Classification of Rational Conformal Field Theories With A Single Critical Exponent

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Based on:

"Classification of Unitary RCFTs with Two Primaries and $c < 25$",
Sunil Mukhi and Brandon Rayhaun, arXiv:2208.05486.

"New Meromorphic CFTs from Cosets",
Arpit Das, Chethan N. Gowdigere and Sunil Mukhi, arXiv: 2207.04061.

"Meromorphic Cosets and the Classification of Three-Character CFT",
Arpit Das, Chethan N. Gowdigere and Sunil Mukhi, arXiv: 2212.03136.

Background:

“Towards a Classification of Two-Character Rational Conformal Field Theories”,
A. Ramesh Chandra and Sunil Mukhi,
JHEP 1904 (2019) 153, arXiv:1810.09472.

“Curiosities above $c = 24$”,
A. Ramesh Chandra and Sunil Mukhi,
SciPost Phys. 6 (2019) 5, 053, arXiv:1812.05109.

“On 2d Conformal Field Theories with Two Characters”,
Harsha Hampapura and Sunil Mukhi,
JHEP 1601 (2016) 005, arXiv: 1510.04478.

“Cosets of Meromorphic CFTs and Modular Differential Equations”,
Matthias Gaberdiel, Harsha Hampapura and Sunil Mukhi,
JHEP 1604 (2016) 156, arXiv: 1602.01022.

And previous work:

"Reconstruction of conformal field theories from modular geometry on the torus",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B318 (1989) 483.

"On the classification of rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Phys. Lett. B213 (1988) 303.

"Differential equations for correlators and characters in arbitrary rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl.Phys. B312 (1989).

Outline

- It is thought that Rational 2d CFT are classified, at least in principle.

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Motivation

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  - What are all RCFTs that have exactly p primaries $\mathbf{1}, \mathbf{\Phi_1}, \mathbf{\Phi_2}, \cdots, \mathbf{\Phi_{p-1}}$?

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- These two developments appeared independent but eventually converged [Gaberdiel-Hampapura- -Mukhi 2016].
- In the last decade there has been progress on both questions, and here I will present some recent results.

- The main result is the complete classification of unitary CFT with $c < 25$ and two primaries $\mathbf{1}$, $\mathbf{\Phi}$ [Mukhi-Rayhaun 2022].

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  - The complete classification of three-character CFT with vanishing Wronskian index [Das-Gowdigere-Mukhi 2022b].

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- It is a special class of QFT having infinitely many symmetry generators $L_n, \bar{L}_n$ satisfying the Virasoro algebra:

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- Unitary CFTs have c real and > 0.

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- The Hilbert space decomposes into towers (modules) over highest-weight states $|\phi_i\rangle$ called primaries satisfying:

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- The remaining states in each tower are called descendants and are spanned by:

$$A_{-n_1}^{\alpha_1} A_{-n_2}^{\alpha_2} \cdots A_{-n_p}^{\alpha_p} \bar{A}_{-m_1}^{\bar{\alpha}_1} \bar{A}_{-m_2}^{\bar{\alpha}_2} \cdots \bar{A}_{-m_q}^{\bar{\alpha}_q} |\phi_i\rangle$$

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- The scaling dimensions of $|\phi_i\rangle$, $h_i, \bar{h}_i$, are the eigenvalues of $L_0, \bar{L}_0$. There is a distinguished primary $|\phi_0\rangle = |0\rangle$ (the vacuum) with $h_0 = \bar{h}_0 = 0$.

- For example, CFTs with a global symmetry under a Lie algebra $\mathfrak{g}$ are invariant under a Kac-Moody (KM) algebra:

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- At any positive integral level k, only finitely many representations are allowed, these are called integrable representations.

- From KM generators one can make a set of Virasoro generators as bilinears:

$$L_n = \frac{1}{2(k+g)} \sum_{m \in \mathbf{Z}} :J^a_{n-m} J^a_m:$$

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- CFTs containing all integrable representations of a KM algebra are called Wess-Zumino-Witten (WZW) or affine theories.

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- This is equivalent to the statement that c, h_i are rational numbers [Anderson-Moore 1988].

Outline

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- The partition function must be modular invariant:

$$Z\left(\tfrac{a\tau+b}{c\tau+d},\tfrac{a\bar\tau+b}{c\bar\tau+d}\right) = Z(\tau,\bar\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL(2,Z)}$$

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- It follows that $\chi(\tau)$ must be modular invariant upto a phase, and hence is a function of the Klein j-invariant:

$$j(q)=q^{-1}+744+196884q+21493760q^2+\cdots$$

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c = 8: & \chi(\tau) = j(\tau)^{\frac{1}{3}} & \mathsf{E}_{8,1} \quad \text{(unique)} \\
c = 16: & \chi(\tau) = j(\tau)^{\frac{2}{3}} & \mathsf{E}_{8,1} \times \mathsf{E}_{8,1},\ \mathsf{D}^{+}_{16,1} \\
c = 24: & \chi(\tau) = j(\tau) + \mathcal{N} & \text{Niemeier lattices} \\
c = 32: & \chi(\tau) = j(\tau)^{\frac{1}{3}}\big(j(\tau) + \mathcal{N}\big) & \text{Even unimodular 32d lattices}
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- These examples correspond to "lattice theories": c free bosons compactified on a torus $\mathbb{R}^c/\Gamma$, where Γ is an even, unimodular lattice.
- Starting from $c \geq 24$, there are more general (non-lattice) possibilities.

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  - Schellekens #34: $\mathsf{A}_{3,1}\mathsf{D}_{7,3}\mathsf{G}_{2,1}$(non-lattice theory)
- These are special modular invariant combinations ("extensions") of characters for the given non-simple KM algebras.

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- The 71st Schellekens theory is also a meromorphic extension, not of KM algebras, but of $(\text{Ising model})^{48}$.
- This extension is called the Monster CFT.
- At $c = 32$ there are around $\sim 10^9$ even, unimodular lattices (and an unknown number of non-lattice theories), so complete classification seems very difficult.

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- Thus:

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- So it is more convenient to classify theories by the number n of independent characters:

$$Z(q,\bar{q}) = \sum_{i=0}^{n-1} \chi_i(q)\bar{\chi}_{\bar{i}}(\bar{q})$$

where $\chi_i(q) = \mathrm{tr}_i\, q^{L_0 - \frac{c}{24}}$ is the trace over holomorphic descendants of ϕ_i.

- Modular invariance of Z $\Longleftrightarrow$ characters go into linear combinations of themselves under SL(2,Z):

$$\chi_i\left(\tfrac{a\tau+b}{c\tau+d}\right)=\sum_{j=0}^{n-1}\varrho_{ij}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\chi_j(\tau),\quad \varrho^\dagger\varrho=1$$

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- Then one has to verify which of these admissible characters correspond to actual CFT.

