Wave-optics limit of the stochastic gravitational wave background

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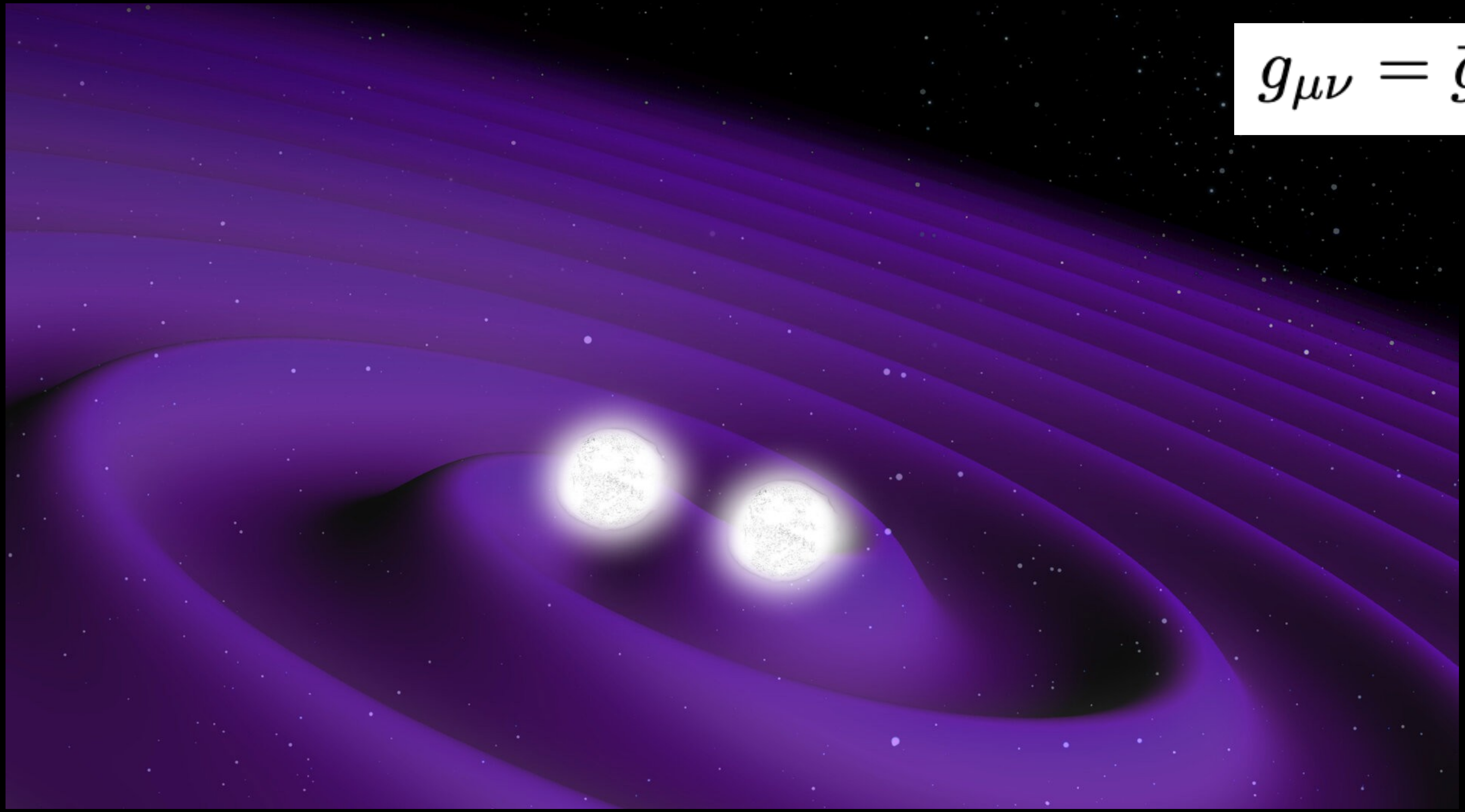
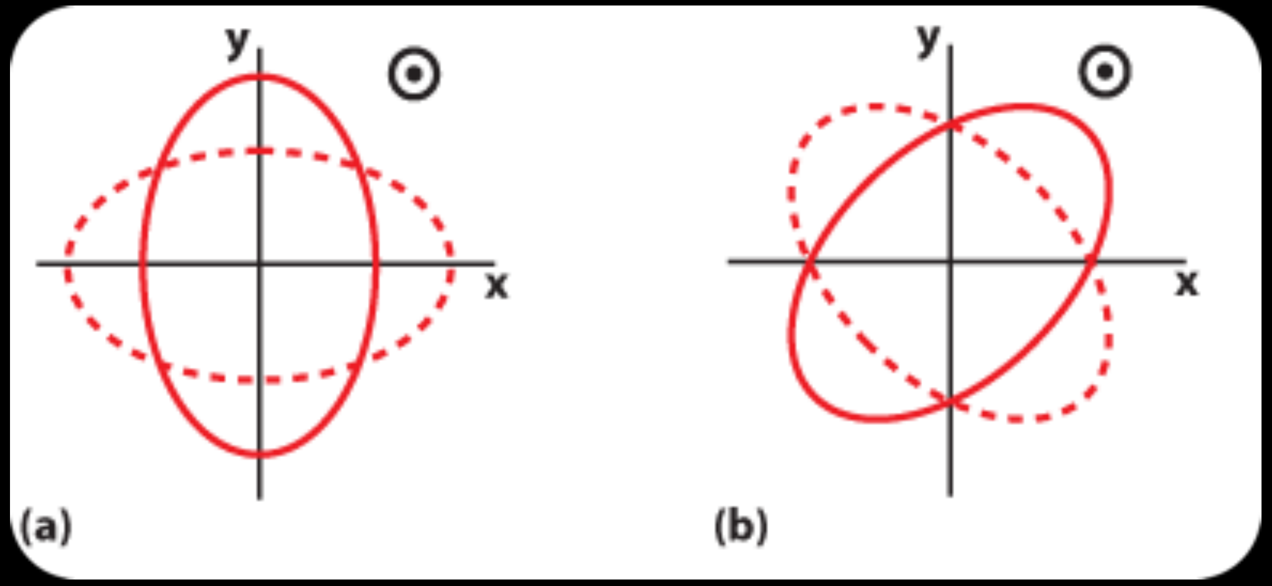
Gravitational waves: a historical detection



$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \alpha\, h_{\mu\nu}$$



We cannot put the Universe in a lab.
Since 2015 entirely new window onto it!

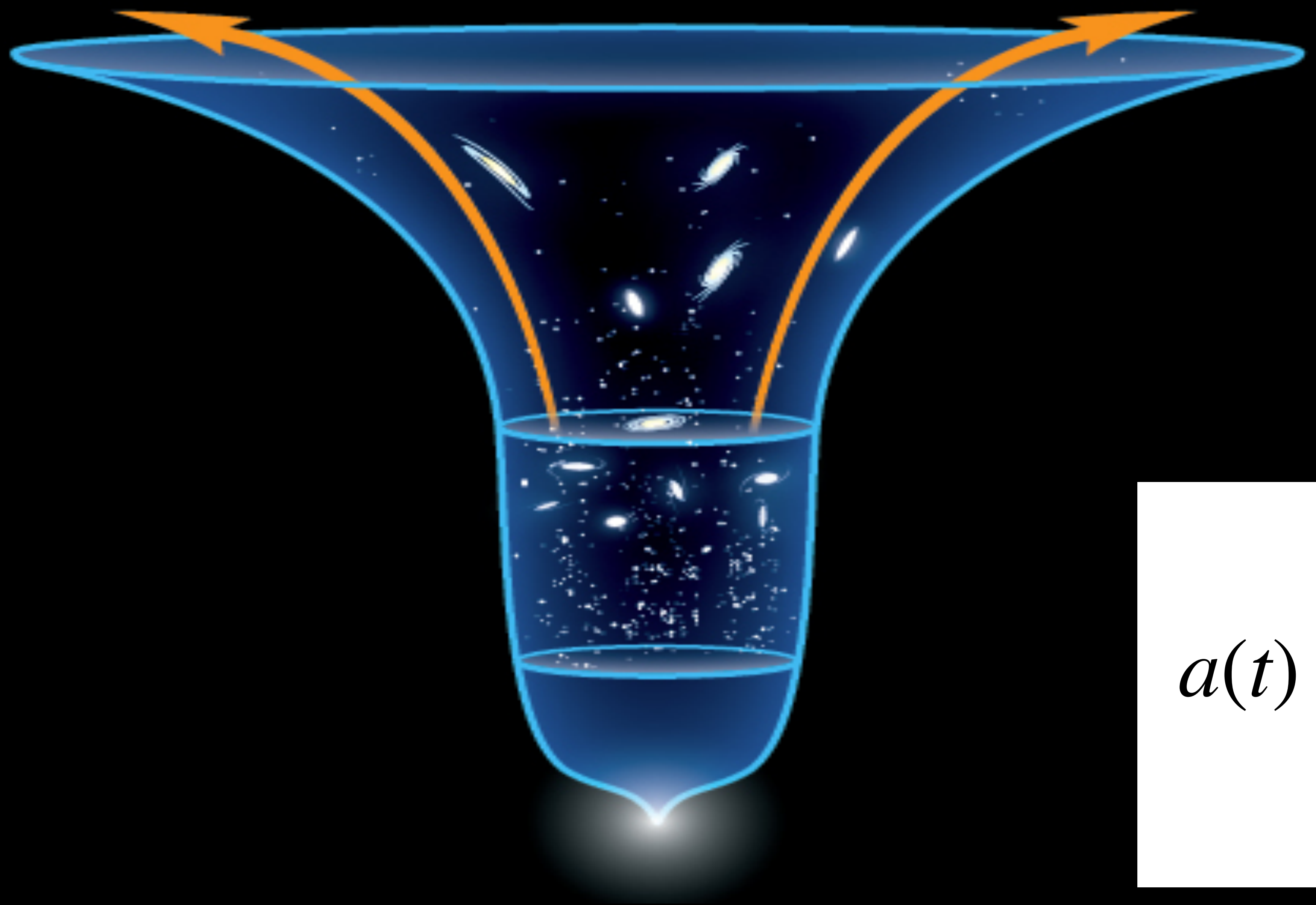
Cosmology in 1 slide

On very large scales ($\gtrsim 100$ Mpc) the Universe is homogeneous and isotropic

$$ds^2 = a^2(\eta)\,\left[-d\eta^2 + d\mathbf{x}^2\right] \quad -2\,\epsilon\, a^2(\eta)\,\phi(x)\,\left[d\eta^2 + d\mathbf{x}^2\right]$$

