black hole microstates

In the bulk we need a metric that goes to empty AdS_3 as $\eta \to 1$ and to a black hole as $\eta \to 0$.

To find the exact metric is of course nontrivial, but it has been done [Bena-Giusto-Russo-Shigemori-Warner]:

$$ds_3^2 = G\,\frac{d\rho^2}{\rho^2+1} - \eta^2(\rho^2+\eta^2)\,d\tau^2 + \eta^2\rho^2\,d\sigma^2 + \eta^2\rho^2 F\,(d\tau + d\sigma)^2$$

$$G = 1 - \frac{1-\eta^2}{\rho^2+1}\left(\frac{\rho^2}{\rho^2+1}\right)^n \quad , \quad F = \frac{1-\eta^2}{\eta^2}\left[1 - \left(\frac{\rho^2}{\rho^2+1}\right)^n\right],$$

as $\eta \to 1$, $G \to 1$ and $F \to 0$, and the resulting metric is just AdS_3.

Heavy states and black hole microstates

As $\eta \to 0$ this goes to

$$ds_3^2 = \frac{r^2}{(r^2 - r_0^2)^2} dr^2 - \frac{(r^2 - r_0^2)^2}{r^2} d\tau^2 + r^2 \left(d\sigma + \frac{r_0^2}{r^2} d\tau \right)^2$$

for

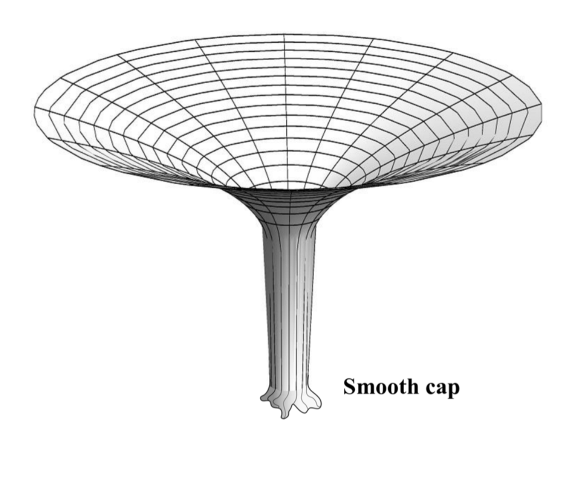
$$r^2 = \eta^2 \rho^2 + n \,, \quad r_0 = n \,.$$

The $n = 0$ case is particularly easy, since the corresponding black hole will have zero area (naked singularity).

For any $n \neq 0$, this is the metric of an extremal BTZ black hole.

Heavy states and black hole microstates

The full geometry looks[2] like BTZ, but has a smooth cap instead of the horizon at $\rho \sim \eta^2$.



[2]Figure stolen from P. Heidmann's PhD thesis.

Heavy states and black hole microstates

Let's start with the easy case: $n = 0$. Again the wave equation reduces to the hypergeometric ODE and we can write down an exact solution [Bombini-Galliani-Giusto-Moscato-Russo]:

$$G(\omega, \ell) = \frac{\Gamma\left(1 - \Delta_L\right)}{\Gamma\left(\Delta_L - 1\right)} \prod_{\pm} \frac{\Gamma\left(\frac{\Delta_L + |\ell|}{2} \pm \frac{\sqrt{\omega^2 - \ell^2(1-\eta^2)}}{2\eta}\right)}{\Gamma\left(\frac{2 - \Delta_L + |\ell|}{2} \pm \frac{\sqrt{\omega^2 - \ell^2(1-\eta^2)}}{2\eta}\right)} .$$

As in AdS_3 G has poles on the real axis at

$$\omega_n = \pm\sqrt{\eta^2 \left(|\ell| + 2n + \Delta_L\right)^2 + \ell^2 \left(1 - \eta^2\right)} .$$

For $\eta = 1$ they reduce to the same poles as in the AdS_3 case.

Heavy states and black hole microstates

These poles are to the energies of states appearing in the OPE

$$\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0) = \sum_{n,\ell} C_{LH[HL]} \sum_m c_m z^{\Delta_{[HL]}-\Delta_L-\Delta_H+m} \mathcal{O}_L(\partial\bar{\partial})^n \partial^\ell \mathcal{O}_H(0) + \text{c.c.}$$

$\mathcal{O}_L(\partial\bar{\partial})^n\partial^\ell\mathcal{O}_H(0)$ are the so called *HL double twist*, and correspond to bound states of the perturbation and the background.

