Emergent Fractons,
EFT for dipole charges
and spontaneous dipole symmetry breaking

Daniele Musso (Universidad de Oviedo, mussodaniele@uniovi.es)

talk at Genova, 5th April 2023

Mainly based upon:

- ***"EFT for dipole charges and spontaneous breaking of dipole symmetry"***
  E.Afxonidis, A.Caddeo, C.Hoyos, D.Musso – (*to appear*)
- ***"Emergent Dipole Gauge Fields and Fractons***"
  A.Caddeo, C.Hoyos, D.Musso – PhysRevD.106.L111903 – 2206.12877 [cond.mat.str.el]
- ***"Fractons in effective field theories for spontaneously broken translations"***
  R.Argurio, C.Hoyos, D.Musso, D.Naegels Phys.Rev.D 104 (2021) 10, 105001 -- 2107.03073 [hep-th]
- ***"Simplest phonons and pseudo-phonons in field theory"***
  D. Musso Eur.Phys.J.C 79 (2019) 12, 986 -- 1810.01799 [hep-th]

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Universidad de Oviedo

ICTEA

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Highlights

- **Dual formulation of elasticity with *vector gauge fields***
  - We fully exploit the fact that elasticity is a low-energy EFT admitting a dual formulation in terms of emergent gauge fields.
  - These gauge the dipole-symmetry algebra realized in the internal material space
  - The internal space is identified with the external space through spontaneous symmetry breaking to the diagonal internal/external translations.

Caddeo, Hoyos, Musso **PhysRevD.106.L111903**

- ***Mobility constraints* for elastic defects from gauge invariance**
  - Elastic defects encoded in the singularities of the displacement fields are the sources of the emergent gauge fields.

- **EFT for dipole-charged fields and spontaneous breaking of dipole symmetry**

Afxonidis, Caddeo, Hoyos, Musso **(*to appear*)**

Key-concepts for the talk

- **EMERGENT**

  Related to a symmetry of a low-energy effective description, which might be spoiled at higher energies.

- **SPACE-DEPENDENT INTERNAL SYMMETRY**

  Such as **charge dipole symmetry** or **subsystem symmetries**. They are real symmetries (*not redundancies*) which act on the states of the Hilbert space.

Origins

The quest for building robust memories for quantum computation spurred research in lattice model with immobile excitations.

There is no local operator in the theory which can move the excitation without creating other exitations. These are **fractons**.

Vijay, Sagar; Haah, Jeongwan; Fu, Liang **Physical Review B. 92 (23): 235136**

The reason for this behavior can be traced back to the presence of **subsystem symmetries**.

It is challenging to find field theories describing the continuum limit of such lattice models.

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \partial_y \phi)^2$$

$$\delta \phi(t,x,y) = f(x)$$

Seiberg Shao **SciPost Phys. 10 2**

Dipole symmetry

Theories for charged matter that conserve the dipole moment give rise to fractons: isolated charged cannot move.

Pretko Radzihovsky **Phys.Rev.Lett. 120 (2018) 19, 195301**

$$\phi \longrightarrow e^{i\alpha}\phi \qquad\qquad \phi \longrightarrow e^{i\beta x}\phi$$

U(1) MONOPOLE DIPOLE

A dipole transformation represents a shift in momentum, again we have UV/IR mixing. Possible relations with Galilean boosts and Galileons.

$$D_{ij} = \partial_i \phi\, \partial_j \phi - \phi\, \partial_i \partial_j \phi \qquad\qquad \mathcal{L} \supset D^*_{ij} D_{ij} + \dots$$

Simplest dipole-covariant combination implies no Gaussian theory.
The action depends only on the difference among momenta.

Pretko **Phys. Rev. B 98, 115134 (2018)**
Afxonidis, Caddeo, Hoyos, Musso (*to appear*)

Motivations

● ► **Accepting the *"fractonic"* challenges to the standard QFT paradigm**:

Mobility constraints, UV/IR mixing, large vacuum degeneracy, ...

Seiberg Shao **SciPost Phys. 10 2**

● ► **Seeking new exotic phases of matter**, with suppressed relaxation properties (*e.g. subdiffusion*).

Glorioso Guo Rodriguez-Nieva Lucas **Nature Physics 18 8**
Grosvenor Hoyos Peña-Benítez Surówka **Frontiers in Physics 9**

● ► **Systematizing and generalizing the elastic theory**

Elastic defect dynamics in standard solids, in amorphous media *(e.g. discompressions)*, in media with lower symmetry *(e.g. broken rotational invariance)*.

Gaa Palle Fernandes Schmalian **Phys. Rev. B 104**

● ► **Fracton coupled to a background geometry**

Obstruction to define tensor gauge fields in curved space and possible gauge anomalies. Possibility to geometrize multipole symmetries in a suitable internal space, thus formulating the theory in terms of standard vector gauge fields which can circumvent the obstruction.

Jain Jensen **SciPost Phys. 12, 142**
Bidussi Hartong Have Musaeus Prohazka **SciPost Phys. 12, 205**
Peña-Benitez **2107.13884 [cond-mat.str-el]**

● ► **Generalization of "low-energy theorem"**

Goldstone counting for spacetime symmetries and evading Coleman

Motivations

●►Relations with gravity

- Theories with dipole symmetry are typically realized through a gauge theory with a two-index, symmetric tensorial gauge field

Pretko **Phys.Rev.B 96 (2017) 3, 035119**

- Natural relation with linearized general relativity

Blasi, Maggiore **Phys.Lett.B 833 (2022) 137304**
Bertolini, Maggiore **Phys.Rev.D 106 (2022) 12, 125008**

- Constructive realization of Mach's principle

Pretko **Phys. Rev. D 96, 024051 (2017)**

- Analogy with models adopted in cosmology (e.g. Galileons)

●►Relations with higher-spin theories

- Multipole symmetries can be formulated in terms of higher tensorial gauge fields

- Borrowing old results (e.g. Aragone-Deser)

Particle-vortex duality

A 2 + 1 superfluid admits a low-energy effective description in terms of an emergent (dual) dynamical gauge field.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$$

Low-energy EFT in terms of the Goldstone boson

$$\tilde{\mathcal{L}} = \partial_\mu\varphi\, J^\mu - \frac{1}{2}J_\mu J^\mu$$

Low-energy EFT in terms of the Goldstone boson *and* a vector auxiliary field. Integration with respect to the auxiliary fields brings us back to the original Lagrangian.

$$\partial_\mu\varphi = J_\mu$$

$$\partial_\mu J^\mu = 0$$

Equation of motion for the Goldstone

$$\boxed{J^\mu \equiv \epsilon^{\mu\nu\rho}\partial_\nu a_\rho}$$

Ansatz in terms of a dual gauge field

$$J^0 = \epsilon^{ij}\partial_i a_j$$

Density of the dual magnetic flux

$$\tilde{\mathcal{L}} \supset \partial_\mu\varphi\,\epsilon^{\mu\nu\rho}\partial_\nu a_\rho \quad \Rightarrow \quad a_0 \quad \text{couples to} \quad \underline{\rho = \epsilon^{ij}\partial_i\partial_j\varphi}$$

The emergent gauge field couples to vortices
(*i.e.* singular configurations of the Goldstone field)

Fracton-elasticity duality *(a sketch)*

- Particle-vortex duality for superfluids is a particular case of *Hubbard-Stratonovich duality*.