$$\epsilon \sim 10^{-5}$$

Scale factor

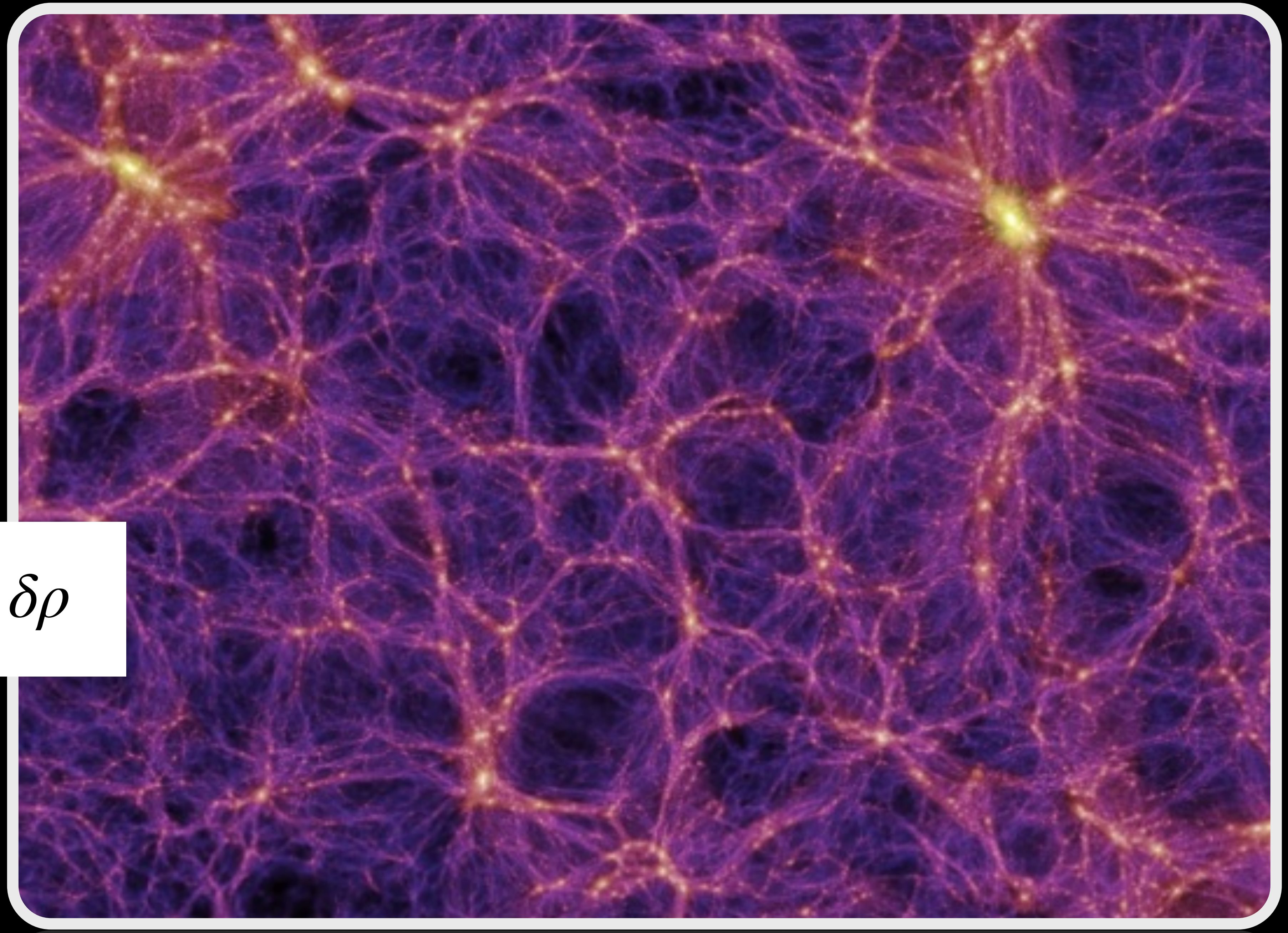


dark matter gravitational potential wells

Einstein Equations
+
. Radiation
. Dark matter
. Λ

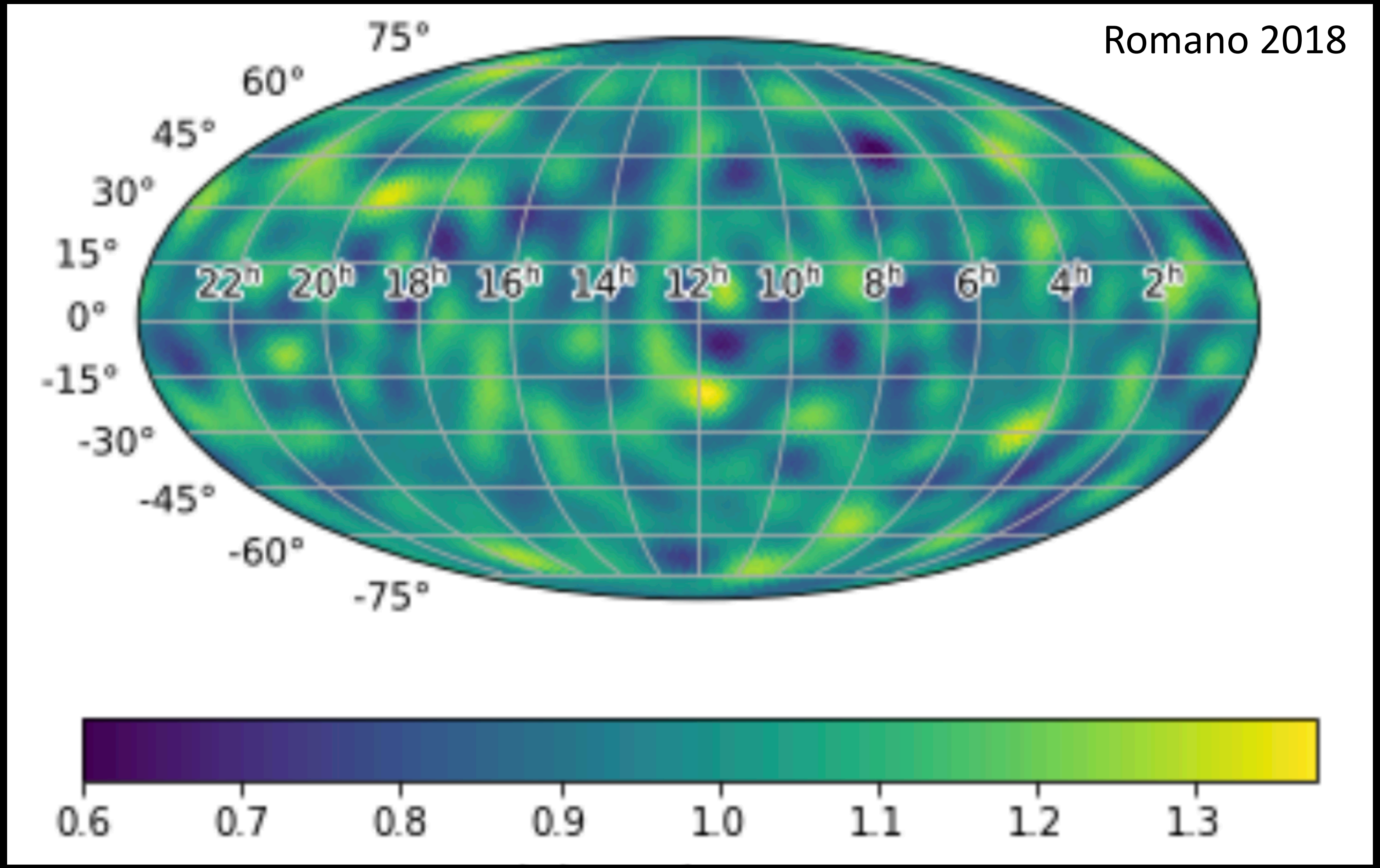
$$a(t) = \begin{cases} \propto t^{1/2} \\ \propto t^{2/3} \\ \propto e^{\sqrt{\Lambda/3}t} \end{cases}$$

$$\nabla^2\phi = 4\pi G a^2 \bar{\rho}\,\delta\rho$$



Goal: primordial fluctuations, growth of cosmic structures, expansion history

Talk's plan:

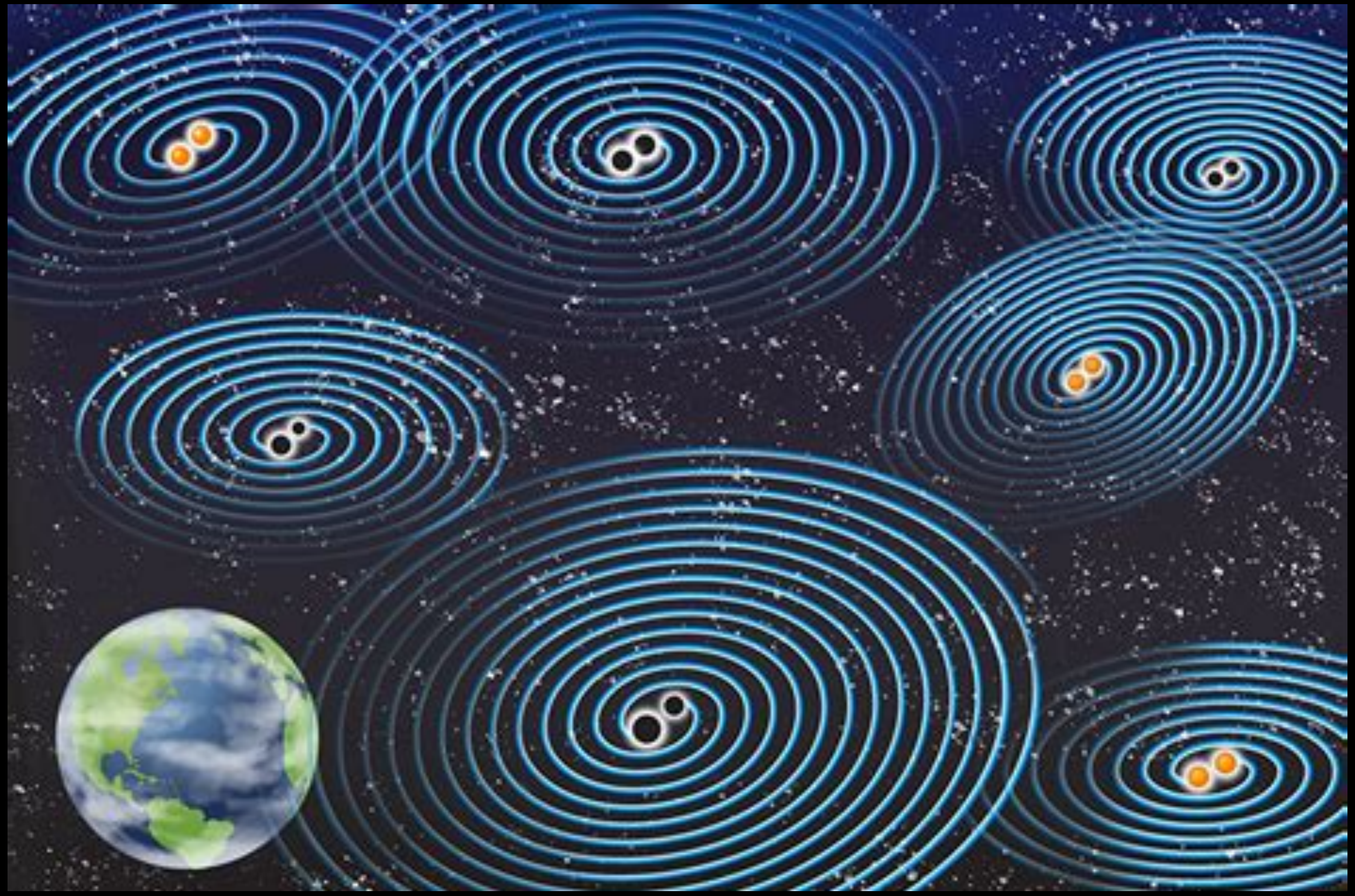


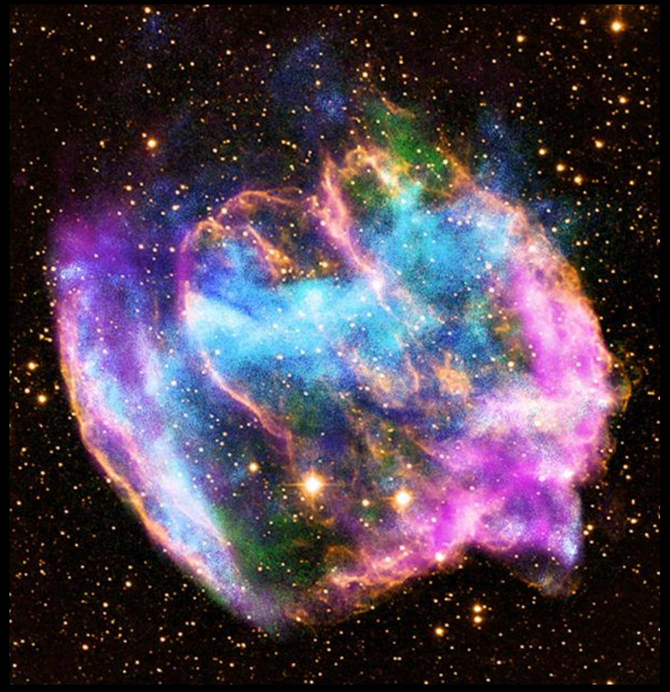
- Introduce SGWB
- Wave optics effects as a tool to investigate cosmology

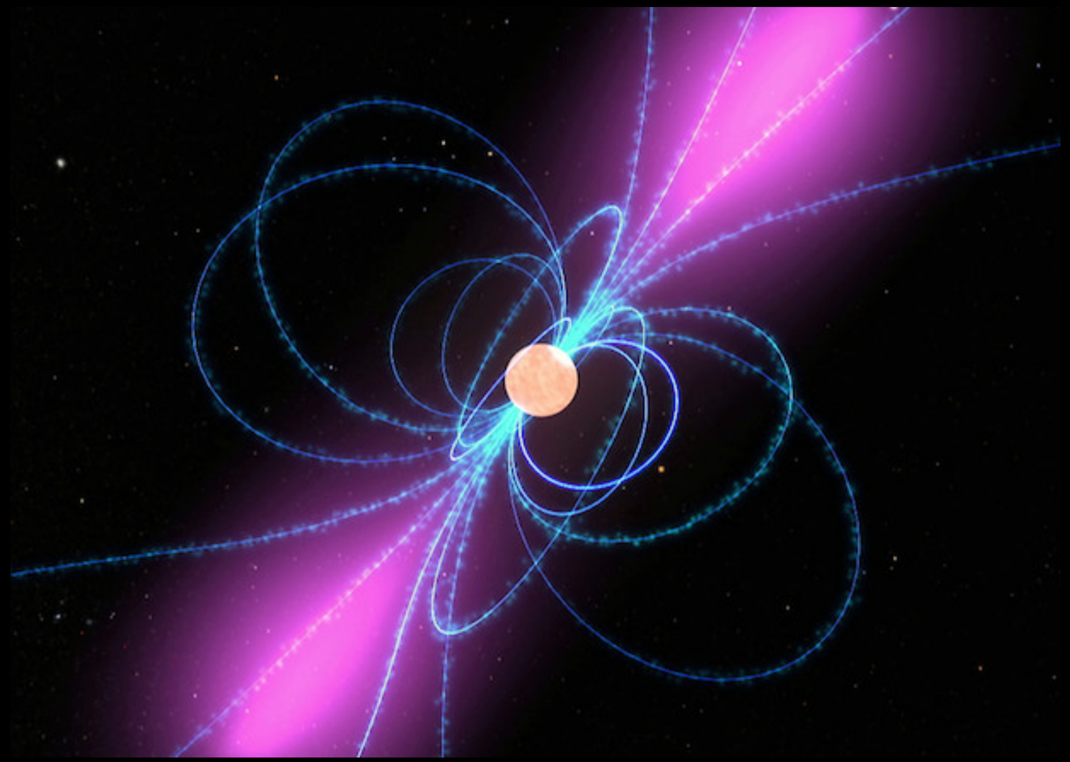
Take home message:
Take this as a new investigation tool

SGWB sources

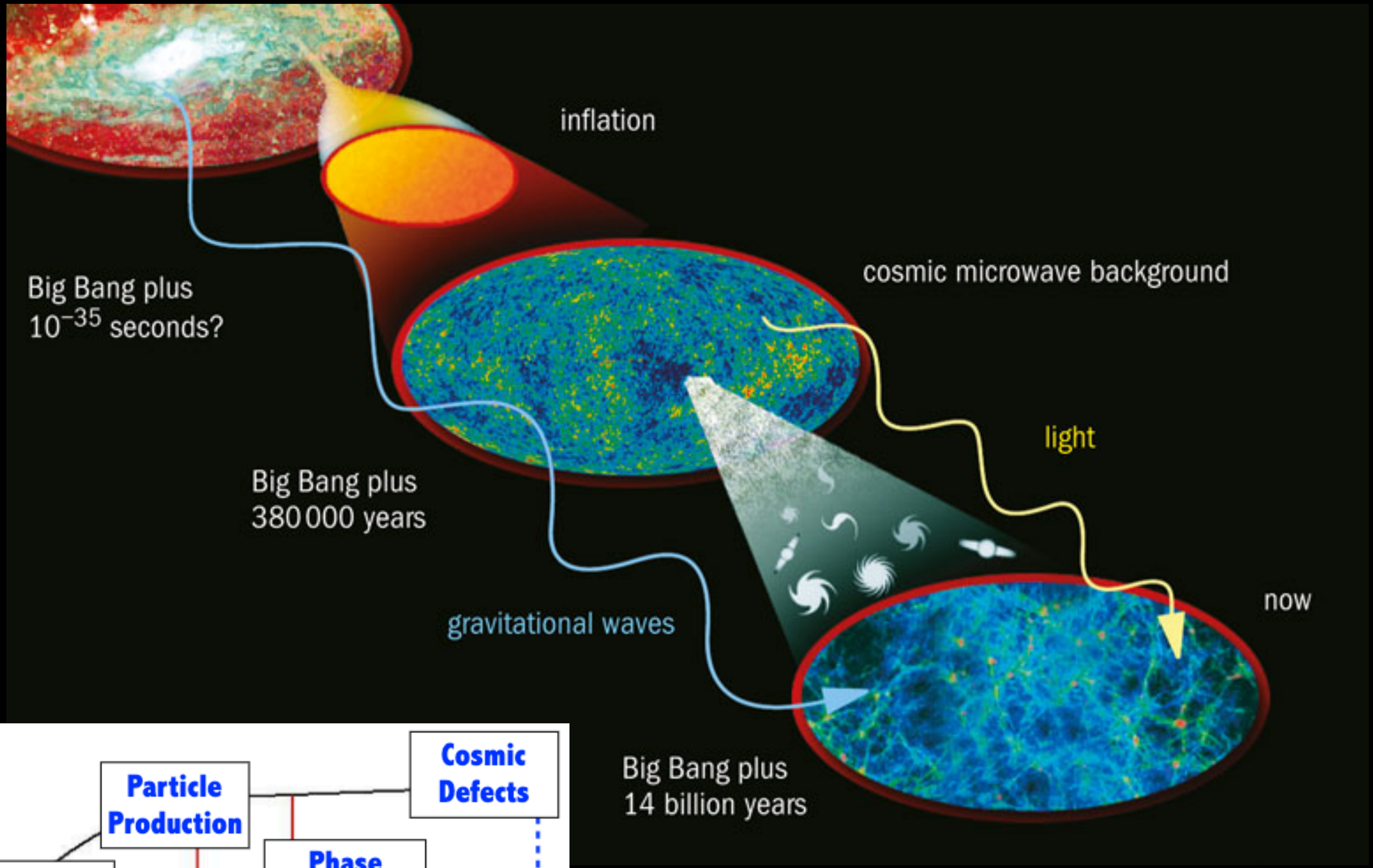
Astrophysical SGWB:
Incoherent superposition of many unresolved GW from astrophysical sources