For $\eta \sim 1$ they are evenly spaced, but when $\eta \to 0$

$$\omega_{n+1} - \omega_n = \frac{2\eta^2(|\ell| + 2n + \Delta)}{\omega_n} = \mathcal{O}\left(\frac{\eta^2}{\omega_n}\right) .$$

As the geometry gets closer to a black hole, poles on the real axis get more dense.

Heavy states and black hole microstates

Let's go back to the dispersive representation of G:

$$G(\omega,\ell)=\sum_n\left(\frac{\omega}{\omega_n}\right)^{\Delta}\frac{\mathrm{Res}(G,\omega_n)}{\omega-\omega_n}\,.$$

When the ω_n become dense we can approximate the sum by an integral

$$G(\omega,\ell)\simeq\int dn\left(\frac{\omega}{\omega_n}\right)^{\Delta}\frac{\mathrm{Res}(G,\omega_n)}{\omega-\omega_n}=\int d\omega_n\frac{\rho(\omega_n)}{\omega-\omega_n}\,.$$

Crucially

$$G(\omega+i\epsilon,\ell)-G(\omega-i\epsilon,\ell)=2i\mathrm{Im}G_R=\pi\rho(\omega)\,.$$

Heavy states and black hole microstates

A branch cut with discontinuity $2i\mathrm{Im}G_R$ formed on the real axis!

For any $\eta \neq 0$,

$$\langle HHLL \rangle \sim \sum_n e^{i\omega_n \tau}\,.$$

However if our resolution doesn't allow us to separate the energy levels..

$$\langle HHLL \rangle \sim \int dn\, e^{i\omega_n \tau}\,.$$

In this easy case the black hole approached as $\eta \to 0$ doesn't have QNMS, and as a result

$$\langle HHLL \rangle \sim \tau^{-\#}\,.$$

Heavy states and black hole microstates

Black hole like behavior emerges as an effective description of our energy spectrum.

Everything happens at $c = \infty$ ($G_N \simeq 0$): the parameter that controls the spectrum is now η.

Let us move to a more complicated example: $n = 1$.

The black hole approached in this case is a extremal BTZ black hole with a nonvanishing horizon.

Heavy states and black hole microstates

Let's take a look at the corresponding wave equation:

$$\psi''(\rho) + \frac{1+3\rho^2}{\rho(1+\rho^2)}\psi'(\rho) + \frac{\rho^2(\ell-\omega)\left[(\ell-\omega)\frac{1+\eta^2\rho^2}{1+\rho^2} - 2\eta^2\ell\right] - \eta^4\ell^2}{\eta^4\rho^2(1+\rho^2)^2}\psi(\rho)$$
$$- \frac{\Delta_L(\Delta_L-2)}{\rho^2+1}\psi(\rho) = 0\,.$$

This ODE is more complicated than the hypergeometric. The prescription to compute G was

$$\begin{aligned}\psi(\rho) &\simeq \rho^{|\ell|}\,, \quad \text{as } \rho \to 0\,, \\ &\simeq \mathcal{A}\rho^{\Delta_L-2} + \mathcal{B}\rho^{-\Delta_L}\,, \quad \text{as } \rho \to \infty\end{aligned}$$

Heavy states and black hole microstates

To read off $\mathcal{A}$, $\mathcal{B}$ a nontrivial analytic continuation is needed.

We knew how to do it for the hypergeometric equation, but for more complicated ODEs the problem is way more complicated.

This made quite difficult a quantitative study of perturbations in these backgrounds, since the only available method was WKB [Bena-Heidmann-Monten-Warner].

We now present a recently discovered method to deal with these ODEs, that gives us better analytic control and allow us to understand the analytic structure of $G(\omega, \ell)$.

An exact solution to wave equations

An exact solution to wave equations

Let's take a closer look to the hypergeometric ODE. It can be rewritten in Schrodinger form

$$(\partial_x^2 + V(x))\psi(x) = 0\,, \quad x = \frac{\rho^2}{1+\rho^2}\,.$$

The potential diverges quadratically at (WLOG) $x = 0, 1, \infty$

$$V(x) \simeq \frac{\#}{(x-x_i)^2}\,.$$

The solution is given by

$$\psi(x) = \sum_n c_n x^n \sim {}_2F_1(\ldots, x)\,,$$

a series centered at $x = 0$.