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$$\Big(D_\tau^2 + \phi_2(\tau) D_\tau + \phi_4(\tau)\Big)\chi(\tau) = 0$$

where

$$D_\tau \equiv \tfrac{1}{2\pi i}\tfrac{\partial}{\partial\tau} - \tfrac{k}{12} E_2(\tau) : \quad \mathcal{M}_k \to \mathcal{M}_{k+2}$$

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- If the coefficient functions ϕ_2, ϕ_4 are modular of weight $2, 4$ respectively then the equation is modular-invariant.

- Suppose we are given a VVMF χ_0, χ_1. A general linear combination of them, χ, satisfies the following equation:

$$\begin{vmatrix} \chi_0 & \chi_1 & \chi \\ D_\tau \chi_0 & D_\tau \chi_1 & D_\tau \chi \\ D_\tau^2 \chi_0 & D_\tau^2 \chi_1 & D_\tau^2 \chi \end{vmatrix} = 0$$

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- Hence:

$$\phi_2 = -\frac{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau^2\chi_0 & D_\tau^2\chi_1 \end{vmatrix}}{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau\chi_0 & D_\tau\chi_1 \end{vmatrix}}, \ \phi_4 = \frac{\begin{vmatrix} D_\tau\chi_0 & D_\tau\chi_1 \\ D_\tau^2\chi_0 & D_\tau^2\chi_1 \end{vmatrix}}{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau\chi_0 & D_\tau\chi_1 \end{vmatrix}}$$

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- Both ϕ_2, ϕ_4 can have poles wherever the denominator, which we call the Wronskian W, has zeroes.

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- For future reference, the Riemann-Roch theorem gives us the useful relation:

$$\ell = \frac{c}{2} - 6h + 1$$

- Now we fix ℓ to various small values and scan the parameter space to look for solutions that are admissible characters.

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- The single parameter μ completely determines the solutions up to overall normalisations.

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- Let's look at two examples.

- MMS equation with $\mu = -\frac{119}{3600}$ gives admissible characters:

$$\begin{aligned}\chi_0(q) = q^{-7/60}(1 &+ 14q + 42q^2 + 140q^3 + 350q^4 + 850q^5 \\ &+ 1827q^6 + 3858q^7 + 7637q^8 + 14756q^9 + \cdots) \\ \chi_1(q) = q^{17/60}(1 &+ \tfrac{34}{7}q + 17q^2 + 46q^3 + 117q^4 + 266q^5 \\ &+ 575q^6 + 1174q^7 + 2311q^8 + 4380q^9 + \cdots)\end{aligned}$$

$c = \frac{14}{5}, h = \frac{2}{5}$. Normalising second character by 7, it becomes admissible. These characters can be identified with the CFT $\mathsf{G}_{2,1}$.

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- MMS equation with $\mu = -\frac{143}{4800}$ gives non-admissible characters:

$$\begin{aligned}\chi_0(q) &= q^{-13/120}(1 + \tfrac{455}{37}q + \tfrac{121784}{3589}q^2 + \tfrac{60836763}{563473}q^3 + \tfrac{4525367613}{17467663}q^4 \\ &\qquad + \tfrac{2893074116179}{4838542651}q^5 + \tfrac{2046920234847579}{1630588873387}q^6 + \cdots) \\ \chi_1(q) &= q^{11/40}(1 + \tfrac{363}{83}q + \tfrac{15849}{1079}q^2 + \tfrac{90512}{2407}q^3 + \tfrac{58528917}{633041}q^4 \\ &\qquad + \tfrac{128150964}{633041}q^5 + \tfrac{102972265445}{242454703}q^6 + \cdots)\end{aligned}$$

Formally $c = \frac{13}{5}, h = \frac{23}{60}$, but clearly this is not a CFT.

- We found a finite and very interesting set of admissible characters, all with $0 < c < 8$, and guessed their identification with various known RCFT:

m_1	c	h	Identification
1	$\frac{2}{5}$	$\frac{1}{5}$	$c=-\frac{22}{5}$ minimal model $(c \leftrightarrow c-24h)$
3	1	$\frac{1}{4}$	$k=1$ SU(2) WZW model
8	2	$\frac{1}{3}$	$k=1$ SU(3) WZW model
14	$\frac{14}{5}$	$\frac{2}{5}$	$k=1$ G_2 WZW model
28	4	$\frac{1}{2}$	$k=1$ SO(8) WZW model
52	$\frac{26}{5}$	$\frac{3}{5}$	$k=1$ F_4 WZW model
78	6	$\frac{2}{3}$	$k=1$ E_6 WZW model
133	7	$\frac{3}{4}$	$k=1$ E_7 WZW model
190	$\frac{38}{5}$	$\frac{4}{5}$	?
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- This brings together several distinct level-1 KM characters, and a few curious entries that have negative fusion rules. Today I will ignore those (they are now called Intermediate Vertex Operator Algebras or IVOA).

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- This leaves just four theories with $(p, \ell) = (2, 0)$, which we identified with the affine theories:

$$\mathsf{A}_{1,1}, \mathsf{G}_{2,1}, \mathsf{F}_{4,1}, \mathsf{E}_{7,1} \qquad \text{“MMS set”}$$

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- Recently this identification was shown to be unique [Mason-Nagatomo-Sakai 2018].

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Meromorphic cosets and classification

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- These four solutions have central charges $24 - c$ and conformal dimensions $2 - h$ relative to the MMS set. Very suggestive!
- For nearly three decades it remained unclear whether these admissible characters were really CFT's. It was finally resolved in [Gaberdiel-Hampapura-Mukhi 2016].

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- These are defined by embedding the KM algebra of the denominator in that of the numerator, satisfying certain embedding conditions.

- We used a variant of the coset construction of RCFT's [Goddard-Kent-Olive 1984,1985].
- The cosets in the physics literature are typically of the form:
$$\frac{\mathcal{V}_1}{\mathcal{V}_2}$$
where $\mathcal{V}_1, \mathcal{V}_2$ are both WZW theories.
- For example:
$$\frac{\mathsf{A}_{1,k} \otimes \mathsf{A}_{1,1}}{\mathsf{A}_{1,k+1}} = \mathcal{M}(k+2, k+3) \text{ (unitary minimal models)}$$
- These are defined by embedding the KM algebra of the denominator in that of the numerator, satisfying certain embedding conditions.
- One has $c_{\text{coset}} = c_{\text{num}} - c_{\text{denom}}$.