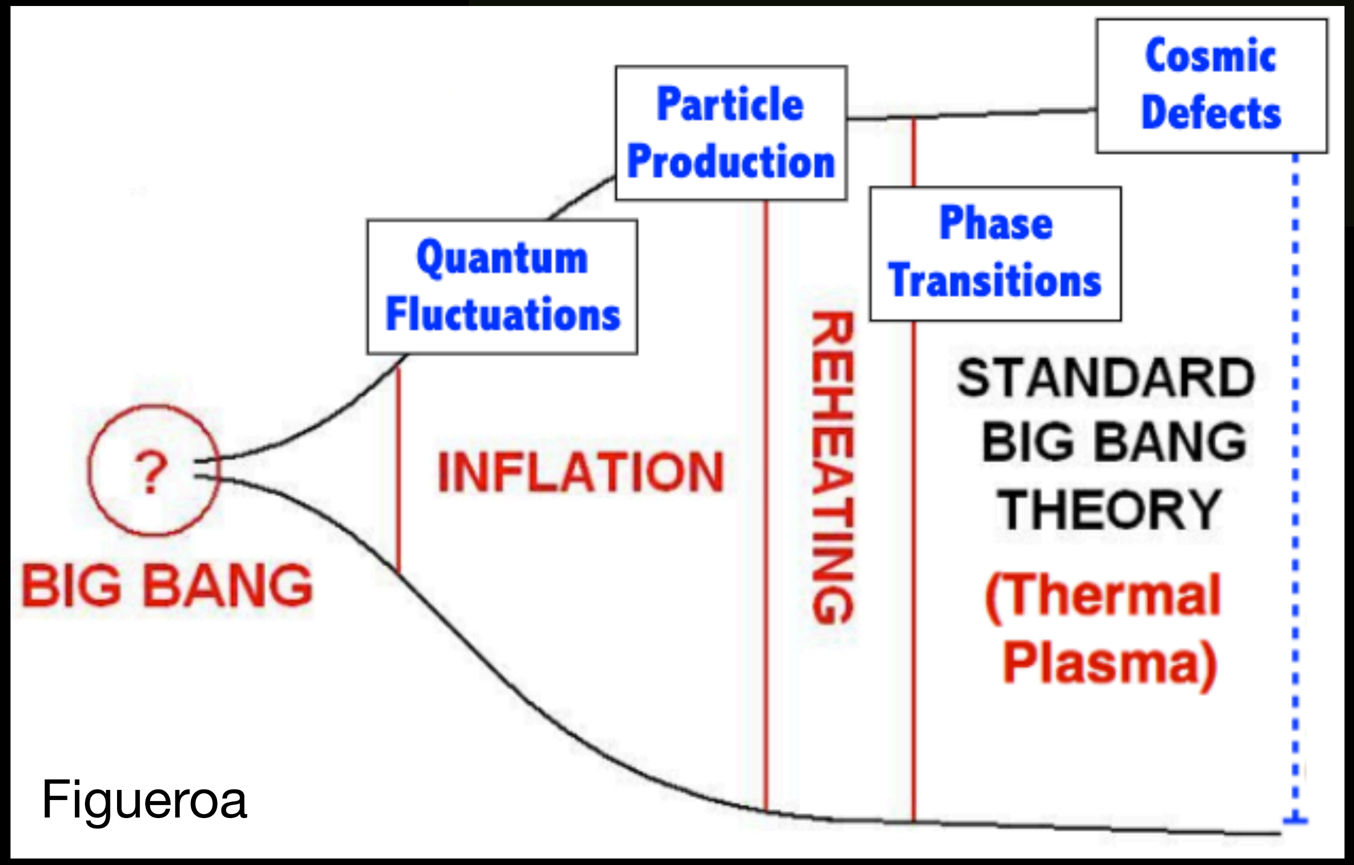




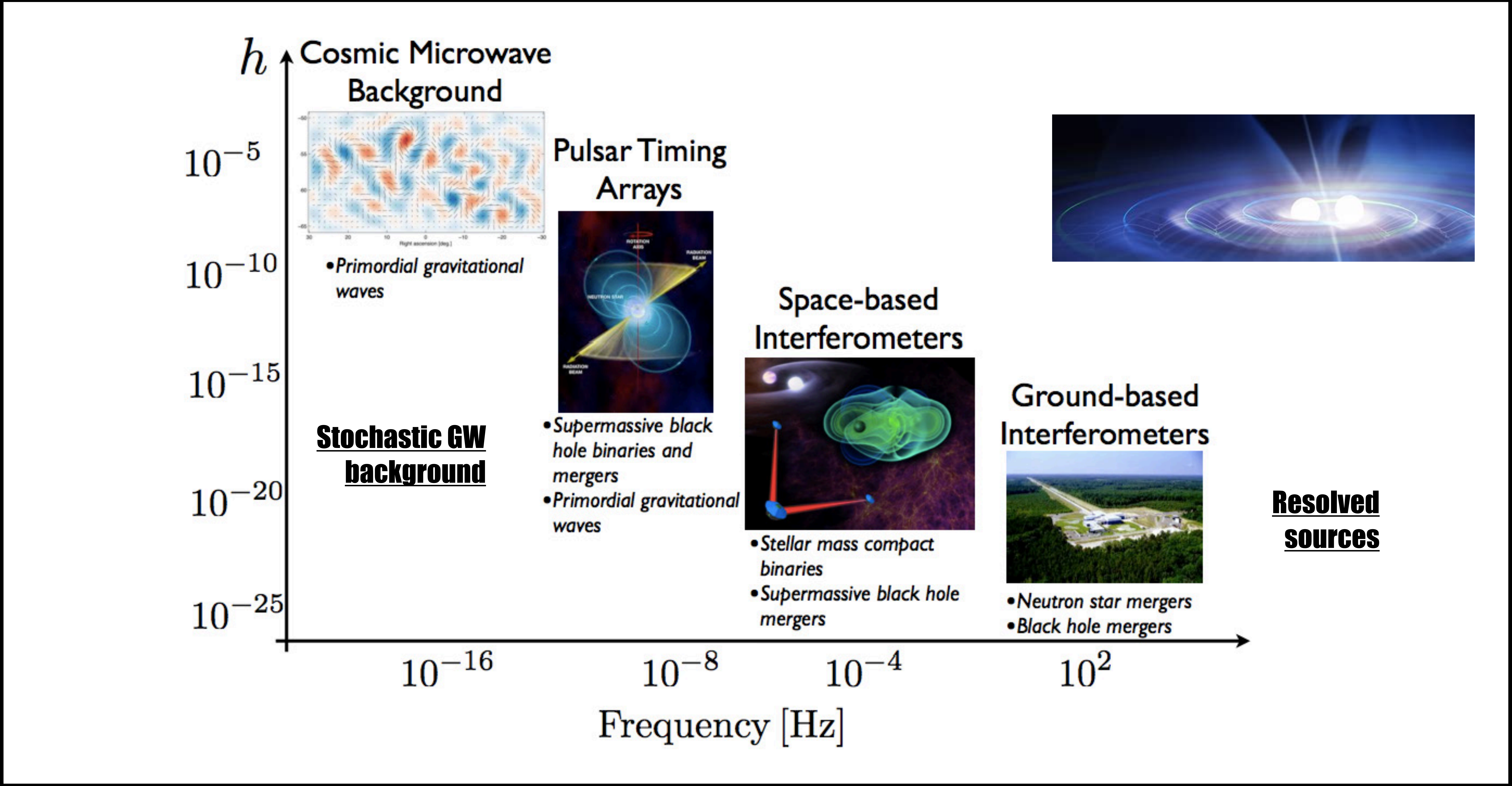


Cosmological SGWB:
Intrinsically stochastic processes





Gravitational waves spectrum

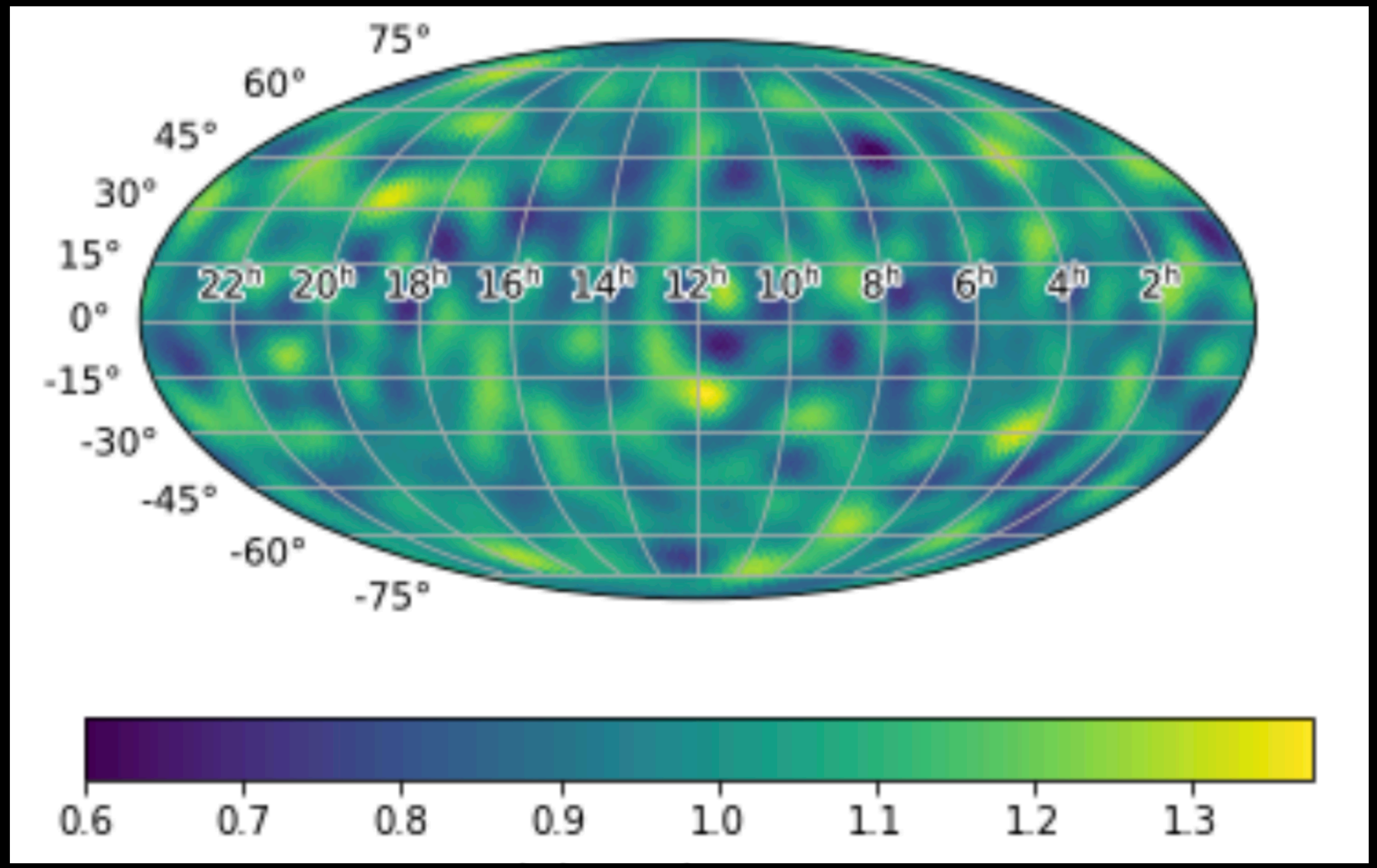


Statistical description of SGWB

Promote GW amplitude to random variable and study n-point function

$$\left\langle H_\lambda(\mathbf{x},\tau_1)\, H_\sigma(\mathbf{y},\tau_2)\right\rangle = \int d^3k\, d^3p\, e^{i(\mathbf{y}\cdot\mathbf{p}-\mathbf{x}\cdot\mathbf{k})} \left\langle H^*_\lambda(\mathbf{k},\tau_1)\, H_\sigma(\mathbf{p},\tau_2)\right\rangle$$

$\lambda, \sigma = L, R$ helicity eigenstates



Romano 2018

Wave equation on FRW

$$(H^{(0)}_{\lambda,\mathbf{k}})'' + 2\mathcal{H}(H^{(0)}_{\lambda,\mathbf{k}})' + k^2 H^{(0)}_{\lambda,\mathbf{k}} = 0$$

Damped waves

Transfer function

$$H^{(0)}_\lambda(\tau,\mathbf{k}) \equiv H_\lambda(\tau_s,\mathbf{k})\,\mathcal{T}^H(\tau,k)$$

Sources statistics

$$\left\langle H^*_\lambda(\mathbf{k},\tau^1_s) H_\lambda(\mathbf{p},\tau^2_s)\right\rangle$$

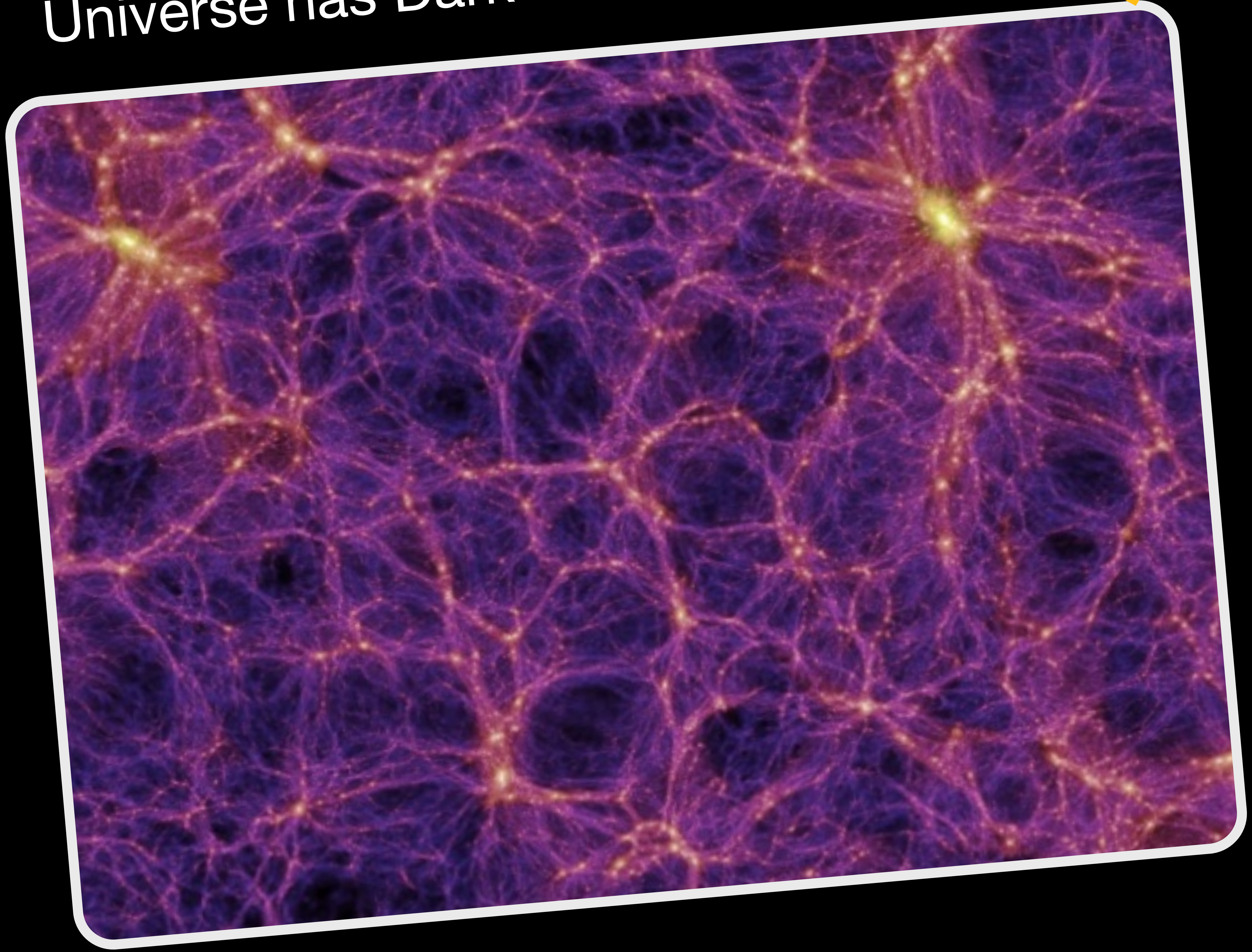
on FRW: SGWB statistics is source statistics

Propagation effects due to structures

$$(H^{(0)}_{\lambda,\mathbf{k}})'' + 2\mathcal{H}(H^{(0)}_{\lambda,\mathbf{k}})' + k^2 H^{(0)}_{\lambda,\mathbf{k}} = 0$$

→ Free propagation: GW is damped with expansion

Universe has Dark Matter structures!