An exact solution to wave equations

Can we expand ${}_2F_1(..,x)$ close to $x \sim 1$? Since the equation is 2nd order, close to 1

$$\psi(x) = \mathcal{A}\, f_1(1-x) + \mathcal{B}\, f_2(1-x)\,.$$

It all boils down to compute $\mathcal{A}$, $\mathcal{B}$. These are the connection coefficients of the ODE.

${}_2F_1$ admits an integral representation

$${}_2F_1(a,b,c,x) \sim \int_0^1 y^{b-1}(1-y)^{c-b-1}(1-xy)^{-a}\, dy$$

that we can continue from $x \sim 0$ to $x \sim 1, \infty$.

An exact solution to wave equations

Going back to our case, the wave equation of the $n = 1$ geometry still looks like

$$(\partial_x^2 + V(x))\psi(x) = 0\,, \quad x = \frac{\rho^2}{1+\rho^2}$$

with

$$V(x) \simeq \frac{\#}{(x - x_i)^2}\,, \quad x_i = 0, 1$$

However close to infinity the divergence takes the form

$$V(x) \simeq \frac{\#}{(x - x_i)^3}\,.$$

An exact solution to wave equations

This is an apparently minor change, but it has dramatic consequences.

This equation is better understood as an ODE with 4 $(x - x_i)^{-2}$ singularities, where 2 singularities collided to produce a higher order one: it is the so called (reduced confluent) Heun equation.

In this case no integral representation is known. Analytic continuation is much harder.

We need a more sophisticated method.

An exact solution to wave equations

General idea: we can compute CFT correlators doing OPE

$$\langle \ldots \mathcal{O}_i(z_i)\mathcal{O}_j(z_j)\ldots\rangle = \sum_k \sum_n c_n (z_i - z_j)^{\alpha_k + n}\,.$$

We can do different OPEs in a given correlators. All the resulting expression have to agree

$$\langle \ldots \mathcal{O}_k(z_\ell)\mathcal{O}_i(z_i)\mathcal{O}_j(z_j)\ldots\rangle = \sum_k \sum_n c_n (z_i - z_j)^{\alpha_k + n} = \sum_k \sum_n c_n (z_i - z_\ell)^{\beta_k + n}$$

This property is called crossing symmetry.

An exact solution to wave equations

Crossing symmetry relates a series centered in $z_i = z_j$ to a series at $z_i = z_\ell$. It is a statement on analytic continuation of a series.

Can we find in CFT a correlator that satisfies the ODE we want to solve?

Can we use crossing symmetry to analytically continue the solution of the ODE say from $z \sim 0$ to $z \sim 1$?

Yes, and yes! [Bonelli-CI-Panea-Tanzini].

Before going on: this CFT has nothing to do with the holographic one!

An exact solution to wave equations

Let's consider the CFT state:

$$\mathcal{O}_{2,1}(0)|0\rangle\,, \quad \Delta_{2,1} = -\frac{1}{2} - \frac{3}{4}b^2\,,$$

where $c = 1 + 6(b + b^{-1})^2$. Then the combination

$$|(b^{-2}L_{-1}^2 + L_{-2})\mathcal{O}_{2,1}(0)|0\rangle|^2 = 0$$

has zero norm. To preserve unitarity we need correlators involving this zero norm state to vanish:

$$\langle\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\dots(b^{-2}L_{-1}^2 + L_{-2})\mathcal{O}_{2,1}(z)\rangle = 0\,.$$

An exact solution to wave equations

Since

$$[L_n, \mathcal{O}(z)] = \left(z^{n+1}\partial_z + \Delta(n+1)z^n\right)\mathcal{O}(z)\,,$$

this turns into a 2nd order differential equation:

$$(\partial_z^2 + V(z, z_i))\langle\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\ldots\mathcal{O}_{2,1}(z)\rangle = 0\,.$$

Each insertion produces a singularity. Playing with the insertions we can produce any singularity we want.

Primary fields produce $(z - z_i)^{-2}$ singularities. We can collide them to excite higher order singularities as the one we encountered.