- For elasticity, the duality maps the stress tensor to the field strength of **tensor gauge fields**.

$$\partial_\mu T^\mu{}_i = 0$$

Momentum conservation equation / Equation of motion for the Goldstone (*phonons*)

$$T^\mu{}_i = \varepsilon^{\mu\nu\rho}\partial_\nu A_{i\rho} = \varepsilon^{\mu\nu\rho}\partial_\nu\left(\delta^0_\rho \Phi_i + \delta^j_\rho A_{ji}\right)$$

Ansatz in terms of *dual* gauge fields

- In ordinary rotation-invariant elasticity, due to the symmetry of the energy-momentum tensor, the spatial components of the dual fields can be expressed in terms of a scalar field and a symmetric rank-two tensor.

$$T^\mu{}_i = \varepsilon^{\mu\nu\rho}\partial_\nu\left(\delta^0_\rho \partial_i \Phi + \delta^j_\rho A_{(ij)}\right)$$

Ansatz for standard elasticity

$$\delta\Phi = \dot\alpha \qquad \delta A_{(ij)} = \partial_i\partial_j\alpha \qquad \delta T^\mu{}_i = 0$$

Gauge variation

Pretko Radzihovsky **Phys. Rev. Lett. 120**
Gromov Surówka **Sci Post Phys 8 4 065**

- **Disclinations** map to **charged particles**. They are immobile due to ***dipole moment conservation***.

Dipole moment conservation

$$\delta\Phi = \dot{\alpha} \qquad \delta A_{(ij)} = \partial_i \partial_j \alpha$$

Gauge variation of the gauge connections

$$\int dt\, dx^2 \;\left(\Phi J^0 + A_{(ij)} J^{ij}\right)$$

$$\Rightarrow \partial_0 J^0 + \partial_i \partial_j J^{ij} = 0$$

Continuity equation derived from gauge invariance
(Ward-Takahashi identity)

The higher-derivative continuity equation implies both **charge** and **dipole moment** conservation.

$$\partial_0 D^{0k} = \partial_0 \int d^2x\, x^k J^0 = -\int d^2x\, x^k \partial_i \partial_j J^{ij} = \int d^2x\, \partial_j J^{kj} = 0$$

Limitations

- We need a more generic theory of elasticity to describe interesting incommensurate and/or lower-symmetry systems.
  - *Quasi-crystals, non-rotationally-invariant systems, twisted bilayer graphene …*
- Non-Gaussian EFT for fields charged under dipole
- When the background geometry is **not** flat, we generically encounter an ***incompatibility*** between *gauge symmetry* and *general covariance*.
  - The Chern-Simons action for tensor gauge fields with covariantized derivatives is not gauge-invariant when the space is curved. Gromov **Phys. Rev. B 122**
  - The field strength for the Pretko scalar model is not gauge-invariant when space is curved

Jensen **SciPost Phys. 12, 142 (2022)**

Similar story in (old) higher-spin literature: when covariantizing the derivatives, in the action for a gauge field with spin greater than 2 there appear terms proportional to the Riemann tensor, this breaks the gauge invariance (Aragone-Deser argument).

Material coordinates

- The fields $X^a(t, \boldsymbol{x})$ map the physical spacetime into the material target space.

- The elastic medium arises from the spontaneous breaking of external, $x^i \to x^i + k^i$, and internal, $X^a \to X^a + \kappa^a$, translations to the diagonal subgroup.

- The symmetry breaking induces an identification among internal and external space.

$$\boxed{X^a(t, \boldsymbol{x}) = \delta^a_i x^i + u^a(t, \boldsymbol{x})}$$

The displacement fields are the Nambu-Goldstone fields of the diagonal symmetry breaking locking internal and external space.

Elastic Modes

$$\mathcal{L} = \frac{1}{2}\delta_{ab}\dot{u}^a\dot{u}^b - \frac{1}{2}C^{i\ \,j}_{\ a\ \,b}\partial_i u^a \partial_j u^b + f_a u^a$$

Non-relativistic low-energy elastic Lagrangian in the presence of point-forces.

$$C^{i\ \,j}_{\ a\ \,b} = G(P_2)^{i\ \,j}_{\ a\ \,b} + H(1-\sigma)(P_1)^{i\ \,j}_{\ a\ \,b} + K(P_0)^{i\ \,j}_{\ a\ \,b}$$

Elastic tensor

"spin" projectors

$$(P_0)^{i\ \,j}_{\ a\ \,b} \equiv \frac{1}{2}\delta^i_a\delta^j_b$$

$$(P_1)^{i\ \,j}_{\ a\ \,b} \equiv \frac{1}{2}\left(\delta^{ij}\delta_{ab} - \delta^i_b\delta^j_a\right) = \frac{1}{2}\epsilon^i_{\ a}\epsilon^j_{\ b}$$

$$(P_2)^{i\ \,j}_{\ a\ \,b} \equiv \frac{1}{2}\left(\delta^{ij}\delta_{ab} + \delta^i_b\delta^j_a - \delta^i_a\delta^j_b\right)$$

$$\epsilon^i_{\ a}\partial_i u^a = \text{const.} \longrightarrow \text{rigid rotation}$$

$$\sigma = 1 \longrightarrow \text{standard elasticity}$$

$$u_{ij} \equiv \partial_{(i}\delta^a_{j)}u_a = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \qquad C^{ijkl} = C^{(ij)(kl)}$$

Dispersion relations for the transverse and longitudinal **elastic modes**, respectively

$$\omega^2 = \frac{G + H(1-\sigma)}{2}\boldsymbol{k}^2 \qquad \omega^2 = \frac{G+K}{2}\boldsymbol{k}^2$$

Elastic dual (Hubbard-Stratonovich)

One introduces auxiliary fields corresponding to the momentum density and stress tensor $\pi_a,\ T_a{}^i$

$$\mathcal{L} = -\frac{1}{2}\delta^{ab}\,\pi_a\pi_b + \frac{1}{2}(C^{-1})_i{}^a{}_j{}^b\,T_a{}^iT_b{}^j + f_a u^a$$
$$+\pi_a\partial_t u^a + T_a{}^i\partial_i u^a - \frac{\sigma}{2}\epsilon_i{}^a\epsilon^j{}_b T_a{}^i\partial_j u^b$$

Integrating the auxiliary fields, one recovers the original elastic action. Instead, the equation of motion for the (*regular part of*) the displacement field gives

$$\dot{\pi}_a + \partial_i\left(T_a{}^i - \frac{\sigma}{2}\epsilon_a{}^i\epsilon^b{}_j T_b{}^j\right) = f_a$$

Energy-momentum conservation equation in the presence of point forces

Monopole-Dipole-Momentum Algebra (MDMA)

$$\boxed{i[P_a, Q_1^b] = \delta_a^b Q_0 \qquad i[P_a, Q_2] = \delta_{ab} Q_1^b}$$

The dipole algebra is a nilpotent algebra of order 3.