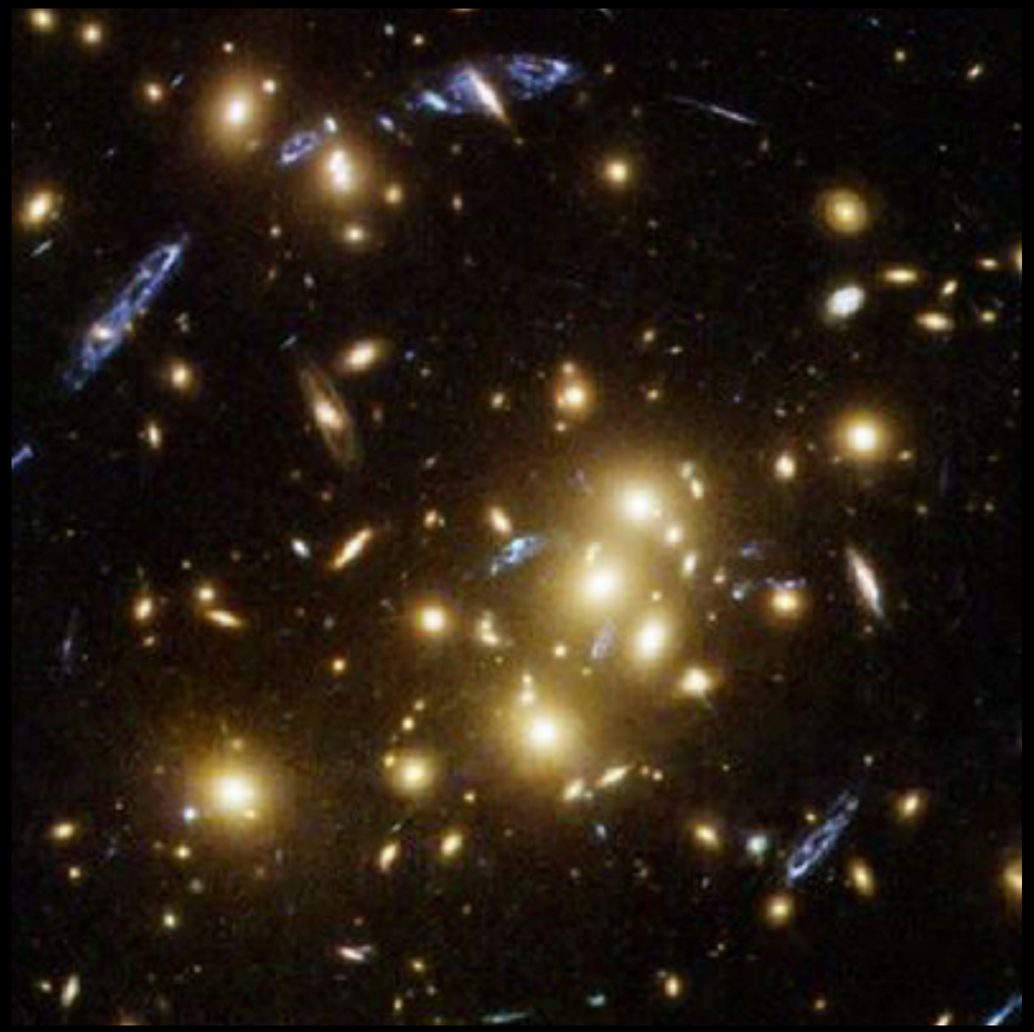


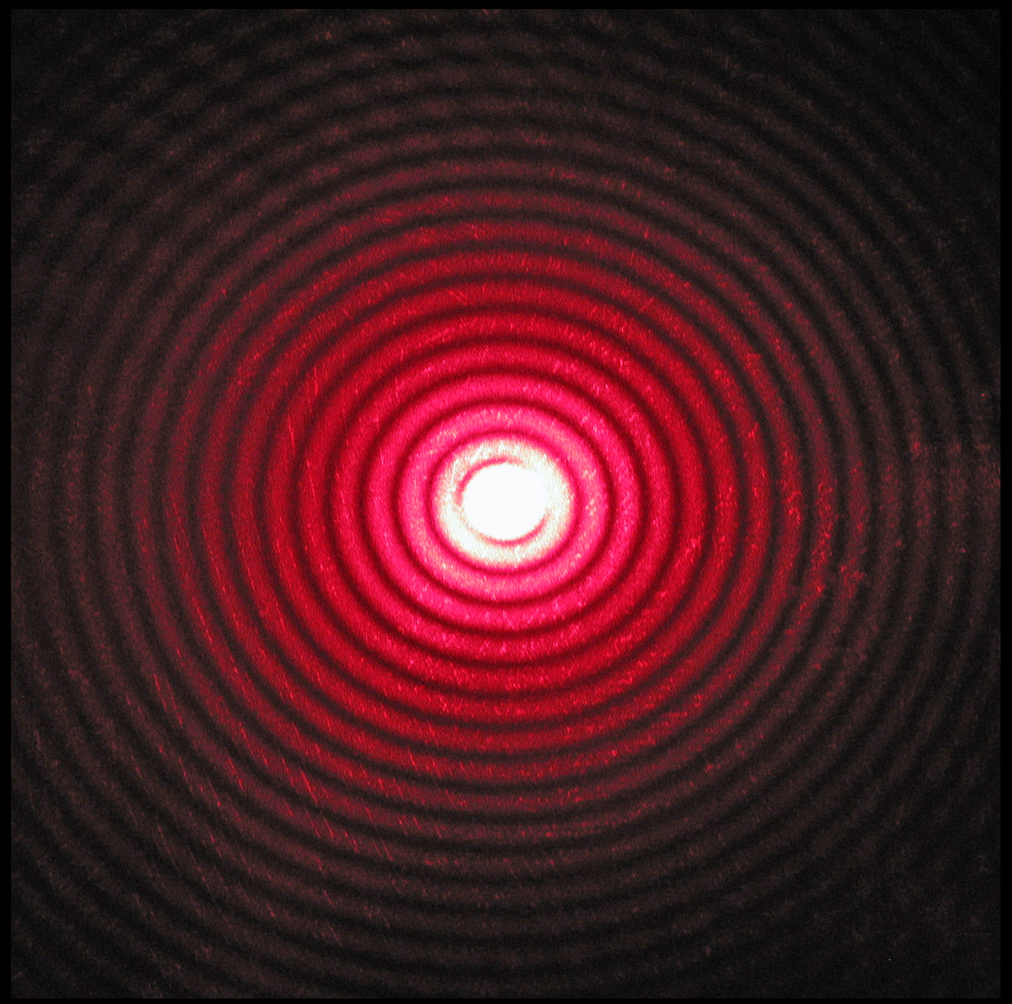
Spacetime:

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)\left[-(1+2\epsilon\phi(x))d\eta^2 + (1-2\epsilon\phi(x))d\mathbf{x}^2\right]$$

GW travels through potential wells. Possible effects?

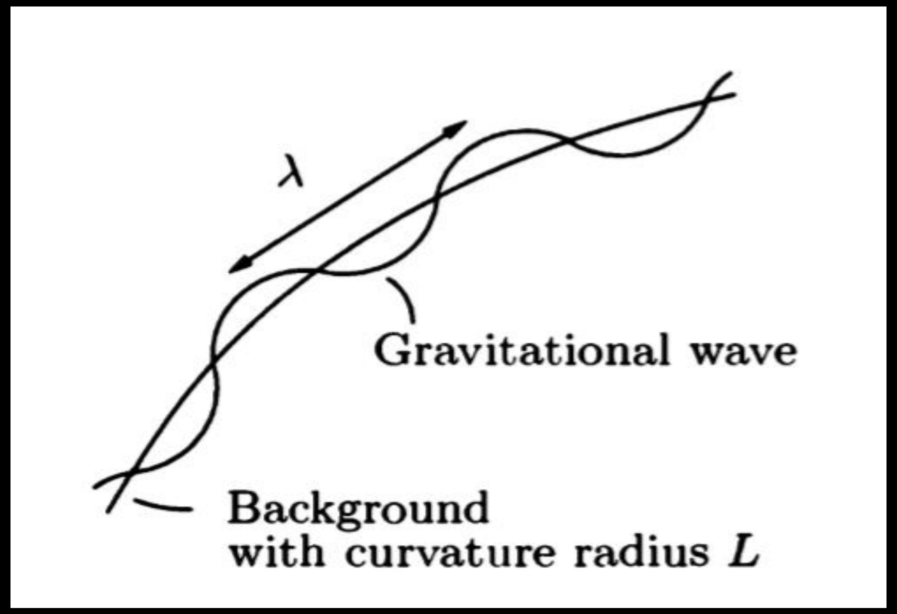
(Can't have pictures of GW, but they work the same as photons)



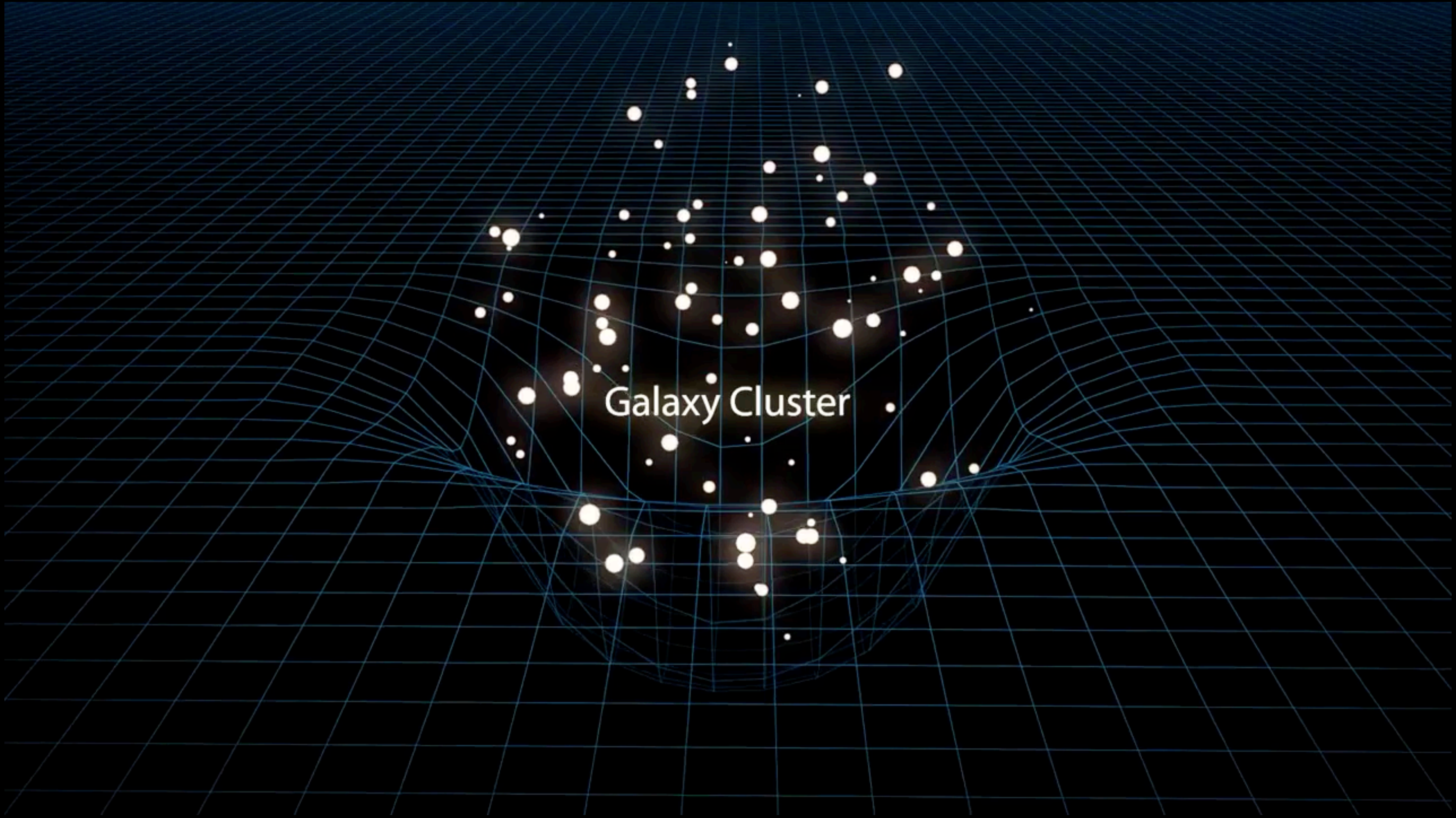


During propagation: interaction between GW and $\phi(x)$

Geometric-optics limit of the SGWB

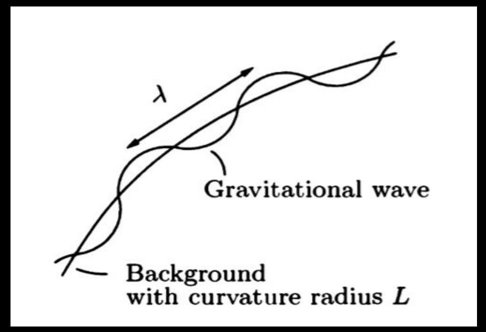


When $\lambda/L \ll 1$: “geometric optics regime” → known in literature!