An exact solution to wave equations

OPE of $\mathcal{O}_{2,1}(z)$ with other fields computes local behavior of solutions of the ODE as $z \sim z_i$.

$$\mathcal{O}_{2,1}(z)\mathcal{O}_i(z_i) = \sum_{\pm} C_{\pm} \sum_n c_n (z - z_i)^{\Delta_{\pm} - \Delta_{2,1} - \Delta_i + n} L_{-n}\mathcal{O}_{\pm}(z_i)\,.$$

The $\pm$ accounts for the two linearly independent solutions of the ODE.

We've said that the constants $C_{\pm}$ are theory dependent. Let's specialize to a theory where they are known exactly: Liouville CFT.

Now doing different OPEs we can construct local solutions close to various singularities.

An exact solution to wave equations

Crossing symmetry tells us

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\ldots\mathcal{O}_{2,1}(z)\rangle = \sum_{\pm n} c_n^{\pm(i)}(z-z_i)^{\alpha_\pm^{(i)}+n} = \sum_n c_n^{(j)}(z-z_j)^{\alpha_\pm^{(j)}+n}$$

This is a statement on the analytic continuation of the solution of the ODE.

Having full control on the normalization of the series we can track down the analytic continuation from one singularity to the other.

An exact solution to wave equations

This allows us to compute connection coefficients of the ODE, the $\mathcal{A}$ and $\mathcal{B}$ we needed.

With this method we can compute them as convergent series expansions in the various z_i.

They admit convenient combinatorial expressions that allow us to compute them very efficiently to high orders with a laptop.

This method is quite efficient, in fact it has already been applied to various backgrounds [Grassi-Aminov-Hatsuda-CI-Bonelli-Panea-Tanzini-Zhiboedov-Dodelson-Bianchi-Morales-Fucito-...]

An exact solution for boundary correlators

An exact solution for boundary correlators

We can now write down an exact expression for $\mathcal{A}$ and $\mathcal{B}$. This will look very different depending whether $\eta \sim 1$ or $\eta \sim 0$, since in the two regimes the ODE has a different structure.

Let's start with $\eta \sim 1$:

$$G(\omega, \ell) = \frac{\Gamma\left(-2a_1\right)\Gamma\left(\frac{1}{2}+a_0+a_1+a\right)\Gamma\left(\frac{1}{2}+a_0+a_1-a\right)}{\Gamma\left(2a_1\right)\Gamma\left(\frac{1}{2}+a_0-a_1+a\right)\Gamma\left(\frac{1}{2}+a_0-a_1-a\right)} e^{-\partial_{a_1} F},$$

$$a_0 = \frac{|\ell|}{2}, \quad a_1 = \frac{\Delta - 1}{2}, \quad L = \frac{i(\ell-\omega)\sqrt{1-\eta^2}}{\eta^2},$$
$$u = \frac{\ell^2(1-\eta^2)+\eta^2-\omega^2}{4\eta^2}.$$

An exact solution for boundary correlators

$$F(a_0, a_1, a, L) = \sum_n c_n L^{2n}$$

is a special function (NS partition function) that admits a combinatorial expression. The last bit of information is contained in a:

$$u = \frac{1}{4} - a^2 + \frac{1}{2} L\partial_L F(a_0, a_1, a, L)\,.$$

This is the Matone relation. a will be given as a whole series in $L^2 \sim 1 - \eta^2$.

G has poles when Γ's in the numerator blow up, that is when

$$\frac{1}{2} + a_0 + a_1 \pm a = -n\,.$$

An exact solution for boundary correlators

This happens when

$$\omega_n = \pm(|\ell| + 2n + \Delta) + \gamma_{n\ell}(1-\eta^2) + \dots$$

Poles of these expressions are real and approximatively evenly spaced.

Let's now take a look at the $\eta \sim 0$ expression:

$$G(\omega,\ell) = (iL)^{2a_1} e^{-\partial_{a_1} F_D} \frac{\Gamma(-2a_1)}{\Gamma(2a_1)} \frac{\frac{(4L)^{\frac{g}{2}} e^{L+\partial_g F_D}}{\Gamma\left(\frac{1-g}{2}-a_1\right)} + \frac{(-4L)^{-\frac{g}{2}} e^{-L-\partial_g F_D}}{\Gamma\left(\frac{1+g}{2}-a_1\right)}}{\frac{(4L)^{\frac{g}{2}} e^{L+\partial_g F_D}}{\Gamma\left(\frac{1-g}{2}+a_1\right)} + \frac{(-4L)^{-\frac{g}{2}} e^{-L-\partial_g F_D}}{\Gamma\left(\frac{1+g}{2}+a_1\right)}}.$$

(g, F_D) take the role of (a, F), now expanded in $L^{-1} \sim \eta^2$.