- However the cosets we need are actually simpler. They are cosets of a meromorphic theory $\mathcal{A}$ by an affine theory $\mathcal{V}$:

$$\mathcal{V}' = \frac{\mathcal{A}}{\mathcal{V}}$$

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- Such cosets can sometimes be defined by embedding the KM algebra of the denominator in that of the numerator, but they exist in greater generality.
- Since $\mathcal{A}$ is meromorphic, the denominator $\mathcal{V}$ and the coset $\mathcal{V}'$ are both $^{(p)}$CFTs for the same p and they satisfy a holomorphic bilinear relation:

$$\sum_{i=0}^{p-1} \chi_i^{\mathcal{V}}(q)\chi_i^{\mathcal{V}'}(q) = \chi^{\mathcal{A}}(q)$$

- For example, recall #34 in Schellekens' list with KM algebra $\mathsf{A}_{3,1}\mathsf{D}_{7,3}\mathsf{G}_{2,1}$.

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- We can now take the quotient:

$$\frac{\mathcal{S}_{\#34}}{\mathsf{G}_{2,1}}$$

by deleting $\mathsf{G}_{2,1}$ from the numerator. This leads to a ${}^{(2)}$CFT with algebra $\mathsf{A}_{3,1}\mathsf{D}_{7,3}$ and $c = 24 - \frac{14}{5} = \frac{106}{5}$.

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- Such "deleting" quotients can be shown to have Wronskian index $\ell = 2$. There are 15 such ${}^{(2)}$CFT, with $c = 17, \frac{94}{5}, \frac{106}{5}, 23$, corresponding to the four MLDE solutions.

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- Thus the $\ell = 2$ admissible solutions are all identified with CFTs.

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- Remarkably this procedure is exhaustive: every theory with two primaries arises by taking cosets of meromorphic theories by the MMS set.
- Let's see where this non-trivial statement comes from.

- The idea is that the characters for any two CFTs $\mathcal{V}, \mathcal{V}'$ with mutually conjugate modular representations ϱ, ϱ' satisfy a bilinear relation to a modular invariant (up to a phase):

$$\sum_{i=0}^{p-1} \chi_i^{\mathcal{V}}(q)\chi_i^{\mathcal{V}'}(q) = \chi^{\mathcal{A}}(q)$$

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- If this condition is also sufficient then all $\mathcal{V}'$ transforming in the conjugate representation ϱ' will arise as cosets in this way.

- Now a CFT is associated to a Modular Tensor Category (MTC), that provides the topological information of a CFT: the modular representation ϱ, the braiding and fusing matrices $\mathcal{B}, \mathcal{F}$, central charge mod 8 and a few other data.

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- The rank of an MTC is the number of primaries of the associated CFT.
- If the MTCs for two CFTs are mutually conjugate (i.e. they pair up to the trivial MTC) then it follows that the bilinear relation on the previous page is a relation between CFTs, not just characters.

- Now in rank 2 (and also 3), the MTC associated to a CFT is uniquely specified by its modular representation [Rowell-Stong-Wang 2007]. (Otherwise a given modular representation could correspond to multiple braiding and fusing data, for example).

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- The result then follows: all CFT with exactly two primaries are the MMS set, or cosets of a meromorphic theory by the MMS set.

- So we simply take all meromorphic CFTs of central charge c and construct all possible cosets by a member of the MMS set:

$$\begin{array}{ccccc} & \mathsf{A}_{1,1}, & \mathsf{G}_{2,1}, & \mathsf{F}_{4,1}, & \mathsf{E}_{7,1} \\ c = & 1 & \frac{14}{5} & \frac{26}{5} & 7 \end{array}$$

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- But the classification of "all meromorphic CFTs of central charge c" exists only up to $c = 24$, and beyond that it is impractical.
- So the best we can do is classify all ${}^{(2)}$CFT with $c < 24$. The possible central charges we will get in this way are:

$$c = c_M - 1,\ c_M - \tfrac{14}{5},\ c_M - \tfrac{26}{5},\ c_M - 7$$

with $c_M = 8, 16, 24$. The maximum value is 23.

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- The minimum and maximum central charges of the MMS set are $1, 7$.
- So meromorphic theories at $c_M = 24, 32$ gives maximum/minimum central charges of $23, 25$ respectively.
- Hence there is no ${}^{(2)}\text{CFT}$ with $23 < c < 25$. So we can push our upper limit to 25.

- From the Riemann-Roch theorem:

$$\ell = \frac{c}{2} - 6h + 1$$

unitary theories with $0 < c < 25$ can only have Wronskian index $0, 2, 4, 6, 8, 10, 12$.

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- From the allowed modular representations one can show there are no admissible characters in this range for $\ell = 6, 10, 12$.
- That only leaves $\ell = 4, 8$. These arise from embeddings of the MMS set in meromorphic theories of $c = 16, 24$ respectively.

- Finally one computes all possible such embeddings.

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- The result is a set of $6 + 98$ theories at $\ell = 4, 8$ respectively.
- As an example, for the Schellekens theory with chiral algebra $\mathsf{A}_{3,1}\,\mathsf{D}_{7,3}\,\mathsf{G}_{2,1}$, we can embed $\mathsf{A}_{1,1}$ in two ways:

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- We cannot embed $A_{1,1}$ into $D_{7,3}$ because the level of the embedding algebra has to be $\geq$ the level of the numerator.

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    - Some theories have both non-Abelian and Abelian factors (not the case in Schellekens theories).
    - There are theories with the same c but different conformal dimension h, and also multiple theories with the same (c, h). For example we find:

$$\text{2 theories with } (c,h) = \left(\tfrac{106}{5}, \tfrac{8}{5}\right)$$

$$\text{27 theories with } (c,h) = \left(\tfrac{106}{5}, \tfrac{3}{5}\right)$$

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- Fortunately we can use a result of [Chandra-Mukhi 2018]. We simply insert the critical exponents and the value of ℓ (both easily determined by the coset construction) into the formulae there to get the characters.
- Thus we know exactly the characters and partition function of all the 123 theories. We can also find their correlators using techniques developed in [Mathur-Mukhi-Sen 1988,1989] and [Muralidhara-Mukhi 2018].

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    - This was resolved in [Betsumiya-Lam-Shimakura 2022] and private communication with the authors. One example is:

$$\frac{\mathsf{D}_{6,5}\mathsf{A}_{1,1}\mathsf{A}'_{1,1}}{\mathsf{A}_{1,1}} \neq \frac{\mathsf{D}_{6,5}\mathsf{A}_{1,1}\mathsf{A}'_{1,1}}{\mathsf{A}'_{1,1}}$$

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  - However this happens in just two cases. In the remaining ones, the multiple copies are permuted by outer automorphisms of the algebra and in this case the two embeddings are equivalent.

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- We computed linearly inequivalent embeddings using suitable software. However in some specific cases, linear equivalence does not imply equivalence.
- The complete set of conditions when this can happen were described in [Minchenko 2006]. We were able to check that for all our cases, linear equivalence corresponds to equivalence of embeddings.