It contains a central extension of the Heisenberg algebra.

The MDMA algebra admits the following representation on functions:

$$P_a = -i\frac{\partial}{\partial x^a} \qquad Q_0 = 1 \qquad Q_1^a = x^a \qquad Q_2 = \frac{x^b x_b}{2}$$

0-th moment generator

1-st moment generator

2-nd moment generator

$$\rho_0 = \int d^2x \, j_0 \qquad \rho_1^a = \int d^2x \, x^a \, j_0 \qquad \rho_2 = \int d^2x \, \frac{\boldsymbol{x}^2}{2} \, j_0$$

Gauging the dipole algebra

$$\mathcal{A}_\mu = V_\mu^a P_a + a_\mu Q_0 + b_{\mu a} Q_1^a + c_\mu Q_2$$

gauge connection

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i\left[\mathcal{A}_\mu, \mathcal{A}_\nu\right]$$

field strength

$$\Lambda = \kappa^a P_a + \lambda_0 Q + \lambda_{1\,a} Q_1^a + \lambda_2 Q_2$$

gauge parameter

$$\delta\mathcal{A}_\mu = \partial_\mu \Lambda + i[\mathcal{A}_\mu, \Lambda]\,, \quad \delta\mathcal{F}_{\mu\nu} = i[\mathcal{F}_{\mu\nu}, \Lambda]$$

gauge transformation (*implicit form*)

$$\delta V_\mu^a = \partial_\mu \kappa^a\,, \quad \delta c_\mu = \partial_\mu \lambda_2\,,$$
$$\delta a_\mu = \partial_\mu \lambda_0 + V_\mu^a \lambda_{1\,a} - b_{\mu a}\kappa^a\,,$$
$$\delta b_{\mu a} = \partial_\mu \lambda_{1\,a} + \delta_{ab}\left(V_\mu^b \lambda_2 - c_\mu \kappa^b\right)$$

gauge transformation of the connection fields (*explicit form*)

$$E_{ia}^{(1)} = \mathcal{F}_{ti\,a}^{(1)} \qquad E_{ia}^{(0)} = \partial_t \mathcal{F}_{ia}^{(0)} - \partial_i \mathcal{F}_{ta}^{(0)}$$
$$B_a^{(1)} = \frac{1}{2}\epsilon^{ij}\mathcal{F}_{ij\,a}^{(1)} \qquad B_a^{(0)} = \epsilon^{ij}\partial_i \mathcal{F}_{ja}^{(0)}$$

Gauge invariant quantitities which generalize the electric and magnetic fields

Ansatz

The dual gauge fields are introduced to the purpose of defining an automatically conserved stress tensor:

$$\dot{\pi}_a + \partial_i \left(T_a{}^i - \frac{\sigma}{2} \epsilon_a{}^b \epsilon^i{}_j T^{b\ j} \right) = 0$$

$$\boxed{\begin{aligned} \pi_a &= \epsilon_a{}^b \left(B_b^{(1)} - \left(1 - \frac{\sigma}{2}\right) B_b^{(0)} \right) \\ T_a{}^i &= -\epsilon_a{}^b \epsilon^{ij} \left(E_{jb}^{(1)} - E_{jb}^{(0)} \right) \end{aligned}}$$

The conservation of the stress tensor corresponds to the Bianchi identities for the generalized electro-magnetic fields.

An explicit analysis of the fluctuation spectrum reveals that the dual theory reproduces the dispersion relations of the elastic modes.

Global symmetries

In the dual formulation, there are *emergent global symmetries* corresponding to the gauge transformations of the charge and its higher moments, which leave the gauge potentials invariant:

$$\delta a_\mu = \partial_\mu \lambda_0 + V_\mu^a \lambda_{1a}$$

$$\delta b_{\mu a} = \partial_\mu \lambda_{1a} + \delta_{ab} V_\mu^b \lambda_2$$

$$\delta c_\mu = \partial_\mu \lambda_2$$

gauge variation of the gauge potentials

moment	λ_0	$\lambda_{1\,a}$	λ_2
0th	const	0	0
1st	$x^i \lambda_{1\,i}$	const_a	0
2nd	$\frac{\boldsymbol{x}^2}{2}\lambda_2$	$\delta_{ai} x^i \lambda_2$	const

For each global symmetry, there is a conserved current.

Note that global symmetries require specific *mixed* combinations of the local symmetries.

Defect currents

The *mixed* action expressed in terms of the desplacement fields and the auxiliary fields at zero external force (before integration):

$$\mathcal{L} = -\frac{1}{2}\delta^{ab}\,\pi_a\pi_b + \frac{1}{2}(C^{-1})_i{}^a{}_j{}^b\, T_a{}^i T_b{}^j \;\; + \pi_a\partial_t u^a + T_a{}^i\partial_i u^a - \frac{\sigma}{2}\epsilon_i{}^a\epsilon^j{}_b T_a{}^i\partial_j u^b$$

We split the displacement fields in a *singular* and a *regular* part:

$$u^a = u^{(s)\,a} + \tilde{u}^a$$

We then integrate the regular part, thus obtaining the dual action:

$$\mathcal{L} \supset \pi_a\partial_t u^{(s)\,a} + T_a{}^i\partial_i u^{(s)\,a} - \frac{\sigma}{2}\epsilon_i{}^a\epsilon^j{}_b T_a{}^i\partial_j u^{(s)\,b}$$

Integrating by parts, we can re-organize the dual action as follows:

$$-a_\mu J_0^\mu - b_{\mu a} J_1^{\mu a} - c_\mu J_2^\mu$$

Dislocation and disclinations

Explicit integration by parts yields:

$$\boxed{\begin{aligned} J_0^t &= -\epsilon^{ab}\epsilon^{ij}\partial_a\partial_i u_{jb}^{(s)} = -\partial_a D^{ta} - \Theta^t \\ J_1^t &= 0 \\ J_2^t &= \partial_i u^{(s)\,i} \end{aligned}}$$

$$\theta^{(s)} \equiv \frac{1}{2}\epsilon^{ij}\partial_i u_j^{(s)} \qquad \text{local rotation}$$

$$\Theta^{\mu} \equiv \epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\theta^{(s)} \qquad \text{disclination current (defect angle)}$$

$$D^{\mu a} \equiv \epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho u^{(s)\,a} \qquad \text{dislocation current}$$

The dipole current is trivial, the singular configurations of the displacement field do not provide any *intrinsic* dipole degree of freedom.