An exact solution for boundary correlators

The structure of G as $\eta \to 0$ looks more complicated. Let's look at its poles:

$$\frac{(4L)^{\frac{g}{2}} e^{L+\partial_g F_D}}{\Gamma\left(\frac{1-g}{2}+a_1\right)}+\frac{(-4L)^{-\frac{g}{2}} e^{-L-\partial_g F_D}}{\Gamma\left(\frac{1+g}{2}+a_1\right)}=0$$

with

$$L \sim i\frac{\ell-\omega}{\eta^2}\,, \quad p \sim -i(\ell+\omega)$$

We find

$$\omega_n \simeq \ell - (n\eta^2)\pi + \mathcal{O}(\eta^4)$$

Again poles become dense in the $\eta \to 0$ limit!

$$\omega_{n+1}-\omega_n \sim \pi\eta^2\,.$$

An exact solution for boundary correlators

Going back to the dispersive representation we find

$$G(\omega, \ell) \simeq \int d\omega_n \frac{\rho(\omega_n)}{\omega - \omega_n} \, .$$

Again this has a cut on the real axis, with discontinuity

$$\rho(\omega) \propto \left(\frac{\ell - \omega}{\eta^2}\right)^{\frac{\Delta-1}{2}} \left(\frac{\Gamma\left(\frac{1}{2}\left(\Delta - i\frac{\ell+\omega}{2}\right)\right)}{\Gamma\left(1 - \frac{1}{2}\left(\Delta + i\frac{\ell+\omega}{2}\right)\right)} - \frac{\Gamma\left(\frac{1}{2}\left(\Delta + i\frac{\ell+\omega}{2}\right)\right)}{\Gamma\left(1 - \frac{1}{2}\left(\Delta - i\frac{\ell+\omega}{2}\right)\right)}\right) .$$

This is precisely the $\mathrm{Im}\, G_R$ of an extremal BTZ black hole we encountered before!

An exact solution for boundary correlators

Now $\rho(\omega) \propto \mathrm{Im}\, G_R$ has poles at the QNMs of the BTZ black hole approached as $\eta \to 0$:

$$\omega_n = -\ell - 2i(2n + \Delta) + \mathcal{O}(\eta^2)\,.$$

This confirms our expectations:
black hole QNMs appear in boundary correlators as resonances behind the cut!

Again, the black hole behavior emerges as an effective description.

An exact solution for boundary correlators

To clarify this point: in position space for any $\eta \neq 0$

$$\langle HHLL \rangle \sim \sum_n e^{i\omega_n t}\,.$$

For small η, destructive interference produces exp decay for small times. Fitting this small time behavior with

$$\sum_n e^{-|\tilde{\omega}_n| t}$$

we will get

$$\tilde{\omega}_n = -\ell - 2i(2n + \Delta) + \mathcal{O}(\eta^2)\,.$$

This is the same result we get approximating the spectrum with a continuum.

An exact solution for boundary correlators

Of course this approximation breaks down at late times: what do QNMs resonances decay into?

It appears that in this framework loss of unitarity in BH physics arises as a result of our inability to resolve the discreteness of the spectrum.

A similar mechanism should be at work when considering quantum corrections to black holes. The energy spacing should scale with

$$\Delta E_n \sim e^{-S} \, .$$

Conclusions and further directions

Conclusions..

- BH-like behavior emerges as a result of approximating the spectrum of the boundary theory with a continuum.
- This is in line with general expectation of what should happen in quantum gravity (finite c). However here everything happened according to the fuzzball proposals.
- We discussed a powerful method to find solutions of Fuchsians ODEs.

..and further directions

- What if we include finite c corrections in our analysis?
- Any connection with chaos?
- A late time calculation more precise than the wkb one?
- Higher dimensional or non extremal microstates?

Thank you for the attention!