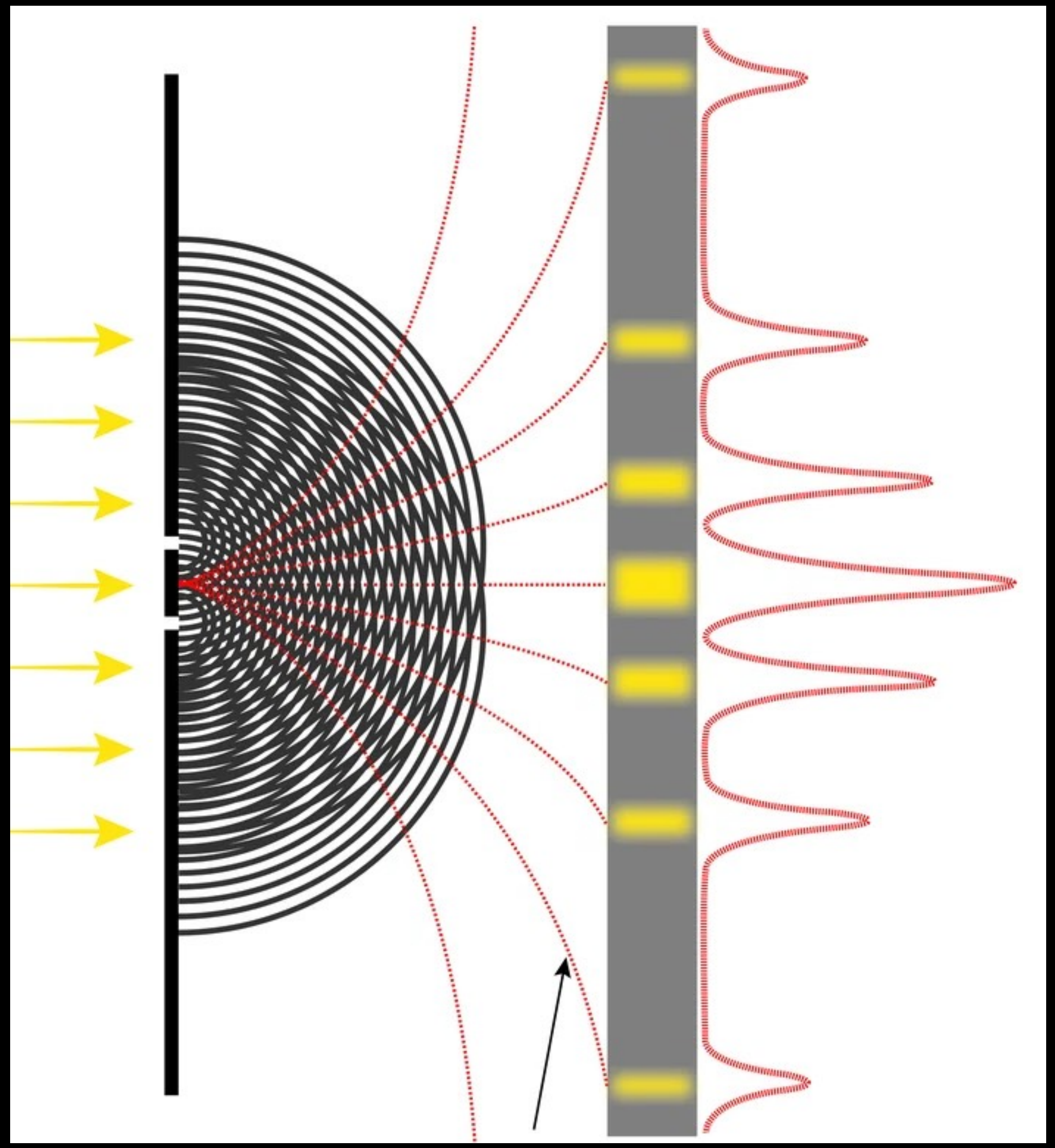


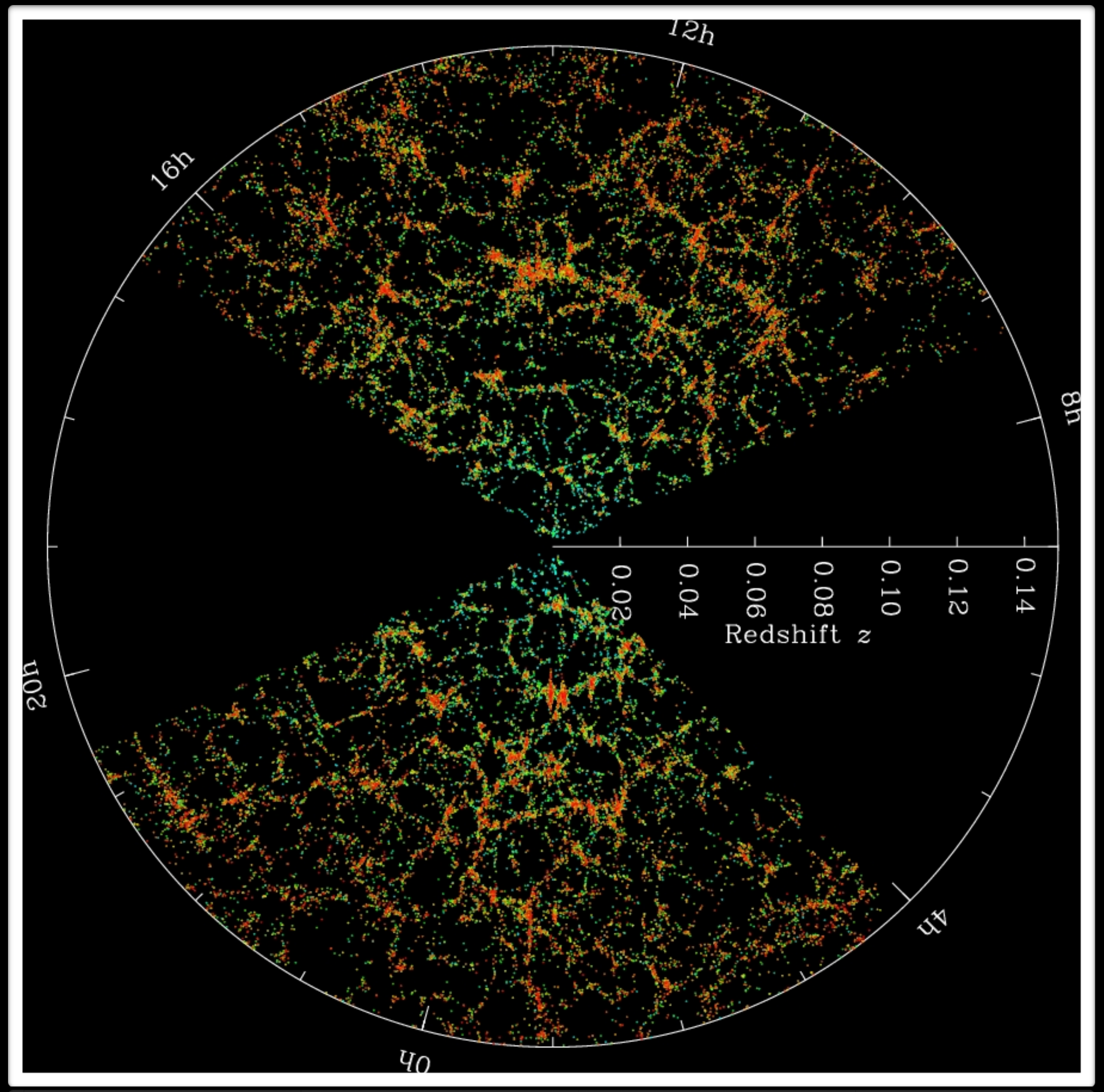
Photon = GW

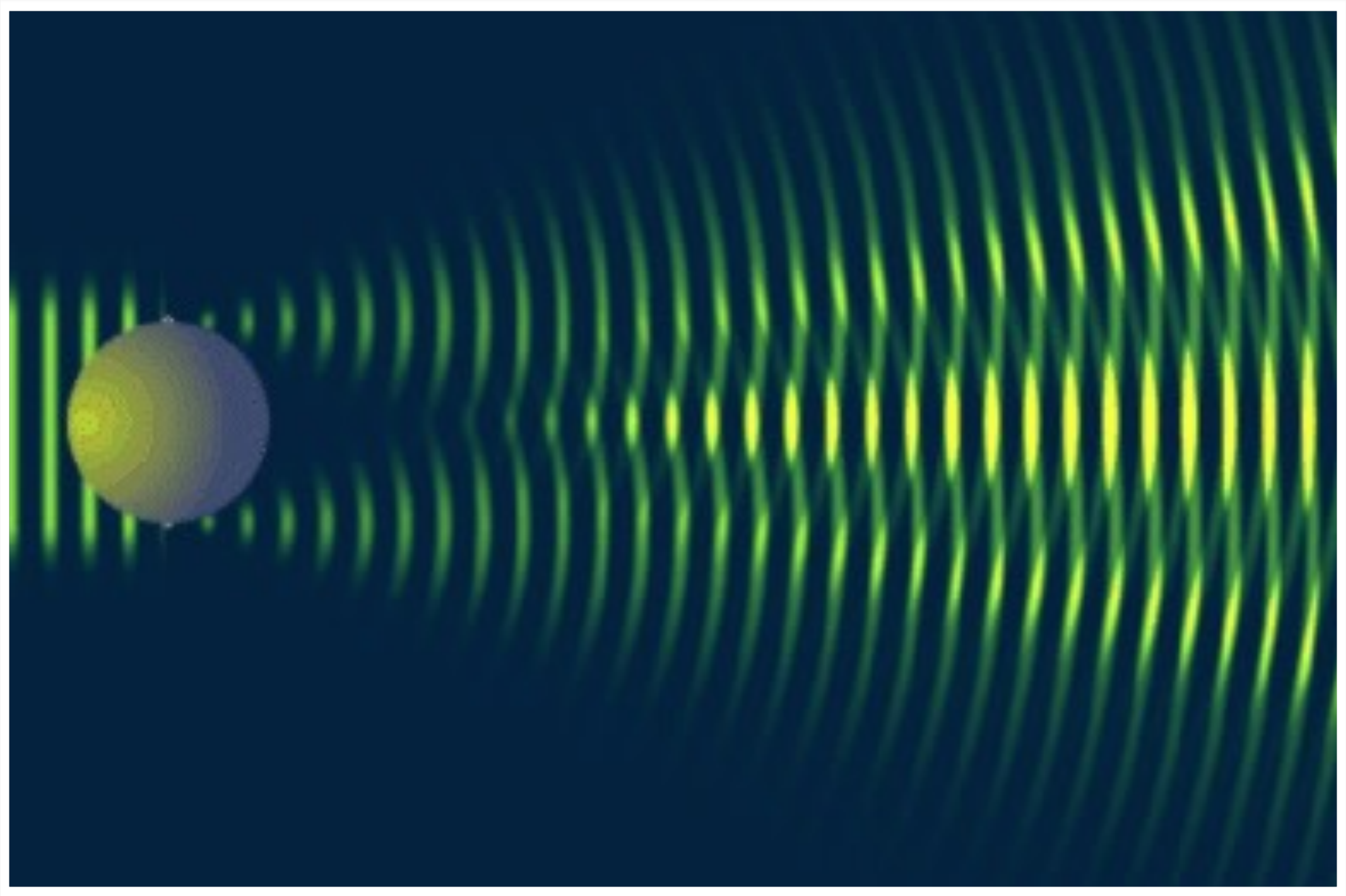
Goal: wave-optics limit of the SGWB



When $\lambda/L \gtrsim 1$ ***diffraction*** and ***interference*** become important







Since SGWB contains all wavelengths, some in wave-optics limit

Strategy and results

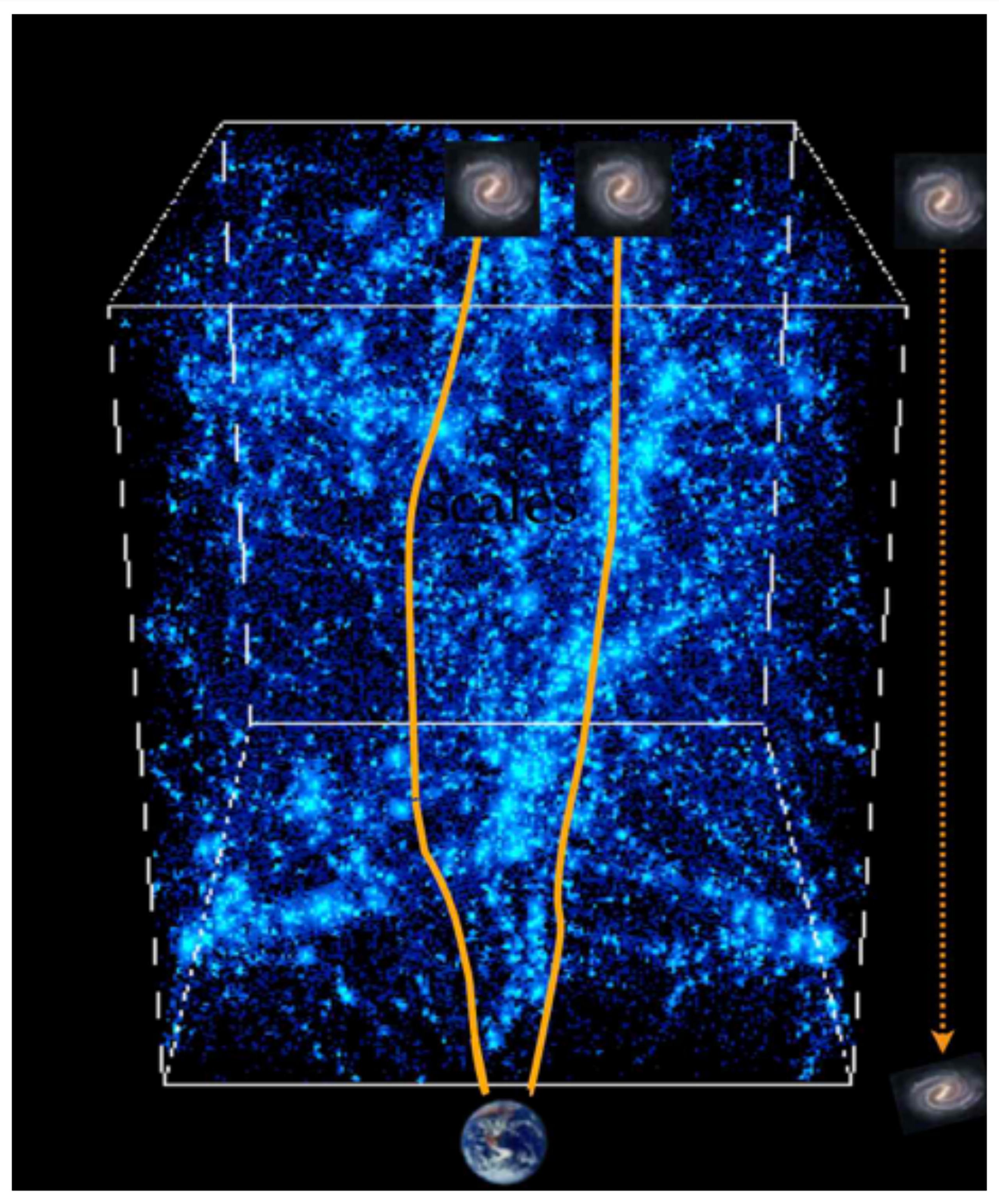
Solve Einstein’s equations

- Linearize Einstein equations and Classical matter approximation
- Add cosmic structures
- Iterative scheme and solutions
- Computation of two point function

And obtain

- New polarization modes
- Wave-optics (WO) effects

Classical Matter approximation



$$ds^2 = a^2(\eta)\left\{\left[-d\eta^2 + d\mathbf{x}^2\right] - 2\epsilon\phi\left[d\eta^2 + d\mathbf{x}^2\right] + \alpha\, H_{\mu\nu}dx^\mu dx^\nu\right\}$$

Linearize in α
Einstein equations:

$$\delta_\alpha\left[G^\mu{}_\nu - \frac{1}{2}T^\mu{}_\nu\right]$$

Linear in α,
Not linear in ϵ !

Classical Matter:

$$\delta_\alpha T^\mu{}_\nu = 0$$

GWs propagating in a
“frozen” perturbed Universe.

Understanding Classical matter approx. #1

$$\mathrm{d}s^2 = a^2(\eta)\Big\{ \left[-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right] - 2\epsilon\phi\left[\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right] + \alpha\, H_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \Big\}$$

ONLY GW: $\alpha \neq 0, \epsilon = 0$

$g_{\mu\nu} =$ FLRW $+ \alpha\, h_{\mu\nu}$

$g_{\mu\nu} =$ FLRW $+ \epsilon\delta g_{\mu\nu}$

ONLY COSMIC WEB: $\alpha = 0, \epsilon \neq 0$

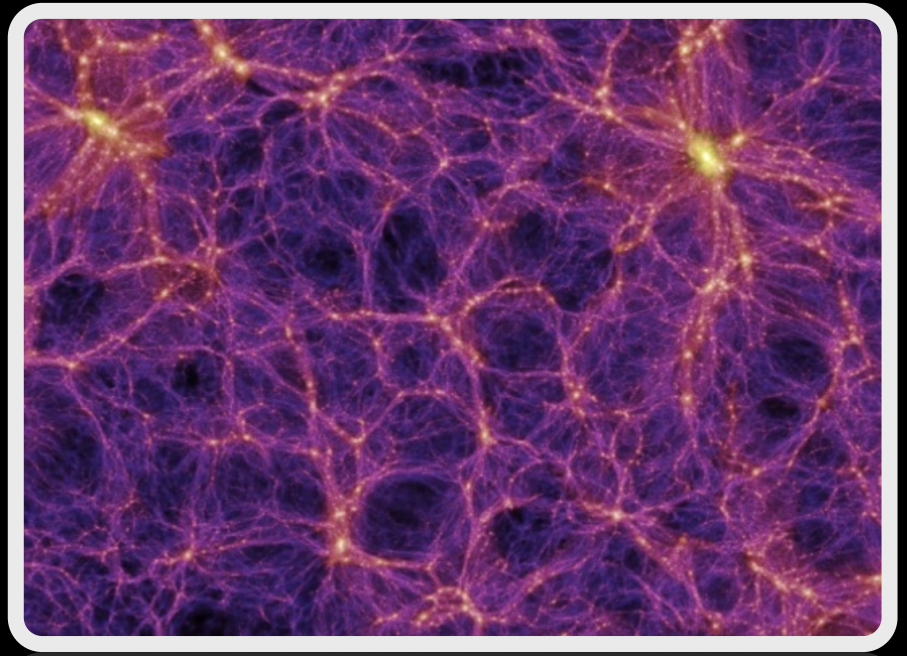
ONLY THE NAME IS DIFFERENT:

$$\epsilon\delta g_{\mu\nu} \leftrightarrow \alpha h_{\mu\nu}$$

APPLY COSMOLOGICAL PERTURBATION TECHNIQUES TO GW!