The second moment current corresponds to a volume defect.

Kleinert **doi.org/10.1142/0356**

Local Ward identities & continuity relations for global currents

Asking invariance with respect to the generic gauge variation, we get the local Ward identities:

$$\partial_\mu J_0^\mu = 0\,, \quad \partial_\mu J_1^{\mu a} = -J_0^a\,, \quad \partial_\mu J_2^\mu = -\delta_{ia} J_1^{ia}$$

The global currents associated to the "*rigid*" gauge transformations are:

$$\mathcal{J}_0^\mu = J_0^\mu \qquad \mathcal{J}_1^{\mu a} = J_1^{\mu a} + x^a J_0^\mu \qquad \mathcal{J}_2^\mu = J_2^\mu + \delta_{ia} x^i J_1^{\mu a} + \frac{\boldsymbol{x}^2}{2} J_0^\mu$$

The Ward identities imply the continuity relations for the global currents.

Appropriately improved local currents satisfy the same ***dipole continuity equation*** already seen in Pretko's model

$$\boxed{\partial_t \widetilde{J}_0^t - \partial_i \partial_j \widetilde{J}_1^{ij} = 0, \qquad \widetilde{J}_1^{i}{}_{i} = 0}$$

Mobility constraint from gauge invariance

When the dipole moment is conserved, an individual charge is constrained to be immobile.

$$S_q = q\int_\gamma a \equiv q\int_\gamma ds\,\dot{x}^\mu(s)\,a_\mu$$

coupling between a charge and the *monopole* gauge field

- Invariant under *monopole* gauge transformation (*same argument as in standard EM*).

$$\delta a_\mu = \partial_\mu \lambda_0$$

- Generically **NOT invariant** under *dipole* gauge transformations.

$$\delta a_\mu = -\delta_\mu^a \lambda_{1a}$$

For a rigid dipole transformation, the variation coincides with the change in the dipole moment as the charge moves along the worldline.

To have dipole gauge invariance, the worldline should be directed along the time direction, this corresponding to an **immobility** constraint.

Intrinsic dipole charge

We can couple a point-like object with *intrinsic* dipole charge to the gauge fields as follows:

$$S_{\boldsymbol{d}} = d^a \int_\gamma A_a^{(1)} = d^a \int_\gamma ds\, \dot{x}^\mu(s)\, A_{\mu a}^{(1)}$$

with

$$A_{\mu a}^{(1)} = b_{\mu a} - \delta_a^\nu \mathcal{F}_{\mu\nu}^{(0)}$$

For a static world-line and time-independent gauge fields, the dipole charge couples to the monopole gauge field as $d^i \partial_i a_t$

Mobility constraint for dipoles

$$S_{\boldsymbol{d}} = d^a \int_\gamma A_a^{(1)} = d^a \int_\gamma ds\, \dot{x}^\mu(s)\, A_{\mu a}^{(1)}$$

$$A_{\mu a}^{(1)} = b_{\mu a} - \delta_a^\nu \mathcal{F}_{\mu\nu}^{(0)}$$

$$\delta b_{\mu a} = \partial_\mu \lambda_{1a} - \delta_{\mu a} \lambda_2$$

$$\delta \mathcal{F}^{(0)} = 0$$

gauge variations reminder

The action is invariant under both monopole and dipole gauge variations.

Enforcing second moment gauge invariance leads to a mobility constraint allowing movement only along the orthogonal direction to the dipole moment (*to be identified with the* ***Burgers vector*** *of a dislocation*)

$$\text{Monopole gauge variation} \quad \delta A_{\mu a}^{(1)} = 0$$

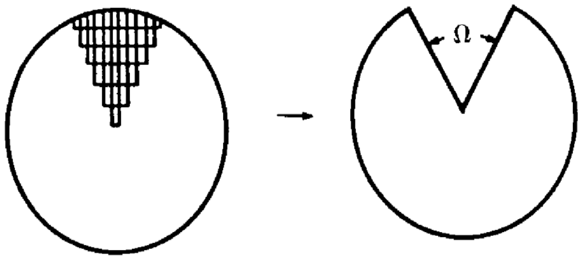
$$\text{Dipole gauge variation} \quad \delta A_{\mu a}^{(1)} = \partial_\mu \lambda_{1a}$$

$$\text{Second moment gauge variation} \quad \delta A_{\mu a}^{(1)} = -\delta_{\mu a} \lambda_2 \longrightarrow \boxed{\boldsymbol{d} \cdot \dot{\boldsymbol{x}} = 0}$$

Dislocation mobility constraint

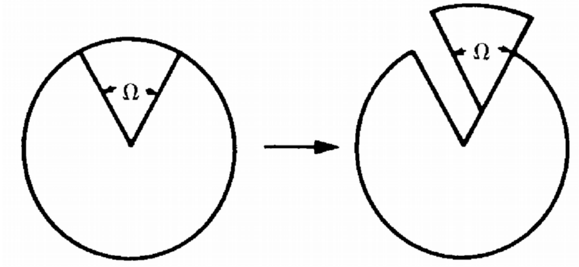
Defects

Disclination



Deficit or excess angle, conical defect.