Theories

No.	Theory	c	h	ℓ	Subalgebra	d
1	$A_{1,1}$	1	$1/4$	0	$A_{1,1}$	2
2	$G_{2,1}$	$14/5$	$2/5$	0	$G_{2,1}$	7
3	$F_{4,1}$	$26/5$	$3/5$	0	$F_{4,1}$	26
4	$E_{7,1}$	7	$3/4$	0	$E_{7,1}$	56
5	$E_{8,1}A_{1,1}$	9	$1/4$	4	$A_{1,1}E_{8,1}$	2
6	$E_{8,1}G_{2,1}$	$54/5$	$2/5$	4	$G_{2,1}E_{8,1}$	7
7	$F_{4,1}E_{8,1}$	$66/5$	$3/5$	4	$F_{4,1}E_{8,1}$	26
8	$D_{16,1}^{+}/G_{2,1}$	$66/5$	$3/5$	4	$B_{12,1}L_{7/10}$	26
9	$E_{7,1}E_{8,1}$	15	$3/4$	4	$E_{7,1}E_{8,1}$	56
10	$D_{16,1}^{+}/A_{1,1}$	15	$3/4$	4	$D_{14,1}A_{1,1}$	56
11	$\mathbf{S}(D_{10,1}E_{7,1}^2)/(E_{7,1} \hookrightarrow E_{7,1})$	17	$5/4$	2	$D_{10,1}E_{7,1}$	1632
12	$\mathbf{S}(A_{17,1}E_{7,1})/(E_{7,1} \hookrightarrow E_{7,1})$	17	$5/4$	2	$A_{17,1}$	1632
13	$E_{8,1}^2A_{1,1}$	17	$1/4$	8	$E_{8,1}^2A_{1,1}$	2
14	$\mathbf{S}(D_{16,1}E_{8,1})/(E_{7,1} \hookrightarrow E_{8,1})$	17	$1/4$	8	$D_{16,1}A_{1,1}$	2
15	$\mathbf{S}(C_{8,1}F_{4,1}^2)/(F_{4,1} \hookrightarrow F_{4,1})$	$94/5$	$7/5$	2	$C_{8,1}F_{4,1}$	4794
16	$\mathbf{S}(E_{7,2}B_{5,1}F_{4,1})/(F_{4,1} \hookrightarrow F_{4,1})$	$94/5$	$7/5$	2	$E_{7,2}B_{5,1}$	4794
17	$\mathbf{S}(E_{6,1}^4)/F_{4,1}$	$94/5$	$2/5$	8	$E_{6,1}^3L_{4/5}$	1
18	$\mathbf{S}(A_{11,1}D_{7,1}E_{6,1})/(F_{4,1} \hookrightarrow E_{6,1})$	$94/5$	$2/5$	8	$A_{11,1}D_{7,1}L_{4/5}$	1
19	$\mathbf{S}(D_{10,1}E_{7,1}^2)/(F_{4,1} \hookrightarrow E_{7,1})$	$94/5$	$2/5$	8	$D_{10,1}E_{7,1}A_{1,3}$	3
20	$\mathbf{S}(A_{17,1}E_{7,1})/(F_{4,1} \hookrightarrow E_{7,1})$	$94/5$	$2/5$	8	$A_{17,1}A_{1,3}$	3
21	$E_{8,1}^3/F_{4,1} \cong G_{2,1}E_{8,1}^2$	$94/5$	$2/5$	8	$E_{8,1}^2G_{2,1}$	7
22	$\mathbf{S}(D_{16,1}E_{8,1})/(F_{4,1} \hookrightarrow E_{8,1})$	$94/5$	$2/5$	8	$D_{16,1}G_{2,1}$	7
23	$\mathbf{S}(E_{6,3}G_{2,1}^3)/G_{2,1}$	$106/5$	$8/5$	2	$E_{6,3}G_{2,1}^2$	15847
24	$\mathbf{S}(D_{7,3}A_{3,1}G_{2,1})/(G_{2,1} \hookrightarrow G_{2,1})$	$106/5$	$8/5$	2	$D_{7,3}A_{3,1}$	15847
25	$\mathbf{S}(D_{6,2}C_{4,1}B_{3,1}^2)/(G_{2,1} \hookrightarrow B_{3,1})$	$106/5$	$3/5$	8	$D_{6,2}C_{4,1}B_{3,1}L_{7/10}$	1
26	$\mathbf{S}(A_{9,2}A_{4,1}B_{3,1})/(G_{2,1} \hookrightarrow B_{3,1})$	$106/5$	$3/5$	8	$A_{9,2}A_{4,1}L_{7/10}$	1
27	$\mathbf{S}(D_{4,1}^6)/G_{2,1}$	$106/5$	$3/5$	8	$D_{4,1}^5L_{1/2}L_{7/10}$	2
28	$\mathbf{S}(A_{5,1}^4D_{4,1})/(G_{2,1} \hookrightarrow D_{4,1})$	$106/5$	$3/5$	8	$A_{5,1}^4L_{1/2}L_{7/10}$	2
29	$\mathbf{S}(D_{8,2}B_{4,1}^2)/G_{2,1}$	$106/5$	$3/5$	8	$D_{8,2}B_{4,1}U_1L_{7/10}$	3
30	$\mathbf{S}(C_{6,1}^2B_{4,1})/(G_{2,1} \hookrightarrow B_{4,1})$	$106/5$	$3/5$	8	$C_{6,1}^2U_1L_{7/10}$	3
31	$\mathbf{S}(A_{7,1}^2D_{5,1}^2)/(G_{2,1} \hookrightarrow D_{5,1})$	$106/5$	$3/5$	8	$A_{7,1}^2D_{5,1}A_{1,2}L_{7/10}$	4
32	$\mathbf{S}(C_{8,1}F_{4,1}^2)/(G_{2,1} \hookrightarrow F_{4,1})$	$106/5$	$3/5$	8	$C_{8,1}F_{4,1}A_{1,8}$	5
33	$\mathbf{S}(E_{7,2}B_{5,1}F_{4,1})/(G_{2,1} \hookrightarrow B_{5,1})$	$106/5$	$3/5$	8	$E_{7,2}A_{1,1}^2F_{4,1}L_{7/10}$	5
34	$\mathbf{S}(E_{7,2}B_{5,1}F_{4,1})/(G_{2,1} \hookrightarrow F_{4,1})$	$106/5$	$3/5$	8	$E_{7,2}B_{5,1}A_{1,8}$	5
35	$\mathbf{S}(D_{6,1}^4)/G_{2,1}$	$106/5$	$3/5$	8	$D_{6,1}^3B_{2,1}L_{7/10}$	6
36	$\mathbf{S}(A_{9,1}^2D_{6,1})/(G_{2,1} \hookrightarrow D_{6,1})$	$106/5$	$3/5$	8	$A_{9,1}^2B_{2,1}L_{7/10}$	6
37	$\mathbf{S}(C_{10,1}B_{6,1})/(G_{2,1} \hookrightarrow B_{6,1})$	$106/5$	$3/5$	8	$C_{10,1}A_{3,1}L_{7/10}$	7
38	$\mathbf{S}(E_{6,1}^4)/G_{2,1}$	$106/5$	$3/5$	8	$E_{6,1}^3A_{2,2}$	8
39	$\mathbf{S}(A_{11,1}D_{7,1}E_{6,1})/(G_{2,1} \hookrightarrow D_{7,1})$	$106/5$	$3/5$	8	$A_{11,1}B_{3,1}E_{6,1}L_{7/10}$	8
40	$\mathbf{S}(A_{11,1}D_{7,1}E_{6,1})/(G_{2,1} \hookrightarrow E_{6,1})$	$106/5$	$3/5$	8	$A_{11,1}D_{7,1}A_{2,2}$	8
41	$\mathbf{S}(D_{8,1}^3)/G_{2,1}$	$106/5$	$3/5$	8	$D_{8,1}^2B_{4,1}L_{7/10}$	10
42	$\mathbf{S}(E_{8,2}B_{8,1})/G_{2,1}$	$106/5$	$3/5$	8	$E_{8,2}D_{5,1}L_{7/10}$	11