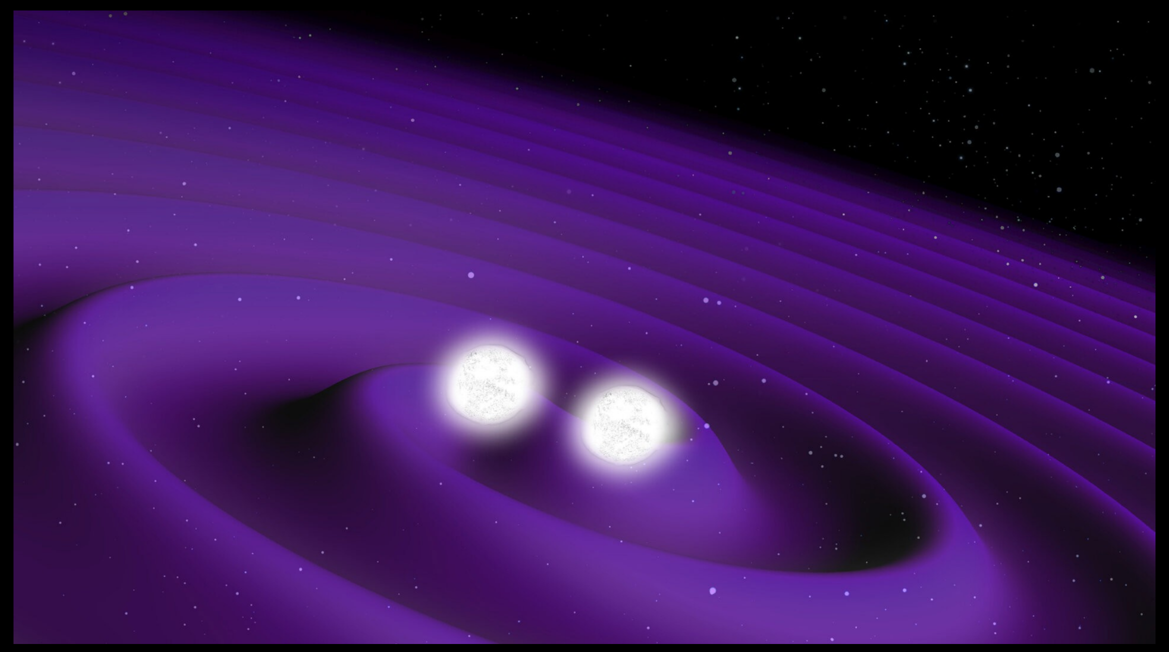
1) Decompose $\alpha h_{\mu\nu}$ and into SCALAR, VECTOR, TENSOR modes
2) Fix a gauge using $\alpha h'_{\mu\nu} = \alpha(\, h_{\mu\nu} - \bar{\nabla}_\mu \xi_\nu - \bar{\nabla}_\nu \xi_\mu\,)$
3) Linearize Einstein's equations

Understanding Classical matter approx. #2



ONLY COSMIC WEB:
$\alpha = 0, \epsilon \neq 0$

ONLY GW:
$\alpha \neq 0, \epsilon = 0$



1) Do SVT and choose Poisson gauge

$$g_{\mu\nu} = a^2(\eta)\left[-(1+2\epsilon\phi)d\eta^2 + 2\epsilon\omega_{0i}dx^i d\eta + \left[(1-2\epsilon\psi)\delta_{ij} + 2\epsilon\gamma_{ij}\right]dx^i dx^j\right]$$

→

$$g_{\mu\nu} = a^2(\eta)\left[-(1+2\alpha H_{00})d\eta^2 + 2\alpha H_{0i}dx^i d\eta + \left[(1-2\alpha H)\delta_{ij} + 2\alpha\gamma_{ij}\right]dx^i dx^j\right]$$

2) Linearize Einstein's equations:

$$\nabla^2\psi = 4\pi G a^2(\delta_\epsilon T^0_0 + 3\mathcal{H}H_f) \quad \rightarrow \quad \nabla^2 H = 4\pi G a^2(\delta_\alpha T^0_0 + 3\mathcal{H}H_f)$$

$$\nabla^2\omega_{0i} = 16\pi G a^2[\delta_\epsilon T^0{}_i]_\perp \quad \rightarrow \quad \nabla^2 H_{0i} = 16\pi G a^2[\delta_\alpha T^0{}_i]_\perp$$

$$(\partial_\tau^2 + 2\mathcal{H}\partial_\tau - \Delta)\gamma_{ij} = 8\pi G a^2\,\delta_{ik}(\delta_\epsilon T^k{}_j)_T \quad \rightarrow \quad (\partial_\tau^2 + 2\mathcal{H}\partial_\tau - \Delta)\gamma_{ij} = 8\pi G a^2\,\delta_{ik}(\delta_\alpha T^k{}_j)_T$$

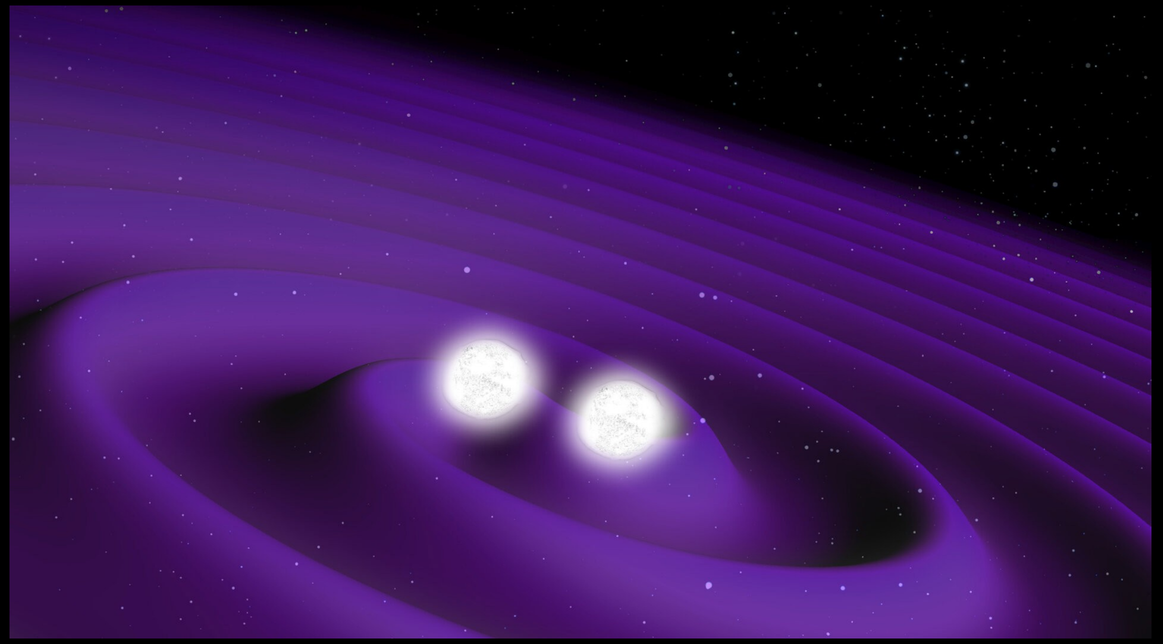
Bertschinger 95'

Symmetries of FLRW guarantee decoupling of S-V-T

Understanding Classical matter approx. #3

ONLY GW:
$\epsilon = 0$

How to isolate tensor modes if background is not FLRW?



$\nabla^2 H = \quad -\partial_i H_f = [\quad]$

$\nabla^2 H_{0i} =$

$(\partial_\tau^2 + 2\mathcal{H}\partial_\tau - \Delta)\gamma_{ij} =$

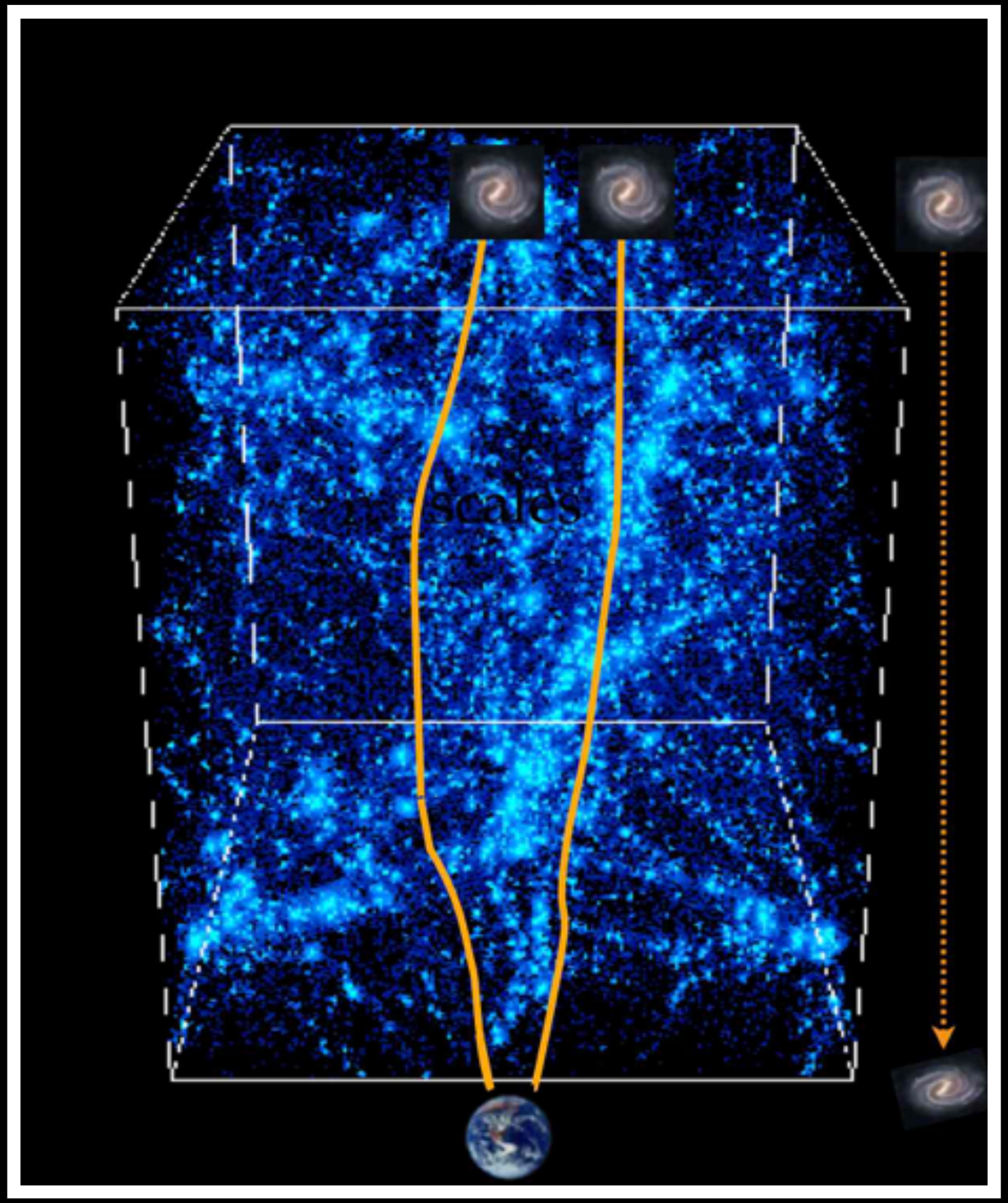
$\delta_\alpha T^\mu{}_\nu = 0$

Sourceless Poisson eqs. → Solution: $H = H_{0i} = 0$ → Only γ_{ij} survives

GR gauge theory → $H, H_{0i} \neq 0$ to ensure $\delta_\alpha[\nabla_\mu T^\mu_\nu] = 0$.

CM approximation isolates T modes by removing sources to S / V modes even when the background is not FLRW

GWs through cosmic structures



Double expansion: ϵ and α

$$\mathrm{d}s^2 = a^2(\eta)\Big\{ \left[-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right] - 2\epsilon\phi\left[\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right] + \alpha\, H_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \Big\}$$

FRW — Cosmic structures — GW

Plug in $\delta_\alpha G_{\mu\nu} = 0$ and keep: $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\epsilon)$ orders

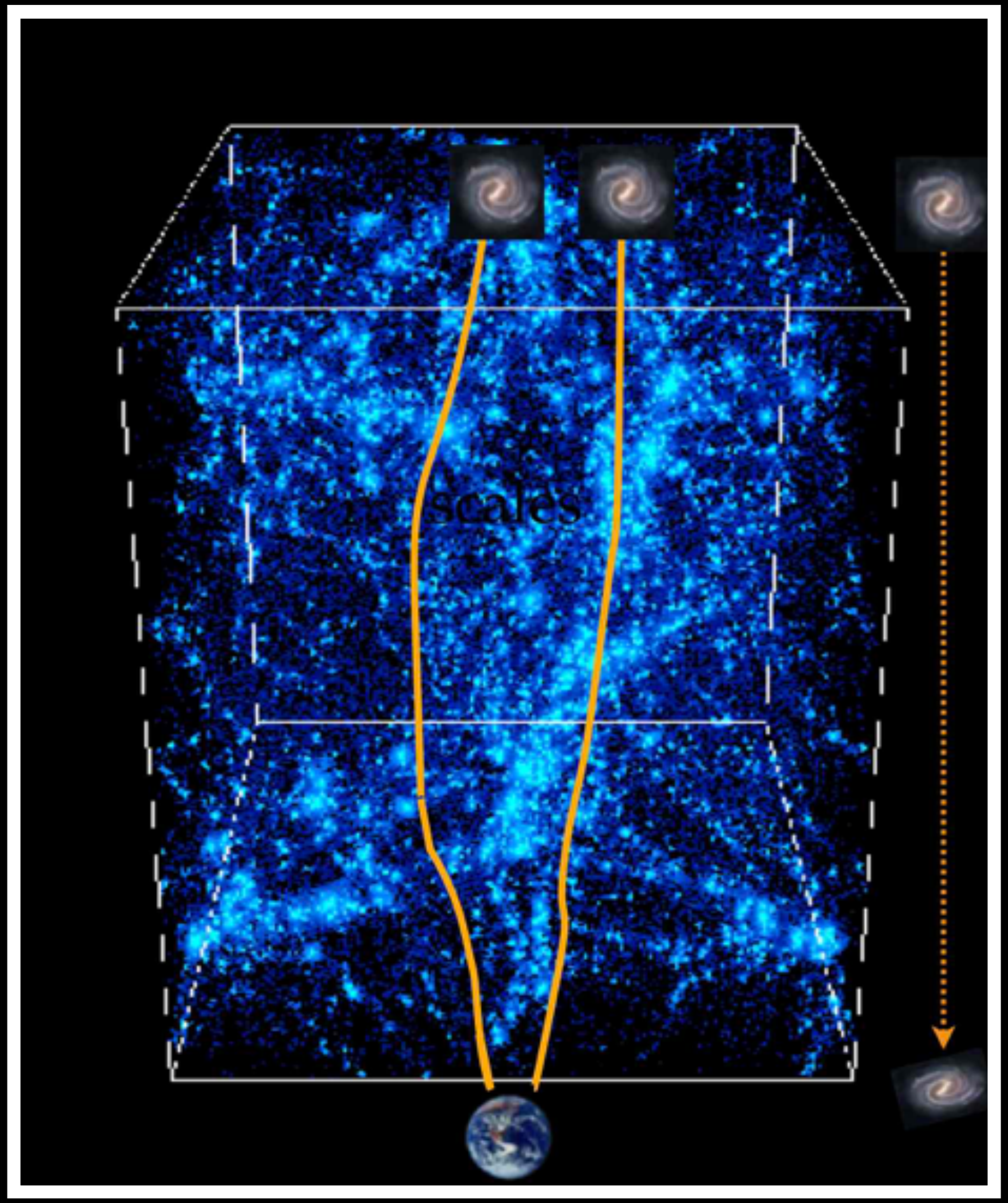
$$[\mathcal{O}_0\, H]_{\mu\nu} + \epsilon\,[\mathcal{O}_1\, H]_{\mu\nu} = 0$$

$\sim \Box\, H_{\mu\nu}, \ldots$ $\qquad$ $\sim \phi\, H_{\mu\nu},\ \partial^k\phi\, \partial_k H_{\mu\nu}, \ldots$

+ Do same for gauge choice: $\bar{\nabla}_\mu h^\mu{}_\nu = 0$

Very long equation (~100 terms), need to find scheme to solve them

Iterative solution



Expand GW in ϵ:

$$H_{\mu\nu} = H^{(0)}_{\mu\nu} + \epsilon H^{(1)}_{\mu\nu}$$

$$[\mathcal{O}_0\, H]_{\mu\nu} + \epsilon\, [\mathcal{O}_1\, H]_{\mu\nu} = 0$$

same order in ϵ

ϵ^0 : $\left[\mathcal{O}_0\, H^{(0)}\right]_{\mu\nu} = 0$ → Free GW

ϵ^1 : $\left[\mathcal{O}_0\, H^{(1)}\right]_{\mu\nu} = -\left[\mathcal{O}_1\, H^{(0)}\right]_{\mu\nu}$ → GW with Source

$$\sim \phi\, H^{(0)}_{\mu\nu},\ \partial^k\phi\, \partial_k H^{(0)}_{\mu\nu}, \ldots$$

New source for $H^{(1)}_{\mu\nu}$: interaction between $H^{(0)}_{\mu\nu}$ and ϕ.