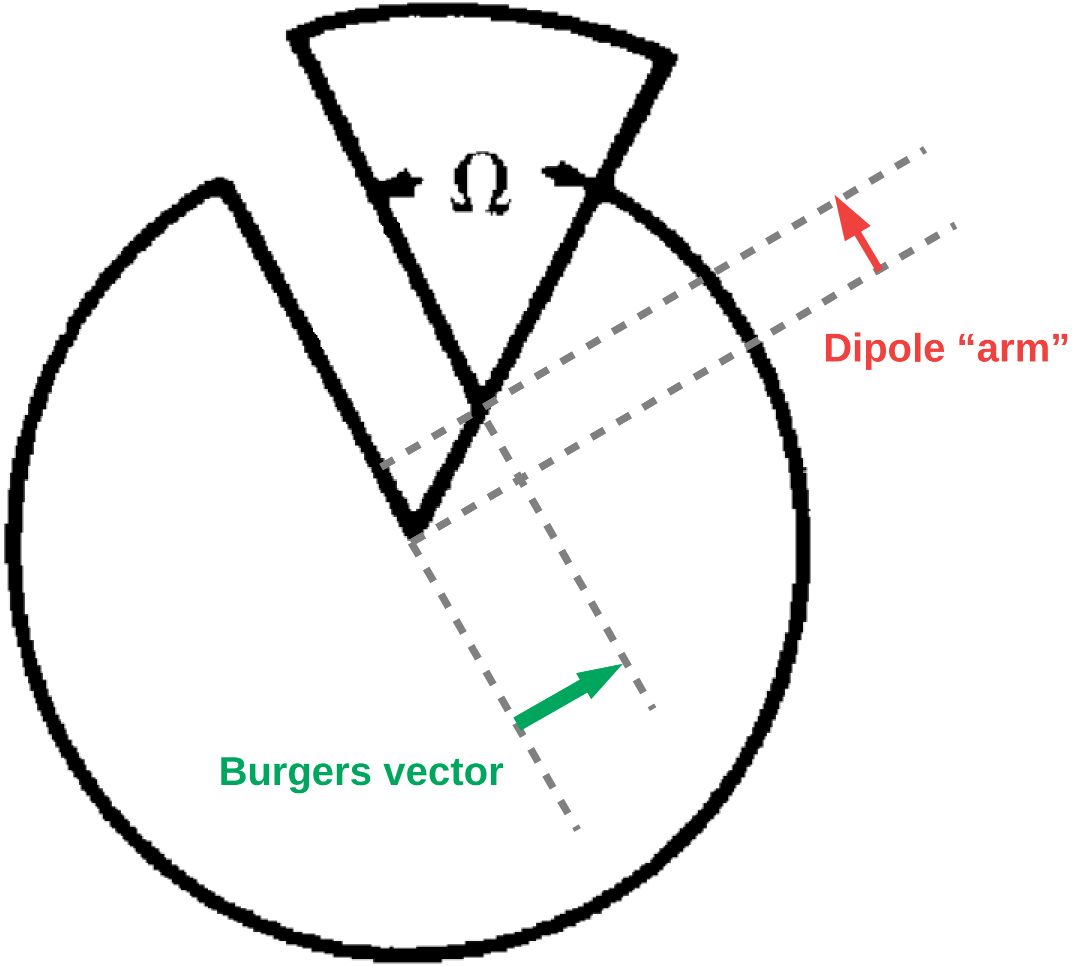
Dislocation



Dislocation seen as a dipole of disclinations. Namely two conical defects of opposite sign.

images taken from Kleinert **doi.org/10.1142/0356**

Dislocations and Burgers vector



Dislocation seen as a dipole of disclinations.

A movement along the Burgers vector needs only a local rearrangment of atoms.

A movement along the dipole direction needs the rearrangment of a whole line of atoms.

Dislocation mobility constraint

$$\boxed{d \cdot \dot{x} = 0}$$

MDMA scalar theory

$$i\,[P_i, Q^j] = \delta_i^j\, Q_0$$

Simple MDMA algebra (without second moment)

$$e^{i\vec{\alpha}\cdot\vec{P}}\,\phi(t,\vec{x})\,e^{-i\vec{\alpha}\cdot\vec{P}} = \phi(t,\vec{x}+\vec{\alpha})$$

$$e^{i\lambda_0 Q_0}\,\phi(t,\vec{x})\,e^{-i\lambda_0 Q_0} = e^{i\lambda_0}\,\phi(t,\vec{x})$$

$$e^{i\vec{\lambda}_1\vec{Q}_1}\,\phi(t,\vec{x})\,e^{i\vec{\lambda}_1\vec{Q}_1} = e^{i\vec{\lambda}_1\cdot\vec{x}}\,\phi(t,\vec{x})$$

Transformation of a scalar field charged under monopole and dipole charges.

Internal space realization

$$e^{i\vec{\kappa}\cdot\vec{P}}\,\phi_{\vec{X}}(t,\vec{x})\,e^{-i\vec{\kappa}\cdot\vec{P}} = \phi_{\vec{X}+\vec{\kappa}}(t,\vec{x})$$

$$e^{i\lambda_0 Q_0}\,\phi_{\vec{X}}(t,\vec{x})\,e^{-i\lambda_0 Q_0} = e^{i\lambda_0}\,\phi_{\vec{X}}(t,\vec{x})$$

$$e^{i\vec{\lambda}_1\cdot\vec{Q}_1}\,\phi_{\vec{X}}(t,\vec{x})\,e^{-i\vec{\lambda}_1\cdot\vec{Q}_1} = e^{i\vec{\lambda}_1\cdot\vec{X}}\,\phi_{\vec{X}}(t,\vec{x})$$

An infinite collection of scalar fields parametrized by the internal spatial coordinates. Momentum generates an internal shift.

$$\boxed{D_\mu\phi_{\vec{X}} \equiv \partial_\mu\phi_{\vec{X}} - i a_\mu\phi_{\vec{X}} - i\vec{b}_\mu\cdot\vec{X}\,\phi_{\vec{X}} - \vec{V}_\mu\cdot\vec{\nabla}_X\phi_{\vec{X}}}$$

$$\delta a_\mu = \partial_\mu\lambda_0 - \vec{V}_\mu\cdot\vec{\lambda}_1 + \vec{b}_\mu\cdot\vec{\kappa}$$

$$\delta\vec{b}_\mu = \partial_\mu\vec{\lambda}_1$$

$$\delta\vec{V}_\mu = \partial_\mu\vec{\kappa}$$

Covariant derivative for the internal transformations and gauge variations for the gauge fields associated to the monopole, dipole and internal translations

Discretizing the internal space

$$X^I \to m^I \qquad \kappa^I \to K^I \qquad m^I, K^I \in \mathbb{Z}$$

Discrete covariant derivative recovering the continuum result in the limit of infinitely fine discretization:

$$D_\mu \phi_{\vec{m}} = \partial_\mu \phi_{\vec{m}} - i a_\mu \phi_{\vec{m}} - i \vec{m} \cdot \vec{b}_\mu \phi_{\vec{m}} - \sum_I v^I_\mu \phi_{\vec{m}} \left[\log \phi^*_{\vec{m}} \phi_{\vec{m}+\hat{I}} - \log \phi^*_{\vec{m}} \phi_{\vec{m}} \right]$$

$$\nabla_{X^I} \phi_{\vec{X}} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \phi_{\vec{X}} \left[\log \phi^*_{\vec{X}} \phi_{\vec{X}+\varepsilon\hat{I}} - \log \phi^*_{\vec{X}} \phi_{\vec{X}} \right]$$

The form of the logarithms in terms of field bilinears is convenient in view of working with Hubbard-Stratonovich auxiliary fields.

Simple Mexican-hat Lagrangian

$$\mathcal{L} = \sum_m \left[-|D_\mu \phi_m|^2 + m_\phi^2 |\phi_m|^2 - \frac{\lambda_\phi}{2} |\phi_m|^4 \right]$$

To keep the analysis as simple as possible, we focus on the case with zero background for the monopole and dipole gauge fields.

$$a_\mu = b_\mu = 0$$

$$D_\mu \phi_m = \partial_\mu \phi_m - V_\mu \phi_m \left[\log \phi_m^* \phi_{m+1} - \log \phi_m^* \phi_m \right]$$

Classical SSB and Goldstone

$$\phi_m(t,x) = \bar{\Phi}_m \, e^{i\vartheta_m(t,x)}$$

Polar parametrization of the scalar field with a coordinate-independent modulus.