43	$\mathbf{S}(A_{15,1}D_{9,1})/(G_{2,1} \hookrightarrow D_{9,1})$	$106/5$	$3/5$	8	$A_{15,1}B_{5,1}L_{7/10}$	12
44	$\mathbf{S}(D_{10,1}E_{7,1}^2)/(G_{2,1} \hookrightarrow D_{10,1})$	$106/5$	$3/5$	8	$B_{6,1}E_{7,1}^2L_{7/10}$	14
45	$\mathbf{S}(D_{10,1}E_{7,1}^2)/(G_{2,1} \hookrightarrow E_{7,1})$	$106/5$	$3/5$	8	$D_{10,1}E_{7,1}C_{3,1}$	14
46	$\mathbf{S}(A_{17,1}E_{7,1})/(G_{2,1} \hookrightarrow E_{7,1})$	$106/5$	$3/5$	8	$A_{17,1}C_{3,1}$	14
47	$\mathbf{S}(D_{12,1}^2)/G_{2,1}$	$106/5$	$3/5$	8	$D_{12,1}B_{8,1}L_{7/10}$	18
48	$E_{8,1}^2F_{4,1} \cong E_{8,1}^3/G_{2,1}$	$106/5$	$3/5$	8	$E_{8,1}^2F_{4,1}$	26
49	$\mathbf{S}(D_{16,1}E_{8,1})/(G_{2,1} \hookrightarrow D_{16,1})$	$106/5$	$3/5$	8	$B_{12,1}E_{8,1}L_{7/10}$	26
50	$D_{16,1}^{+}F_{4,1} \cong \mathbf{S}(D_{16,1}E_{8,1})/(G_{2,1} \hookrightarrow E_{8,1})$	$106/5$	$3/5$	8	$D_{16,1}F_{4,1}$	26
51	$\mathbf{S}(D_{24,1})/G_{2,1}$	$106/5$	$3/5$	8	$B_{20,1}L_{7/10}$	42
52	$\mathbf{S}(A_{1,1}^{24})/A_{1,1}$	23	$7/4$	2	$A_{1,1}^{23}$	32384
53	$\mathbf{S}(A_{3,2}^4A_{1,1}^4)/A_{1,1}$	23	$7/4$	2	$A_{3,2}^4A_{1,1}^3$	32384
54	$\mathbf{S}(A_{5,3}D_{4,3}A_{1,1}^3)/A_{1,1}$	23	$7/4$	2	$A_{5,3}D_{4,3}A_{1,1}^2$	32384
55	$\mathbf{S}(A_{7,4}A'_{1,1}A_{1,1}^2)/A_{1,1}$	23	$7/4$	2	$A_{7,4}A_{1,1}^2$	32384
56	$\mathbf{S}(A_{7,4}A'_{1,1}A_{1,1}^2)/A'_{1,1}$	23	$7/4$	2	$A_{7,4}A_{1,1}^2$	32384
57	$\mathbf{S}(D_{5,4}C_{3,2}A_{1,1}^2)/A_{1,1}$	23	$7/4$	2	$D_{5,4}C_{3,2}A_{1,1}$	32384
58	$\mathbf{S}(D_{6,5}A_{1,1}A'_{1,1})/A_{1,1}$	23	$7/4$	2	$D_{6,5}A_{1,1}$	32384
59	$\mathbf{S}(D_{6,5}A_{1,1}A'_{1,1})/A'_{1,1}$	23	$7/4$	2	$D_{6,5}A_{1,1}$	32384
60	$\mathbf{S}(C_{5,3}G_{2,2}A_{1,1})/A_{1,1}$	23	$7/4$	2	$C_{5,3}G_{2,2}$	32384
61	$\mathbf{S}(A_{2,1}^{12})/A_{1,1}$	23	$3/4$	8	$A_{2,1}^{11}U_1$	2
62	$\mathbf{S}(D_{4,2}^2B_{2,1}^4)/A_{1,1}$	23	$3/4$	8	$D_{4,2}^2C_{2,1}^3A_{1,1}$	2
63	$\mathbf{S}(A_{5,2}^2B_{2,1}A_{2,1}^2)/(A_{1,1} \hookrightarrow A_{2,1})$	23	$3/4$	8	$A_{5,2}^2C_{2,1}A_{2,1}U_1$	2
64	$\mathbf{S}(A_{5,2}^2B_{2,1}A_{2,1}^2)/(A_{1,1} \hookrightarrow B_{2,1})$	23	$3/4$	8	$A_{5,2}^2A_{1,1}A_{2,1}^2$	2
65	$\mathbf{S}(A_{8,3}A_{2,1}^2)/A_{1,1}$	23	$3/4$	8	$A_{8,3}A_{2,1}U_1$	2
66	$\mathbf{S}(E_{6,4}C_{2,1}A_{2,1})/(A_{1,1} \hookrightarrow B_{2,1})$	23	$3/4$	8	$E_{6,4}A_{1,1}A_{2,1}$	2
67	$\mathbf{S}(E_{6,4}C_{2,1}A_{2,1})/(A_{1,1} \hookrightarrow A_{2,1})$	23	$3/4$	8	$E_{6,4}C_{2,1}U_1$	2
68	$\mathbf{S}(A_{3,1}^8)/A_{1,1}$	23	$3/4$	8	$A_{3,1}^7A_{1,1}U_1$	4
69	$\mathbf{S}(D_{5,2}^2A_{3,1}^2)/A_{1,1}$	23	$3/4$	8	$D_{5,2}^2A_{3,1}A_{1,1}U_1$	4
70	$\mathbf{S}(E_{6,3}G_{2,1}^3)/A_{1,1}$	23	$3/4$	8	$E_{6,3}G_{2,1}^2A_{1,3}$	4
71	$\mathbf{S}(A_{7,2}C_{3,1}^2A_{3,1})/(A_{1,1} \hookrightarrow A_{3,1})$	23	$3/4$	8	$A_{7,2}C_{3,1}^2A_{1,1}U_1$	4
72	$\mathbf{S}(A_{7,2}C_{3,1}^2A_{3,1})/(A_{1,1} \hookrightarrow C_{3,1})$	23	$3/4$	8	$A_{7,2}C_{3,1}B_{2,1}A_{3,1}$	4
73	$\mathbf{S}(D_{7,3}A_{3,1}G_{2,1})/(A_{1,1} \hookrightarrow G_{2,1})$	23	$3/4$	8	$D_{7,3}A_{3,1}A_{1,3}$	4
74	$\mathbf{S}(D_{7,3}A_{3,1}G_{2,1})/(A_{1,1} \hookrightarrow A_{3,1})$	23	$3/4$	8	$D_{7,3}G_{2,1}A_{1,1}U_1$	4
75	$\mathbf{S}(C_{7,2}A_{3,1})/A_{1,1}$	23	$3/4$	8	$C_{7,2}A_{1,1}U_1$	4
76	$\mathbf{S}(A_{4,1}^6)/A_{1,1}$	23	$3/4$	8	$A_{4,1}^5A_{2,1}U_1$	6
77	$\mathbf{S}(C_{4,1}^4)/A_{1,1}$	23	$3/4$	8	$C_{4,1}^3C_{3,1}$	6
78	$\mathbf{S}(D_{6,2}C_{4,1}B_{3,1}^2)/(A_{1,1} \hookrightarrow C_{4,1})$	23	$3/4$	8	$D_{6,2}C_{3,1}B_{3,1}^2$	6
79	$\mathbf{S}(D_{6,2}C_{4,1}B_{3,1}^2)/(A_{1,1} \hookrightarrow B_{3,1})$	23	$3/4$	8	$D_{6,2}C_{4,1}B_{3,1}A_{1,2}A_{1,1}$	6
80	$\mathbf{S}(A_{9,2}A_{4,1}B_{3,1})/(A_{1,1} \hookrightarrow A_{4,1})$	23	$3/4$	8	$A_{9,2}A_{2,1}B_{3,1}U_1$	6
81	$\mathbf{S}(A_{9,2}A_{4,1}B_{3,1})/(A_{1,1} \hookrightarrow B_{3,1})$	23	$3/4$	8	$A_{9,2}A_{4,1}A_{1,2}A_{1,1}$	6
82	$\mathbf{S}(D_{4,1}^6)/A_{1,1}$	23	$3/4$	8	$D_{4,1}^5A_{1,1}A_{1,1}A_{1,1}$	8