Free solution

0^{th} Einstein eqs.

$$\left[\mathcal{O}_0\, H^{(0)}\right]_{\mu\nu} = 0$$

→ 00,0i components

$$\Box H^{(0)} + \ldots = 0$$
$$\Box H^{(0)}_{0i} + \ldots = 0$$

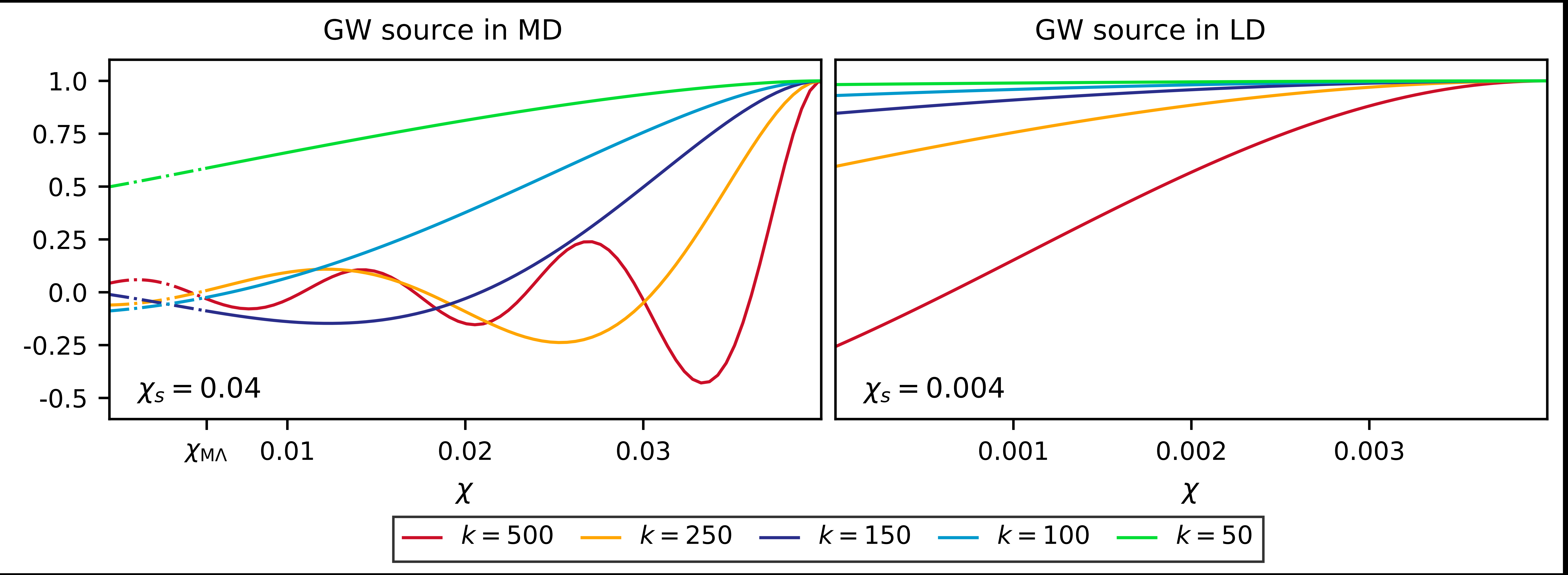
→ No scalar / vector sources in GR

$$H^{(0)} = H^{(0)}_{0i} = 0$$

In covariant gauge: “constraint equations” aren’t first order PDE.

Only TT components $\neq 0$

$$(H^{(0)}_{\lambda,\mathbf{k}})'' + 2\mathcal{H}(H^{(0)}_{\lambda,\mathbf{k}})' + k^2 H^{(0)}_{\lambda,\mathbf{k}} = 0$$



First order solution: new polarization modes

1^{st} Einstein eqs.

$$\left[\mathcal{O}_0\, H^{(1)}\right]_{\mu\nu} = -\left[\mathcal{O}_1\, H^{(0)}\right]_{\mu\nu}$$

$$H^{(1)}_{\mu\nu} = \left[\begin{array}{c|c} 0 & H^{(1)}_{0i} \\ \hline H^{(1)}_{i0} & \gamma^{(1)}_{ij} + E^{(1)}_{ij} + \frac{1}{3}\delta_{ij} H^{(1)} \end{array}\right]$$

00,0i components

$$\Box H^{(1)} + \ldots = -4\, H^{(0)}_{ij}\, \partial^i \partial^j \phi$$

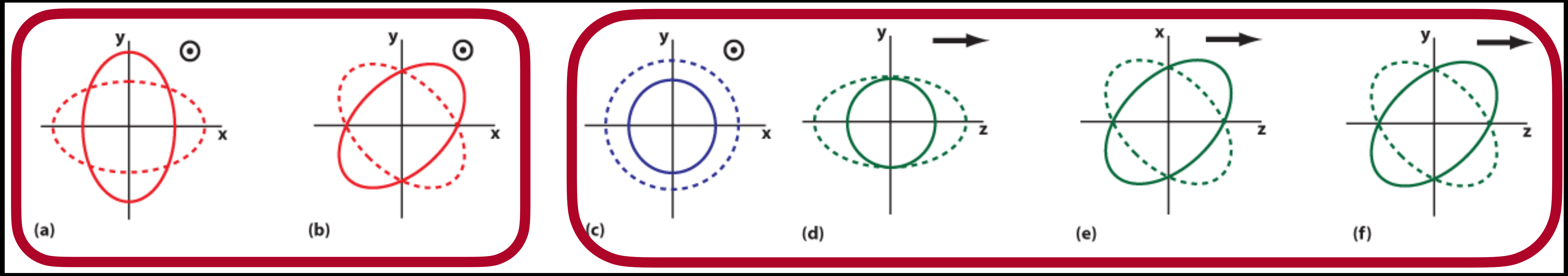
$$\Box H^{(1)}_{0i} + \ldots = -\frac{2}{a}\partial^k \phi\, (a H^{(0)}_{ki})'$$

$$\partial^j E^{(1)}_{ij} = (H^{(1)}_{0i})' + 4\mathcal{H} H^{(1)}_{0i} + \frac{\partial_i H^{(1)}}{6}$$

$$H^{(1)}, H^{(1)}_{0i}, E_{ij} \neq 0$$

- New polarization
- Not new d.o.f

Compute perturbed Riemann tensor for geodesic deviation equation



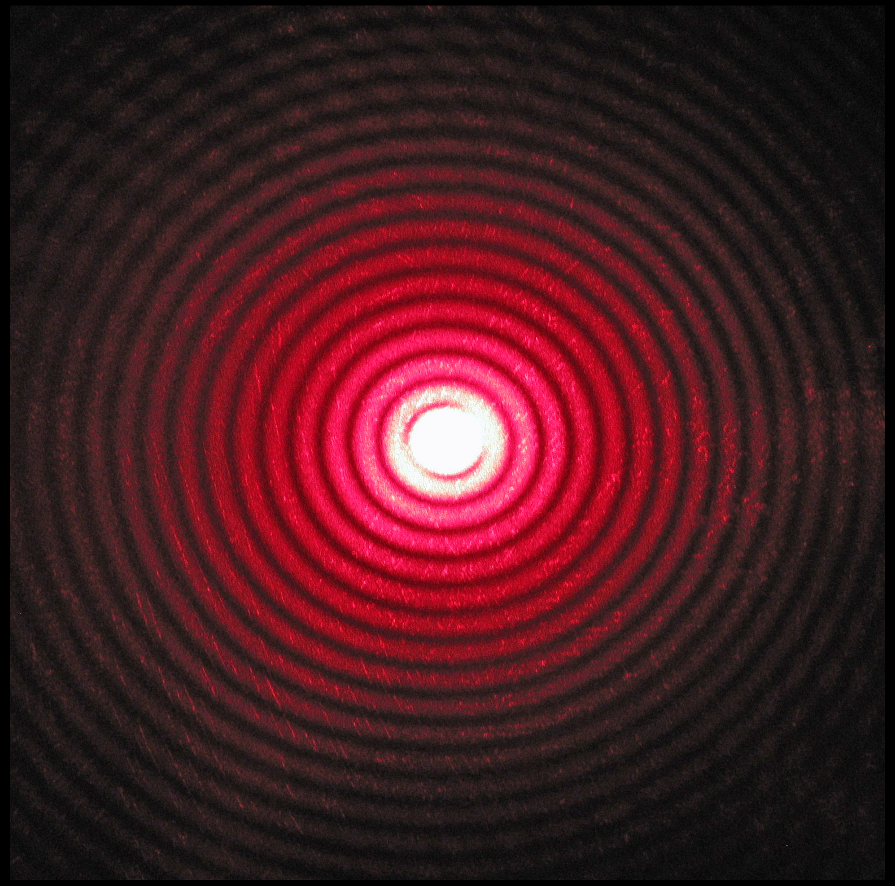
Components of $H^{(0)}_{ij}$ along $\partial_i \phi$ source scalar & vector modes in $H^{(1)}_{ij}$

First order solution: transverse-traceless part

1^{st} Einstein eqs.

$$\left[\mathcal{O}_0\, H^{(1)}\right]_{\mu\nu} = -\left[\mathcal{O}_1\, H^{(0)}\right]_{\mu\nu}$$

$$H^{(1)}_{\mu\nu} = \left[\begin{array}{c|c} 0 & H^{(1)}_{0i} \\ \hline H^{(1)}_{i0} & \gamma^{(1)}_{ij} + E^{(1)}_{ij} + \frac{1}{3}\delta_{ij} H^{(1)} \end{array}\right]$$



Extract TT part of Einstein eqs. Use Green's function to solve.

$$(\gamma^{(1)}_{\lambda,\mathbf{k}})'' + 2\mathcal{H}(\gamma^{(1)}_{\lambda,\mathbf{k}})' + k^2\gamma^{(1)}_{\lambda,\mathbf{k}} = \mathcal{S}_{\lambda,\mathbf{k}}$$

$$\mathcal{S}_{\lambda,\mathbf{k}} = -\frac{1}{2}\sum_{\sigma=L,R}\int d^3p\, H^{(0)}_{\sigma,\mathbf{p}}\left[\phi''_{\mathbf{k}-\mathbf{p}} + 2\mathcal{H}\phi'_{\mathbf{k}-\mathbf{p}} + \phi_{\mathbf{k}-\mathbf{p}}(p^2+k^2)\right][\mathcal{R}(\hat{p},\hat{k})]^{\sigma}{}_{\lambda}$$

Sum over different modes: interference

$$H_\lambda(\tau_s,\mathbf{k})\,\mathcal{T}^H(\tau,k)$$

$$\mathcal{F}(\mathbf{p},\mathbf{k}-\mathbf{p},\tau)\phi^{in}_{\mathbf{k}-\mathbf{p}}$$

Rotation matrix: Diagonal only if $\hat{p} \parallel \hat{k}$

Statistical description of SGWB

Promote amplitude to random variable, compute 2-point function

2x2 matrix

$$\left\langle \gamma^{(1)*}_{\lambda}(\mathbf{k}_1, \tau_1)\, \gamma^{(1)}_{\sigma}(\mathbf{k}_2, \tau_2) \right\rangle = \int_{\tau_1^s}^{\tau_1} d\tau_1' \int_{\tau_2^s}^{\tau_2} d\tau_2'\, g^*_{k_1}(\tau_1, \tau_1')\, g_{k_2}(\tau_2, \tau_2') \left\langle \mathcal{S}^*_{\lambda}(\mathbf{k}_1, \tau_1')\, S_{\sigma}(\mathbf{k}_2, \tau_2') \right\rangle$$

Green's function

$$\left\langle \mathcal{S}^*_{\lambda}(\mathbf{k}_1, \tau_1')\, S_{\sigma}(\mathbf{k}_2, \tau_2') \right\rangle = \sum_{\lambda'\sigma'} \int d^3p_1 d^3p_2\, \mathcal{T}^H_{p_1}(\tau_1')\, \mathcal{T}^H_{p_2}(\tau_2')\, \mathcal{F}^*(\mathbf{p}_1, \mathbf{k}_1 - \mathbf{p}_1, \tau_1')\, \mathcal{F}(\mathbf{p}_2, \mathbf{k}_2 - \mathbf{p}_2, \tau_2') \times$$
$$\times \left[\mathcal{R}^*(\hat{p}_1, \hat{k}_1)\right]^{\lambda'}_{\lambda} \left[\mathcal{R}(\hat{p}_2, \hat{k}_2)\right]^{\sigma'}_{\sigma} \left\langle \phi^{\tau_{in}*}_{\mathbf{k}_1 - \mathbf{p}_1}\, \phi^{\tau_{in}}_{\mathbf{k}_2 - \mathbf{p}_2}\, H^{(0)*}_{\lambda'}(\mathbf{p}_1, \tau_1^s)\, H^{(0)}_{\sigma'}(\mathbf{p}_2, \tau_2^s) \right\rangle$$

Free GW

Growth of cosmic structures

Statistics of GW sources and cosmic structures

Assume gaussian random fields and use Wick's theorem:
sum of 3 products of two 2-point functions

Cosmological SGWB

Assume: unpolarized 0^{th} order SGWB, statistical homogeneity and isotropy and uncorrelated GW and ϕ:

$$\left\langle H^{(0)*}_{\lambda,\mathbf{p}_1}(\tau_1^s)\, H^{(0)}_{\sigma,\mathbf{p}_2}(\tau_2^s)\right\rangle = \delta_{\lambda\sigma}\,\delta^3(\mathbf{p}_1-\mathbf{p}_2)\,\frac{\mathrm{I}^{(0)}(p_1,\tau_1^s,\tau_2^s)}{2}$$

$$\left\langle \phi^{\tau_{in}*}_{\mathbf{k}}\, \phi^{\tau_{in}}_{\mathbf{p}}\right\rangle = \delta^3(\mathbf{k}-\mathbf{p})\,\frac{2\pi^2}{k^3}\,P^{\phi}_{in}(k)$$

Scale dependent features?

$$\left\langle \phi^{\tau_{in}}_{\mathbf{k}_2-\mathbf{p}_2}\, H^{(0)}_{\lambda,\mathbf{p}_2}(\tau_2^s)\right\rangle = 0$$

WO effects are frequency dependent: multi-band analysis!

$$\left\langle S^*_\lambda(\mathbf{k},\tau_1')\, S_\sigma(\mathbf{k},\tau_2')\right\rangle = \pi^2\int d^3p \left\{\begin{array}{c}\text{Deterministic evolution:}\\ \text{Transfer functions \&}\\ \text{Rotation matrix}\end{array}\right\}_{\lambda\sigma} \times \frac{P^{\phi}_{in}(|\mathbf{k}_1-\mathbf{p}|)}{|\mathbf{k}_1-\mathbf{p}|^3} \times \frac{\mathrm{I}^{(0)}(p,\tau_1^s,\tau_2^s)}{2}$$

Stokes parameters

Decompose 2-point function on Pauli matrices basis

$$\langle (\gamma_\lambda^1)^* \gamma_\sigma^1 \rangle = \frac{1}{2} \begin{bmatrix} I+V & Q-iU \\ Q+iU & I-V \end{bmatrix}$$

$$I = |\gamma_R|^2 + |\gamma_L|^2$$

- Intensity of SGWB

$$V = |\gamma_R|^2 - |\gamma_L|^2$$

- Circular polarization due to different amounts of R/L modes

$$\begin{aligned} Q &= 2\mathrm{Re}(\gamma_R^* \gamma_L) \\ U &= 2\mathrm{Im}(\gamma_R^* \gamma_L) \end{aligned}$$

- Linear polarization due to phase difference between R/L modes

Polarization in GO is transported trivially: must include WO effects!