The minima of the potential

$$V = \sum_{m=1}^{N} \left[V_x^2 \, \bar{\Phi}_m^2 \left| \log\left(\frac{\bar{\Phi}_{m+1}}{\bar{\Phi}_m}\right) + i\left(\vartheta_{m+1} - \vartheta_m\right) \right|^2 - m_\phi^2 \bar{\Phi}_m^2 + \frac{\lambda_\phi}{2} \bar{\Phi}_m^4 \right]$$

are given by

$$\bar{\Phi}_{m+1} = \bar{\Phi}_m = \bar{\Phi} = \frac{m_\phi}{\sqrt{\lambda_\phi}} \qquad \vartheta_{m+1} = \vartheta_m = \vartheta$$

The classical SSB gives rise to a standard Nambu-Goldstone mode with linear dispersion relation

$$\boxed{S_\theta = N \int d^2x \left[(\partial_t \vartheta)^2 - (\partial_x \vartheta)^2 \right]}$$

Quantum case

To study a quantum case we adopt a Hubbard-Stratonovich strategy.

To this purpose it is useful to modify the model to include up to quartic terms in the field bilinears.

We consider only positive masses, so the classical model is in this case stable without breaking the monopole/dipole symmetries. In fact, we seek to obtain slight simmetry-breaking effects through quantum corrections.

$$
\mathcal{L} = \sum_m \left(-|D_\mu \phi_m|^2 - m_\phi^2 \, \phi_m^\dagger \phi_m \right) - \frac{m_\sigma^2}{2N} \left(\sum_m \phi_m^* \phi_m \right)^2
$$

$$
- \frac{\lambda_\sigma}{4N^3} \left(\sum_m \phi_m^* \phi_m \right)^4 - \frac{m_\chi^2}{N} \left| \sum_m \phi_{m+1} \phi_m^\dagger \right|^2 - \frac{\lambda_\chi}{2N^3} \left| \sum_m \phi_{m+1} \phi_m^* \right|^4
$$

Hubbard-Stratonovich

The Hubbard-Stratonovich fields are introduced by means of Lagrange multipliers which fix them to the field bilinears

$$\mathcal{L}_{HS} = \sum_{\mu} \left(-|D_{\nu}\phi_{\mu}|^2 - m_{\phi}^2 \phi_{\mu}^{\dagger}\phi_{\mu} \right) - \frac{m_{\sigma}^2}{2N}\left(\sum_{\mu}\sigma_{\mu}\right)^2$$

$$-\frac{\lambda_{\sigma}}{4N^3}\left(\sum_{\mu}\sigma_{\mu}\right)^4 - \frac{m_{\chi}^2}{N}\left|\sum_{\mu}\chi_{\mu}\right|^2 - \frac{\lambda_{\chi}}{2N^3}\left|\sum_{\mu}\chi_{\mu}\right|^4$$

$$+\sum_{\mu}\left[\tau_{\sigma_{\mu}}\left(\sigma_{\mu} - \phi_{\mu}^{*}\phi_{\mu}\right) + \tau^{*}_{\chi_{\mu}}\left(\chi_{\mu} - \phi_{\mu+1}\phi_{\mu}^{*}\right) + \tau_{\chi}\left(\chi_{\mu}^{*} - \phi_{\mu+1}^{*}\phi_{\mu}\right)\right]$$

$$D_{\nu}\phi_{\mu} = \partial_{\nu}\phi_{\mu} - iV_{\nu}\theta_{\mu} - V_{\nu}\phi_{\mu}\log\frac{|\chi_{\mu}|}{\sigma_{\mu}}$$

Covariant derivative in terms of the Hubbard-Stratonovich fields

Internal space homogeneity

Ansatz independent from n $\quad \chi_n = \bar{\chi} \quad \sigma_n = \bar{\sigma} \quad \tau_{\chi_n} = \tau_\chi \quad \tau_{\sigma_n} = \tau_\sigma$

$$\mathcal{L}_{HS} = \sum_n \left[-|D_\mu \phi_n|^2 - (m_\phi^2 + \tau_\phi)\, \phi_n^* \phi_n - \tau_\chi^* \phi_{n+1} \phi_n^* - \tau_\chi \phi_{n+1}^* \phi_n \right]$$

$$-N \left(\frac{m_\sigma^2}{2} \bar{\sigma}^2 + \frac{\lambda_\sigma}{4} \bar{\sigma}^4 + m_\chi^2 |\bar{\chi}|^2 + \frac{\lambda_\chi}{2} |\bar{\chi}|^4 - \tau_\chi^* \chi - \tau_\chi \chi^* - \tau_\sigma \bar{\sigma} \right)$$

The action is quadratic in the original fields, hence we can adopt standard effective field theory methods (heat kernel) to obtain a low-energy effective action for the Hubbard-Stratonovich fields and the associated Lagrange multipliers.

The cut-off of the low-energy theory is the mass of the fields phi that we are integrating away.

Nambu-Goldstone

We found a locally stable solution with non-trivial chi field. This is charged under the dipole charge but not on the monopole. So we break only the former.

We don’t worry about metastability because the large-N limit suppresses tunneling to other possible vacua.

The resulting low-energy Lagrangian for the Nambu-Goldstone field is non-standard:

$$\mathcal{L}_\theta = \frac{V_\chi^2 N}{240\pi \tilde{m}_\phi^4}\left[10\,\tilde{m}_\phi^2 (\partial_t \theta)^2 - (\partial_t \partial_x \theta)^2 + (\partial_t^2 \theta)^2\right]$$

In Fourier space the lowest frequency solution of the dispersion equation

$$\omega^2\left[\omega^2 + 10\,\tilde{m}_\phi^2 - q^2\right] = 0$$

Gives a “fractonic” dispersion relation $\omega^2 = 0$

Avoiding Coleman

Spontaneous symmetry breaking requires a stable order parameter against quantum fluctuations.

$$\rho(t,x) \equiv \frac{\langle e^{i(\vartheta(t,x)-\vartheta(0,0))}\rangle}{\langle e^{i\vartheta(t,x)}\rangle\langle e^{i\vartheta(0,0)}\rangle}$$

Relying upon the cluster-decomposition principle:

$$\langle e^{i(\vartheta(0,x)-\vartheta(0,0))}\rangle \to \langle e^{i\vartheta(0,x)}\rangle\langle e^{i\vartheta(0,0)}\rangle$$

The presence of SSB is associated to $\lim_{x\to\infty} \rho(0,x) = 1$

The dipole-symmetry Nambu-Goldstone mode satisfies this criterion **independently of the large-N limit.**

Conclusions ✓ & future prospects ★

Elastic fractons can be described in terms of a gauged dipole symmetry formulated in terms of ordinary vector connections ✓

- *Mobility constraints arise from gauge invariance* ✓
- *Possibility to couple the theory to coupled backgrounds* ★

Emergent fractons are a convenient way to describe generalized elastic theories at low energy, with the possibility of accounting for the defect dynamics ✓

- *They encompass crystalline and amorphous media, with or withour rotational symmetry* ✓
- *Description of the defects' dynamics* ★

We worked out an explicit dynamical model breaking spontaneously the dipole symmetry while preserving the monopole symmetry ✓

- *Coupling the elastic gauge fields to the scalar model and possible descriptions of dynamical defects* ★

THANKS!