No.	Theory	c	h	ℓ	Subalgebra	d
83	$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{5,1})$	23	$3/4$	8	$\mathsf{A}_{5,1}^3\mathsf{A}_{3,1}\mathsf{D}_{4,1}\mathsf{U}_1$	8
84	$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{4,1})$	23	$3/4$	8	$\mathsf{A}_{5,1}^4\mathsf{A}_{1,1}^3$	8
85	$\mathbf{S}(\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{5,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{5,1})$	23	$3/4$	8	$\mathsf{E}_{6,2}\mathsf{C}_{4,1}\mathsf{A}_{5,1}$	8
86	$\mathbf{S}(\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{5,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{5,1})$	23	$3/4$	8	$\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{3,1}\mathsf{U}_1$	8
87	$\mathbf{S}(\mathsf{E}_{7,3}\mathsf{A}_{5,1})/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{E}_{7,3}\mathsf{A}_{3,1}\mathsf{U}_1$	8
88	$\mathbf{S}(\mathsf{A}_{6,1}^4)/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{A}_{6,1}^3\mathsf{A}_{4,1}\mathsf{U}_1$	10
89	$\mathbf{S}(\mathsf{D}_{8,2}\mathsf{B}_{4,1}^2)/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{D}_{8,2}\mathsf{B}_{4,1}\mathsf{B}_{2,1}\mathsf{A}_{1,1}$	10
90	$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{6,1})$	23	$3/4$	8	$\mathsf{C}_{6,1}\mathsf{C}_{5,1}\mathsf{B}_{4,1}$	10
91	$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{4,1})$	23	$3/4$	8	$\mathsf{C}_{6,1}^2\mathsf{B}_{2,1}\mathsf{A}_{1,1}$	10
92	$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{7,1})$	23	$3/4$	8	$\mathsf{A}_{7,1}\mathsf{A}_{5,1}\mathsf{D}_{5,1}^2\mathsf{U}_1$	12
93	$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{5,1})$	23	$3/4$	8	$\mathsf{A}_{7,1}^2\mathsf{A}_{3,1}\mathsf{A}_{1,1}\mathsf{D}_{5,1}$	12
94	$\mathbf{S}(\mathsf{D}_{9,2}\mathsf{A}_{7,1})/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{D}_{9,2}\mathsf{A}_{5,1}\mathsf{U}_1$	12
95	$\mathbf{S}(\mathsf{A}_{8,1}^3)/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{A}_{8,1}^2\mathsf{A}_{6,1}\mathsf{U}_1$	14
96	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{8,1})$	23	$3/4$	8	$\mathsf{C}_{7,1}\mathsf{F}_{4,1}^2$	14
97	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{F}_{4,1})$	23	$3/4$	8	$\mathsf{C}_{8,1}\mathsf{F}_{4,1}\mathsf{C}_{3,1}$	14
98	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{5,1})$	23	$3/4$	8	$\mathsf{E}_{7,2}\mathsf{B}_{3,1}\mathsf{A}_{1,1}\mathsf{F}_{4,1}$	14
99	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{F}_{4,1})$	23	$3/4$	8	$\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{C}_{3,1}$	14
100	$\mathbf{S}(\mathsf{D}_{6,1}^4)/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{D}_{6,1}^3\mathsf{D}_{4,1}\mathsf{A}_{1,1}$	16
101	$\mathbf{S}(\mathsf{A}_{9,1}^2\mathsf{D}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{9,1})$	23	$3/4$	8	$\mathsf{A}_{9,1}\mathsf{A}_{7,1}\mathsf{D}_{6,1}\mathsf{U}_1$	16
102	$\mathbf{S}(\mathsf{A}_{9,1}^2\mathsf{D}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{6,1})$	23	$3/4$	8	$\mathsf{A}_{9,1}^2\mathsf{D}_{4,1}\mathsf{A}_{1,1}$	16
103	$\mathbf{S}(\mathsf{C}_{10,1}\mathsf{B}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{10,1})$	23	$3/4$	8	$\mathsf{C}_{9,1}\mathsf{B}_{6,1}$	18
104	$\mathbf{S}(\mathsf{C}_{10,1}\mathsf{B}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{6,1})$	23	$3/4$	8	$\mathsf{C}_{10,1}\mathsf{B}_{4,1}\mathsf{A}_{1,1}$	18
105	$\mathbf{S}(\mathsf{E}_{6,1}^4)/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{E}_{6,1}^3\mathsf{A}_{5,1}$	20
106	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{11,1})$	23	$3/4$	8	$\mathsf{A}_{9,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1}\mathsf{U}_1$	20
107	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{7,1})$	23	$3/4$	8	$\mathsf{A}_{11,1}\mathsf{D}_{5,1}\mathsf{A}_{1,1}\mathsf{E}_{6,1}$	20
108	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{6,1})$	23	$3/4$	8	$\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{A}_{5,1}$	20
109	$\mathbf{S}(\mathsf{A}_{12,1}^2)/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{A}_{12,1}\mathsf{A}_{10,1}\mathsf{U}_1$	22
110	$\mathbf{S}(\mathsf{D}_{8,1}^3)/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{D}_{8,1}^2\mathsf{D}_{6,1}\mathsf{A}_{1,1}$	24
111	$\mathbf{S}(\mathsf{E}_{8,2}\mathsf{B}_{8,1})/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{E}_{8,2}\mathsf{B}_{6,1}\mathsf{A}_{1,1}$	26
112	$\mathbf{S}(\mathsf{A}_{15,1}\mathsf{D}_{9,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{15,1})$	23	$3/4$	8	$\mathsf{A}_{13,1}\mathsf{D}_{9,1}\mathsf{U}_1$	28
113	$\mathbf{S}(\mathsf{A}_{15,1}\mathsf{D}_{9,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{9,1})$	23	$3/4$	8	$\mathsf{A}_{15,1}\mathsf{D}_{7,1}\mathsf{A}_{1,1}$	28
114	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{10,1})$	23	$3/4$	8	$\mathsf{D}_{8,1}\mathsf{A}_{1,1}\mathsf{E}_{7,1}^2$	32
115	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{7,1})$	23	$3/4$	8	$\mathsf{D}_{10,1}\mathsf{E}_{7,1}\mathsf{D}_{6,1}$	32
116	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{17,1})$	23	$3/4$	8	$\mathsf{A}_{15,1}\mathsf{E}_{7,1}\mathsf{U}_1$	32
117	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{7,1})$	23	$3/4$	8	$\mathsf{A}_{17,1}\mathsf{D}_{6,1}$	32
118	$\mathbf{S}(\mathsf{D}_{12,1}^2)/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{D}_{12,1}\mathsf{D}_{10,1}\mathsf{A}_{1,1}$	40
119	$\mathbf{S}(\mathsf{A}_{24,1})/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{A}_{22,1}\mathsf{U}_1$	46
120	$\mathsf{E}_{8,1}^2\mathsf{E}_{7,1}\cong\mathsf{E}_{8,1}^3/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{E}_{8,1}^2\mathsf{E}_{7,1}$	56
121	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{16,1})$	23	$3/4$	8	$\mathsf{D}_{14,1}\mathsf{A}_{1,1}\mathsf{E}_{8,1}$	56
122	$\mathsf{D}_{16,1}^+\mathsf{E}_{7,1}\cong\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{8,1})$	23	$3/4$	8	$\mathsf{D}_{16,1}\mathsf{E}_{7,1}$	56
123	$\mathbf{S}(\mathsf{D}_{24,1})/\mathsf{A}_{1,1}$	23	$3/4$	8	$\mathsf{D}_{22,1}\mathsf{A}_{1,1}$	88