Cosmological SGWB: Stokes parameters

$$I^{(1)}(\mathbf{k}) = \int d^3p \, \frac{P^{\phi}_{in}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^3} \times \mathrm{I}^{(0)}(p) \times \left(4Y^0_0(\theta,\varphi) + \frac{8\sqrt{5}}{7}Y^2_0(\theta,\varphi) + \frac{2}{21}Y^4_0(\theta,\varphi)\right) \times \left\{ \text{Deterministic evolution: Transfer functions, Green's function} \right\}$$

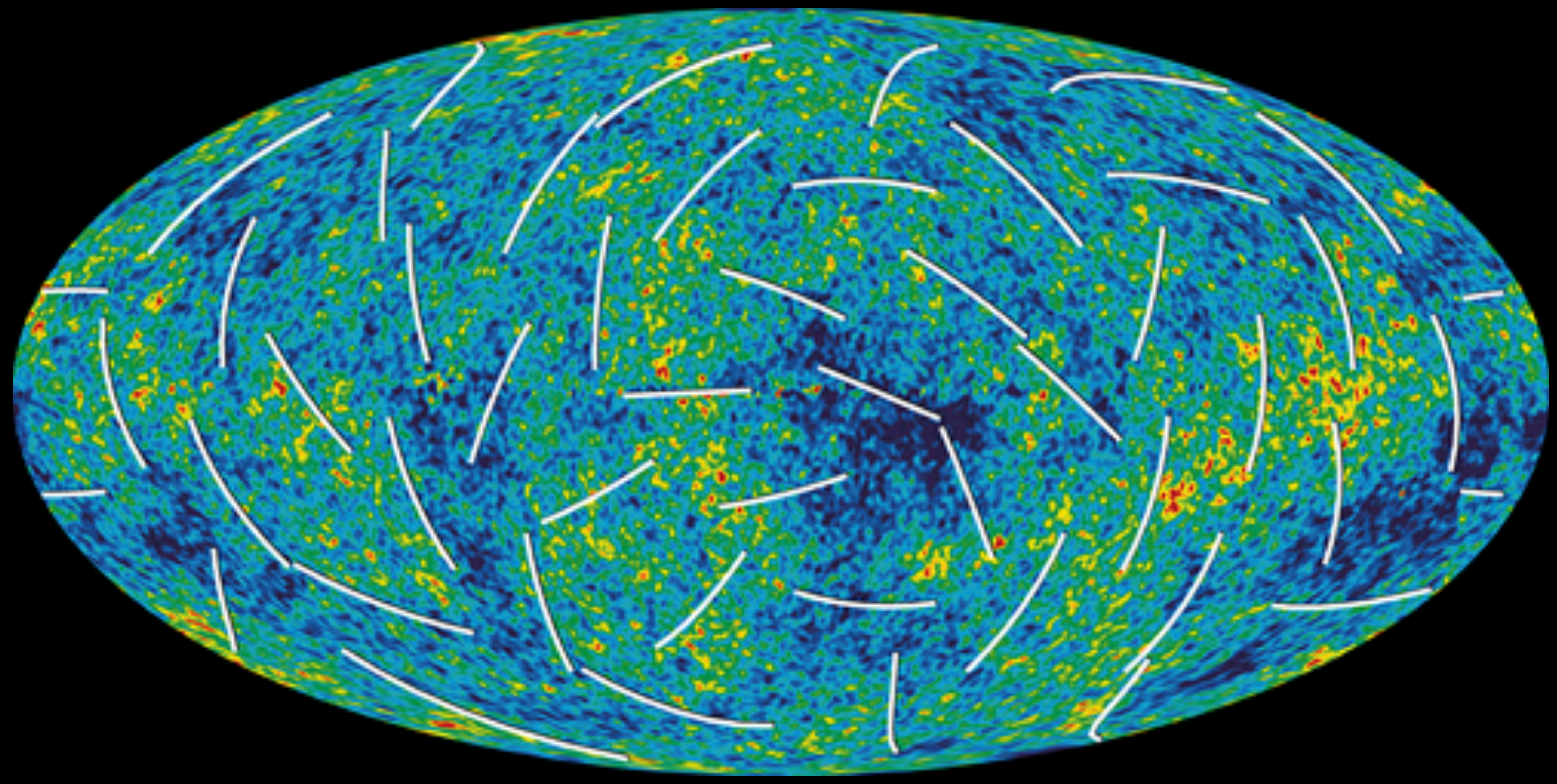
$$V^{(1)}(\mathbf{k}) = 0$$

$$\sim e^{\pm 4i\phi}$$

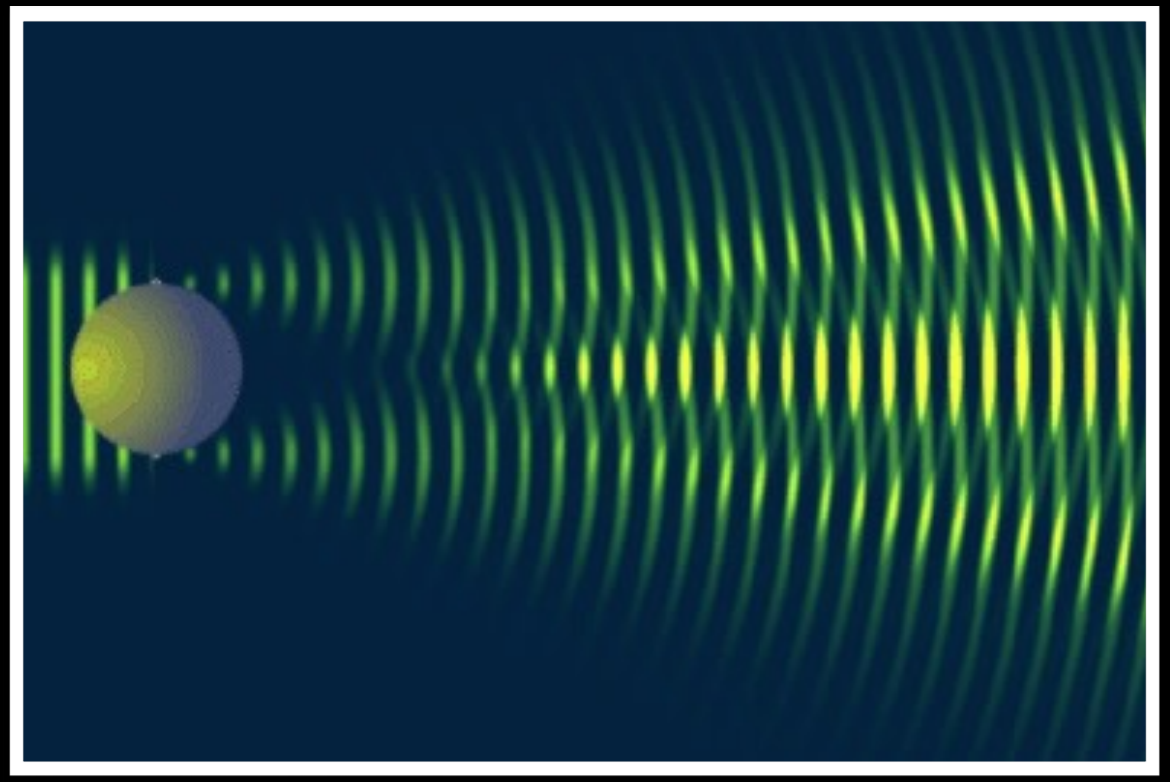
$$(Q^{(1)} \pm iU^{(1)})(\mathbf{k}) = \int d^3p \, \frac{P^{\phi}_{in}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^3} \times \mathrm{I}^{(0)}(p) \times \left(\sqrt{\frac{40}{63}}Y^4_{\mp 4}(\theta,\varphi)\right) \times \left\{ \text{Deterministic evolution: Transfer functions, Green's function} \right\}$$

<u>Interaction of SGWB with cosmic web:</u>

1. Can modulate amplitude of SGWB
2. Do not produce V
3. Can produce Q/U polarization modes

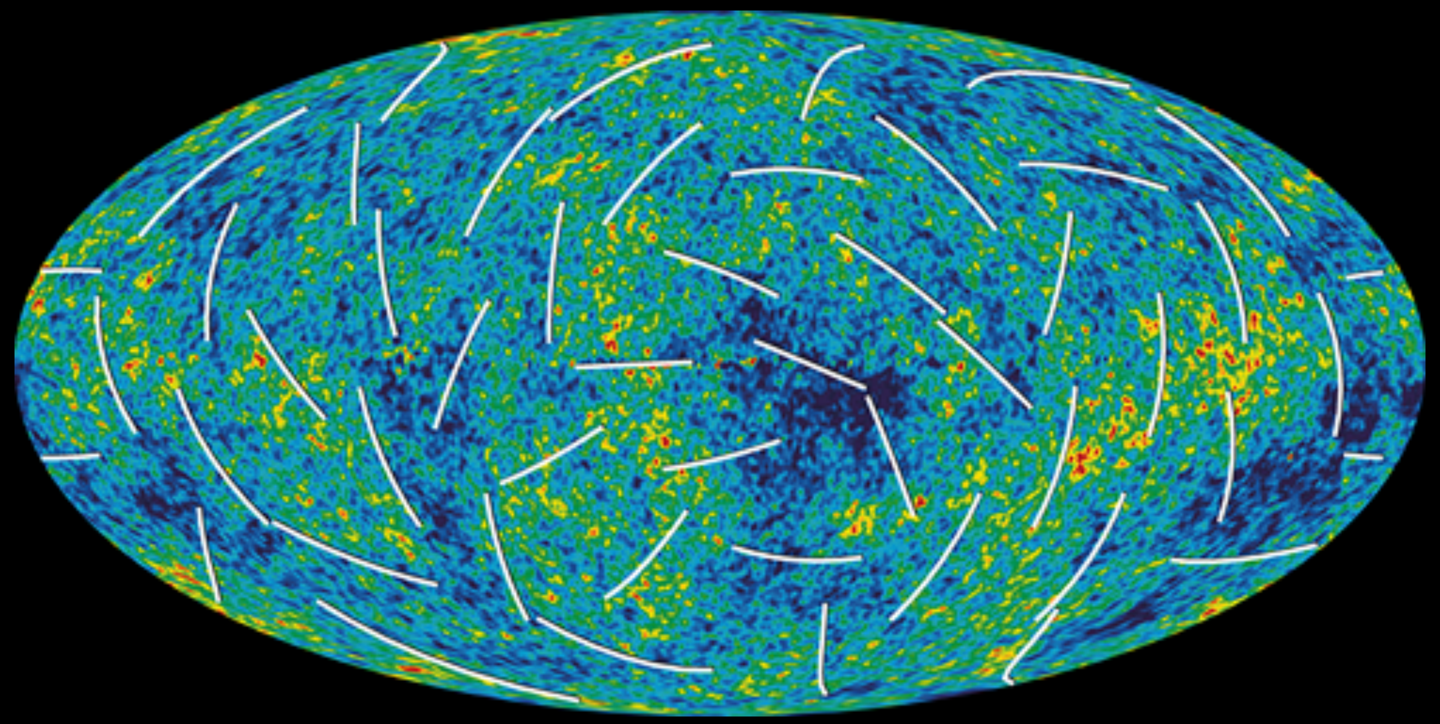


Conclusions



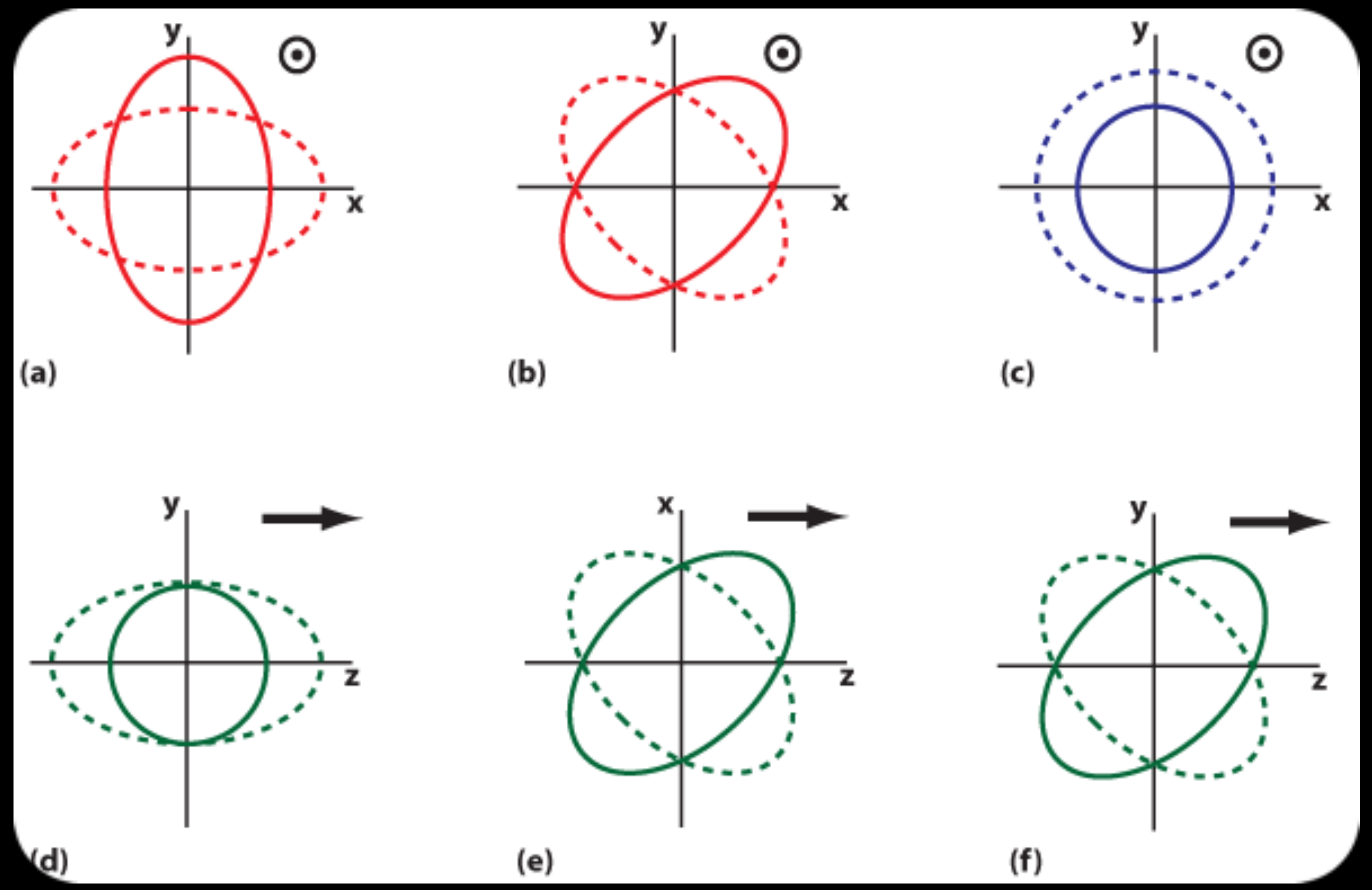
GWs are a new powerful window on the Universe.

They offer new theoretical challenges so we must learn how to use them before drawing any conclusions.



Why use AG, 2210.05718:

- Valid across the entire GW spectrum
- In the wave-optics limit propagation effects are frequency dependent: multi-band analysis to probe different scales
- New investigation channel: scalar and vector modes + Q/U



Thank you!