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talk at Genova, 5th April 2023

Mainly based upon:

- ***"EFT for dipole charges and spontaneous breaking of dipole symmetry"***
  E.Afxonidis, A.Caddeo, C.Hoyos, D.Musso – (*to appear*)
- ***"Emergent Dipole Gauge Fields and Fractons***"
  A.Caddeo, C.Hoyos, D.Musso – PhysRevD.106.L111903 – 2206.12877 [cond.mat.str.el]
- ***"Fractons in effective field theories for spontaneously broken translations"***
  R.Argurio, C.Hoyos, D.Musso, D.Naegels Phys.Rev.D 104 (2021) 10, 105001 -- 2107.03073 [hep-th]
- ***"Simplest phonons and pseudo-phonons in field theory"***
  D. Musso Eur.Phys.J.C 79 (2019) 12, 986 -- 1810.01799 [hep-th]

$$D_{ij} = \partial_i \phi \, \partial_j \phi - \phi \, \partial_i \partial_j \phi \xrightarrow{\text{FOURIER}} -\frac{1}{2}\left(qp + pq - q^2 - p^2\right) \tilde{\phi}(q) \, \tilde{\phi}(p)$$

$$= \frac{1}{2}(q-p)^2 \, \tilde{\phi}(q) \, \tilde{\phi}(p)$$

Recovering Pretko's theory

- When external forces are present, the conservation equation reduces to:

$$E_i^{(2)} \equiv \mathcal{F}_{ti}^{(2)} = f_i$$

which represents a constraint that can be implemented at the level of the effective action through a Lagrange multiplier. It makes the gauge field c_μ non-dynamical.

- When external forces are absent: $f_i = 0 \quad \rightarrow \quad c_\mu = 0$

- Imposing a symmetric energy-momentum tensor $\epsilon^{ij} T_{ij} = 0$

$$\left.\begin{aligned} b_{[ij]} &= 0 \\ a_i &= 0 \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} a_0 &\leftrightarrow \Phi \\ b_{ij} = b_{(ij)} &\leftrightarrow A_{ij} \end{aligned}\right.$$

Pretko's scalar and symmetric tensor gauge fields

Gradient Mexican hat

Musso **Eur.Phys.J.C 79 12**

Consider the following higher-derivative scalar Lagrangian where all the coefficients are positive:

$$\mathcal{L} = \partial_0\phi^*\partial_0\phi + A\,\partial_i\phi^*\partial_i\phi - B\left(\partial_i\phi^*\partial_i\phi\right)^2 + G\,\partial_i\phi^*\partial_i\phi^*\partial_j\phi\partial_j\phi$$

static potential

$$V\left(\partial\phi^*, \partial\phi\right) \equiv \underline{-A\,\partial_i\phi^*\partial_i\phi} + \underline{B\left(\partial_i\phi^*\partial_i\phi\right)^2} - G\,\partial_i\phi^*\partial_i\phi^*\partial_j\phi\partial_j\phi$$

Competition between a quadtratic **unstable** term and a quartic **stabilizer**.

$$\phi(t,\vec{x}) = \bar{\phi}(\vec{x}) \equiv \rho\, e^{ik\cdot x} \longrightarrow \rho^2\, k^2 = \frac{1}{2}\frac{A}{B-G}$$

time-independent vacuum Ansatz

vacuum solutions

$$\phi(t,\vec{x}) = \bar{\phi}(\vec{x}) + \delta\phi(t,\vec{x}) \equiv \left[\rho + a(t,\vec{x}) + ib(t,\vec{x})\right] e^{ik\cdot x}$$

fluctuation Ansatz

Fractonic dispersion relations

$$q_l \equiv \frac{q \cdot k}{k}$$
$$q_t^2 \equiv q^2 - q_l^2$$

transverse and longitudinal momentum

$$\boxed{\omega^2 = 4G\rho^2\, q_l^2 q_t^2 + \dots}$$

Gapless Nambu-Goldstone mode for broken translations (*phonon*) featuring a Seiberg-Shao fractonic dispersion relation with **UV/IR mixing**. For trivial G, it reduces to a non-propagating (*immobile*) mode.

$$\omega^2 = 2Ak^2 + 2A\, q_l^2 + 4G\, \rho^2 k^2 q_t^2 + \dots$$

Gapped Higgs mode.

Can these models provide a UV-completion of Seiberg-Shao and tame the UV/IR mixing by means of the scale of the symmetry-breaking?

?

Seiberg Shao **SciPost Phys. 10 2**

Similar, scale-invariant Ginzburg-Landau models feature fractonic modes with either subdimensional propagation (*e.g. lineons*) or no propagation at all. They are related to an emergent symmetry under higher moment charges, leading to the trivialization of some elastic coefficients.

Argurio, Hoyos, Musso, Naegels **Phys.Rev.D 104 10**

Partial gauge fixing

$$\epsilon^{ij} T_{ij} = 0 \longrightarrow 0 = \epsilon^{ij}\left(E^{(1)}_{ij} - E^{(0)}_{ij}\right) = -\epsilon^{ij}\partial_t\left(\partial_i a_j + b_{ij}\right)$$

The constraint is satisfied for:

$$b_{ij} = b_{(ij)} + \partial_i \lambda_j \qquad a_i = \partial_i \alpha - \lambda_i \qquad a_t = \phi + \partial_t \alpha$$

We can partially fix the gauge asking:

$$b_{ij} = b_{(ij)} \qquad a_i = 0$$

The remaining gauge freedom consists in:

$$\lambda_i = \partial_i \beta \qquad \alpha = \beta$$

This coincides with the gauge freedom of Pretko's model

Gauging the dipole algebra (details)

$$\mathcal{F}_{\mu\nu} = \mathcal{V}^a_{\mu\nu} P_a + \mathcal{F}^{(0)}_{\mu\nu} Q_0 + \mathcal{F}^{(1)}_{\mu\nu\,a} Q_1^a + \mathcal{F}^{(2)}_{\mu\nu} Q_2$$

field strength decomposed on the algebra generators

$$\begin{aligned}
\mathcal{V}^a_{\mu\nu} &= \partial_\mu V^a_\nu - \partial_\nu V^a_\mu\,, \quad \mathcal{F}^{(2)}_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu \\
\mathcal{F}^{(0)}_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu + V^a_\mu b_{\nu a} - V^a_\nu b_{\mu a}\,, \\
\mathcal{F}^{(1)}_{\mu\nu\,a} &= \partial_\mu b_{\nu a} - \partial_\nu b_{\mu a} + \delta_{ab}\left(V^b_\mu c_\nu - V^b_\nu c_\mu\right)
\end{aligned}$$