A close-up of a few entries:

$\mathbf{S}(\mathsf{E}_{6,3}\mathsf{G}_{2,1}^3)/\mathsf{G}_{2,1}$	$^{106}/_5$	$^8/_5$	2	$\mathsf{E}_{6,3}\mathsf{G}_{2,1}^2$	15847
$\mathbf{S}(\mathsf{D}_{7,3}\mathsf{A}_{3,1}\mathsf{G}_{2,1})/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{G}_{2,1})$	$^{106}/_5$	$^8/_5$	2	$\mathsf{D}_{7,3}\mathsf{A}_{3,1}$	15847
$\mathbf{S}(\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}^2)/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{B}_{3,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}\mathsf{L}_{7/10}$	1
$\mathbf{S}(\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{B}_{3,1})/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{B}_{3,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{L}_{7/10}$	1
$\mathbf{S}(\mathsf{D}_{4,1}^6)/\mathsf{G}_{2,1}$	$^{106}/_5$	$^3/_5$	8	$\mathsf{D}_{4,1}^5\mathsf{L}_{1/2}\mathsf{L}_{7/10}$	2
$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{D}_{4,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{A}_{5,1}^4\mathsf{L}_{1/2}\mathsf{L}_{7/10}$	2
$\mathbf{S}(\mathsf{D}_{8,2}\mathsf{B}_{4,1}^2)/\mathsf{G}_{2,1}$	$^{106}/_5$	$^3/_5$	8	$\mathsf{D}_{8,2}\mathsf{B}_{4,1}\mathsf{U}_1\mathsf{L}_{7/10}$	3
$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{B}_{4,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{C}_{6,1}^2\mathsf{U}_1\mathsf{L}_{7/10}$	3
$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{D}_{5,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}\mathsf{A}_{1,2}\mathsf{L}_{7/10}$	4
$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{F}_{4,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{C}_{8,1}\mathsf{F}_{4,1}\mathsf{A}_{1,8}$	5

Three character case, in brief

- The three-character case has been studied for $\ell = 0$ in several papers [Mathur-Mukhi-Sen 1989, Hampapura-Mukhi 2015, Gaberdiel-Hampapura-Mukhi 2016, Franc-Mason 2020, Mukhi-Poddar-Singh 2020] but very little is known for $\ell > 0$.

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- Following this, in [Das-Gowdigere-Mukhi 2022b] we were finally able to identify all the actual CFT within this set.
- This completes the classification of three-character CFT with vanishing Wronskian index (no restriction on central charge).

#	c	(h_1, h_2)	m_1	$\mathcal{W}$	Chiral Algebra	#(primaries)
1.	$\frac{2r+1}{2}$	$(\frac{1}{2}, \frac{2r+1}{16})$	$2r^2+r$	**I**	$B_{r,1}$	3
2.	r	$(\frac{1}{2}, \frac{r}{8})$	$2r^2-r$	**I**	$D_{r,1}$ $(r \neq 8, 16)$	4
3.	$\frac{12}{5}$	$(\frac{1}{5}, \frac{3}{5})$	6	$\mathbf{III_2}$	$\mathcal{E}_3[A_{1,8}]$	4
4.	4	$(\frac{2}{5}, \frac{3}{5})$	24	**I**	$A_{4,1}$	5
5.	$\frac{28}{5}$	$(\frac{2}{5}, \frac{4}{5})$	28	**I**	$G_{2,1}^{\otimes 2}$	4
6.	$\frac{52}{5}$	$(\frac{3}{5}, \frac{6}{5})$	104	**I**	$F_{2,1}^{\otimes 2}$	4
7.	12	$(\frac{2}{3}, \frac{4}{3})$	156	**I**	$E_{6,1}^{\otimes 2}$	9
8.	$\frac{68}{5}$	$(\frac{4}{5}, \frac{7}{5})$	136	$\mathbf{III_{22}}$	$\mathcal{E}_3[C_{8,1}]$	4
9.	14	$(\frac{3}{4}, \frac{3}{2})$	266	**I**	$E_{7,1}^{\otimes 2}$	4
10.	15	$(\frac{7}{8}, \frac{3}{2})$	255	GHM_{255}	$\mathcal{E}_3[A_{15,1}]$	4
11.	$\frac{31}{2}$	$(\frac{15}{16}, \frac{3}{2})$	248	**I**	$E_{8,2}$	3
12.	17	$(\frac{9}{8}, \frac{3}{2})$	221	GHM_{221}	$\mathcal{E}_3[A_{11,1}E_{6,1}]$	4
13.	$\frac{35}{2}$	$(\frac{19}{16}, \frac{3}{2})$	210	GHM_{210}	$\mathcal{E}_3[C_{10,1}]$	3
14.	18	$(\frac{5}{4}, \frac{3}{2})$	198	GHM_{198}	$\mathcal{E}_3[D_{6,1}^{\otimes 3}]$	4
15.					$\mathcal{E}_3[A_{9,1}^{\otimes 2}]$	4
16.	$\frac{92}{5}$	$(\frac{6}{5}, \frac{8}{5})$	92	$\mathbf{III_{37}}$	$\mathcal{E}_3[E_{6,3}G_{2,1}]$	4
17.	$\frac{37}{2}$	$(\frac{21}{16}, \frac{3}{2})$	185	GHM_{185}	$\mathcal{E}_3[E_{7,2}F_{4,1}]$	3
18.	19	$(\frac{11}{8}, \frac{3}{2})$	171	GHM_{171}	$\mathcal{E}_3[A_{7,1}^{\otimes 2}D_{5,1}]$	4
19.	$\frac{39}{2}$	$(\frac{23}{16}, \frac{3}{2})$	156	GHM_{156}	$\mathcal{E}_3[B_{4,1}D_{8,2}]$	3
20.					$\mathcal{E}_3[C_{6,1}^{\otimes 2}]$	3
21.	20	$(\frac{4}{3}, \frac{5}{3})$	80	$\mathbf{V_{39}}$	$\mathcal{E}_3[A_{2,1}^{\otimes 10}]$	9
22.					$\mathcal{E}_3[A_{5,2}^{\otimes 2}C_{2,1}]$	9
23.					$\mathcal{E}_3[A_{8,3}]$	9

24.	20	$(\frac{7}{5}, \frac{8}{5})$	120	GHM_{120}	$\mathcal{E}_3[A_{4,1}^{\otimes 5}]$	5
25.					$\mathcal{E}_3[A_{9,2}B_{3,1}]$	5
26.	$\frac{41}{2}$	$(\frac{3}{2}, \frac{25}{16})$	123	GHM_{123}	$\mathcal{E}_3[D_{6,2}C_{4,1}B_{3,1}]$	3
27.					$\mathcal{E}_3[A_{9,2}A_{4,1}]$	3
28.	21	$(\frac{3}{2}, \frac{13}{8})$	105	GHM_{105}	$\mathcal{E}_3[A_{3,1}^{\otimes 7}]$	4
29.					$\mathcal{E}_3[A_{3,1}D_{5,2}^{\otimes 2}]$	4
30.					$\mathcal{E}_3[A_{7,2}C_{3,1}^{\otimes 2}]$	4
31.					$\mathcal{E}_3[D_{7,3}G_{2,1}]$	4
32.					$\mathcal{E}_3[C_{7,2}]$	4
33.	$\frac{43}{2}$	$(\frac{3}{2}, \frac{27}{16})$	86	GHM_{86}	$\mathcal{E}_3[C_{2,1}^{\otimes 3}D_{4,2}^{\otimes 2}]$	3
34.					$\mathcal{E}_3[A_{5,2}^{\otimes 2}A_{2,1}^{\otimes 2}]$	3
35.					$\mathcal{E}_3[A_{2,1}E_{6,4}]$	3
36.	22	$(\frac{3}{2}, \frac{7}{4})$	66	$\mathbf{III_{45}}$	$\mathcal{E}_3[A_{1,1}^{\otimes 22}]$,	4
37.					$\mathcal{E}_3[A_{3,2}^{\otimes 4}A_{1,1}^{\otimes 2}]$	4
38.					$\mathcal{E}_3[A_{5,3}D_{4,3}A_{1,1}]$	4
39.					$\mathcal{E}_3[A_{7,4}A_{1,1}]$	4
40.					$\mathcal{E}_3[D_{5,4}C_{3,2}]$	4
41.					$\mathcal{E}_3[D_{6,5}]$	4
42.	$\frac{45}{2}$	$(\frac{3}{2}, \frac{29}{16})$	45	GHM_{45}	$\mathcal{E}_3[A_{1,2}^{\otimes 15}]$	3
43.					$\mathcal{E}_3[A_{3,4}^{\otimes 3}]$	3
44.					$\mathcal{E}_3[A_{5,6}C_{2,3}]$	3
45.					$\mathcal{E}_3[D_{5,8}]$	3
46.	23	$(\frac{3}{2}, \frac{15}{8})$	23	$\mathbf{III_{50}}$	$\mathcal{E}_3[D_{1,1}^{\otimes 23}]$	4
47.	$\frac{47}{2}$	$(\frac{3}{2}, \frac{31}{16})$	0	**IV**	Baby Monster	3

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- In this way we wrote down entire families of new (non-lattice) meromorphic CFT at $c = 8N$ for arbitrarily large N.

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Discussion

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- Relation to penumbral moonshine – relation between VVMF's and certain types of finite groups [Duncan-Harvey-Rayhaun 2021].

Thank you