coefficients of the field strength given in terms of the gauge fields

$$\begin{aligned}
\delta V^a_\mu &= \partial_\mu \kappa^a\,, \\
\delta a_\mu &= \partial_\mu \lambda_0 + V^a_\mu \lambda_{1\,a} - b_{\mu a}\kappa^a\,, \\
\delta b_{\mu a} &= \partial_\mu \lambda_{1\,a} + \delta_{ab}\left(V^b_\mu \lambda_2 - c_\mu \kappa^b\right)\,, \\
\delta c_\mu &= \partial_\mu \lambda_2\,.
\end{aligned}$$

$$\begin{aligned}
\delta\mathcal{F}^{(0)}_{\mu\nu} &= \mathcal{V}^a_{\mu\nu}\lambda_{1\,a} - \mathcal{F}^{(1)}_{\mu\nu\,a}\kappa^a\,, \\
\delta\mathcal{F}^{(1)}_{\mu\nu\,a} &= \delta_{ab}\left(\mathcal{V}^b_{\mu\nu}\lambda_2 - \mathcal{F}^{(2)}_{\mu\nu}\kappa^b\right)\,.
\end{aligned}$$

Global current J1

$$V^a_\mu = -\delta^a_\mu$$

$$\lambda_2 = 0$$

$$\lambda_1 = \text{const.}$$

$$\lambda_0 = \delta^a_i x^i \lambda_{1a}$$

$$\Longrightarrow$$

$$\delta a_\mu = \partial_\mu \lambda_0 + V^a_\mu \lambda_{1a} = \partial_\mu \lambda_0 - \delta^a_\mu \lambda_{1a} = 0$$

$$\delta b_{\mu a} = \partial_\mu \lambda_{1a} + \delta_{ab} V^b_\mu \lambda_2 = \partial_\mu \lambda_{1a} - \delta_{ab} \delta^b_\mu \lambda_2 = 0$$

$$\delta c_\mu = \partial_\mu \lambda_2 = 0$$

$$\begin{aligned} \delta \mathcal{L} &= -\delta^a_\mu \lambda_{1a} J^{(0)}_\mu + \partial_\mu \lambda_{1a} J^{\mu a}_{(1)} + \partial_\mu \left(\delta^a_i x^i \lambda_{1a} \right) J^\mu_{(0)} \\ &= -\lambda_{1a} \partial_\mu \left(J^{\mu a}_{(1)} + \delta^a_i x^i J^\mu_{(0)} \right) \end{aligned}$$

Improvements

Improved currents

$$\tilde{J}^{\mu}_{(0)} = J^{\mu}_{(0)} + \partial_\alpha \Psi^{\alpha\mu}_{(1)}$$
$$\tilde{J}^{\mu}_{(1)} = J^{\mu a}_{(1)} - \Psi^{\mu a}_{(1)} + \partial_\alpha \Psi^{\alpha\mu a}_{(2)}$$
$$\tilde{J}^{\mu}_{(2)} = J^{\mu}_{2} - \Psi^{\mu\nu a}_{(2)} \delta_{\nu a}$$

with

$$\Psi^{\mu\nu}_{(1)} = -\Psi^{\nu\mu}_{(1)}$$
$$\Psi^{\mu\nu a}_{(2)} = -\Psi^{\nu\mu a}_{(2)}$$

Through the improving terms we can ask: $\tilde{J}^{\mu}_{(2)} = 0$ $\quad \tilde{J}^{ta}_{(1)} = 0$ $\quad \tilde{J}^{ia}_{(1)} = \tilde{J}^{(ia)}_{(1)}$

$$\boxed{\left.\begin{array}{rcl} \partial_\mu J^{\mu}_{(0)} & = & 0 \\ \partial_\mu J^{\mu a}_{(1)} + J^{a}_{(0)} & = & 0 \\ \partial_\mu J^{\mu a}_{(2)} + \delta_{ia} J^{ia}_{(1)} & = & 0 \end{array}\right\} \Rightarrow \left\{ \begin{array}{rcl} \partial_t \tilde{J}^{t}_{(0)} - \partial_i \partial_a \tilde{J}^{(ia)}_{(1)} & = & 0 \\ \delta_{ia} \tilde{J}^{(ia)}_{(1)} & = & 0 \end{array}\right.}$$

Dislocation as a disclination dipole

$$\Theta^0 = \frac{1}{2}\epsilon^{ij}\epsilon^{kl}\partial_i\partial_j\partial_k u_l^{(s)} = \frac{1}{4}\epsilon^{ij}\epsilon^{kl}\partial_i\left(\delta_{jk}\partial^2 + \epsilon_{jk}\epsilon^{mn}\partial_m\partial_n\right)u_l^{(s)}$$
$$= -\frac{1}{4}\partial_i\partial^2 u_i^{(s)} - \frac{1}{4}\epsilon^{il}\epsilon^{mn}\partial_i\partial_m\partial_n u_l^{(s)} = -\frac{1}{4}\partial^2\partial_i u_i^{(s)} - \frac{1}{4}\epsilon^{il}\epsilon^{mn}\partial_i\partial_m\partial_n u_l^{(s)}$$

Thus, in the absence of vacancies/insertions,

$$\Theta^0 = -\frac{1}{4}\epsilon^{il}\epsilon^{mn}\partial_i\partial_m\partial_n u_l^{(s)}$$

$$\boxed{\begin{aligned} d^j &\equiv \int d^d x\, x^j \Theta^0 = -\frac{1}{4}\int d^d x\, x^j \epsilon^{il}\epsilon^{mn}\partial_i\partial_m\partial_n u_l^{(s)} \\ &= \frac{1}{4}\int d^d x\, \epsilon^{jl}\epsilon^{mn}\partial_m\partial_n u_l^{(s)} = \frac{1}{4}\epsilon^{jl}D^0{}_l \end{aligned}}$$

The dislocation density is proportional to the Burgers vector.

Particle with second moment charge

We can couple a point-like object with second moment charge to the gauge fields as follows:

$$S_{q_2} = q_2 \int_\gamma A^{(2)} = q_2 \int ds\, \dot{x}^\mu(s) A^{(2)}_\mu$$

with

$$A^{(2)}_\mu = \frac{1}{2}\delta^{\nu a}\left(\mathcal{F}^{(1)}_{\mu\nu a} - \partial_\mu \mathcal{F}^{(0)}_{\nu a} + \partial_\nu \mathcal{F}^{(0)}_{\mu a}\right)$$

Evaluating the action for a static particle shows explicitly that the particle acts as a point-like source for the second moment of the charge, in fact it couples to

$$c_t + \tfrac{1}{2}\boldsymbol{\partial}^2 a_t$$