Normal Forms for non linear betatronic motion

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Many thanks to Prof. G. Turchetti & Dr. M. Giovannozzi

Outline

The equation of motion

4-D phase space: $\mathbf{x} = (x, p_x, y, p_y)$. The Hamiltonian function

$$\mathcal{H}(\mathbf{x}; s) = \underbrace{\frac{{p_x}^2 + {p_y}^2}{2} + \left(\frac{1}{\rho(s)^2} - k_1(s)\right)\frac{x^2}{2} + k_1(s)\frac{y^2}{2}}_{\text{Linear Motion}}$$

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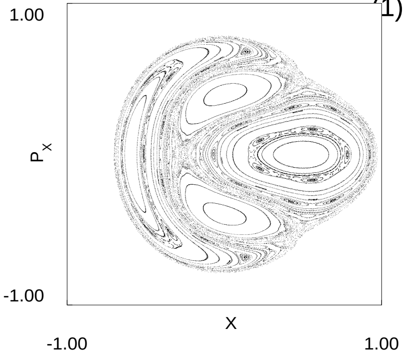

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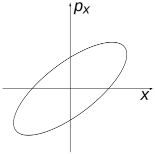

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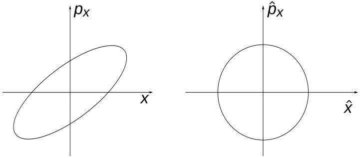

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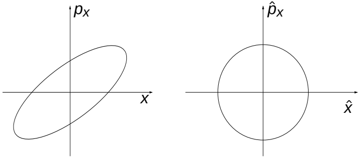

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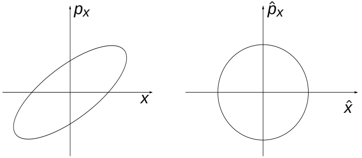

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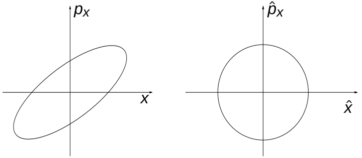

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The theory 1/2...the construction...

C-S theory extension.

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C-S theory extension. In complex coordinate $z_1 = \hat{x} - i\hat{p}_x$, $z_2 = \hat{y} - i\hat{p}_y$ and $\mathbf{z} = (z_1, z_2)$ after $\mathbf{z} \to \beta_x^{3/2}/2K_2\mathbf{z}$ Eq. (3) reads

$$\mathbf{z}' = \mathbf{F}\left(\mathbf{z_1}, \mathbf{z_1^*}, \mathbf{z_2}, \mathbf{z_2^*}\right) \Rightarrow \begin{cases} z_1' = e^{i\omega_1}\left(z_1 - \dfrac{i}{4}\left[\left(z_1 + z_1^*\right)^2 - \beta\left(z_2 + z_2^*\right)^2\right]\right) \\ z_2' = e^{i\omega_2}\left(z_2 + \dfrac{i}{2}\beta\left(z_1 + z_1^*\right)\left(z_2 + z_2^*\right)\right) \end{cases} \tag{4}$$

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Normal forms coordinate $\xi = (\xi_1, \xi_1^*, \xi_2, \xi_2^*)$ and NORMAL FORM $\mathbf{U}(\xi)$ such that $\Delta_\alpha \mathbf{U} \equiv \mathbf{U}\left(e^{i\alpha}\xi, e^{-i\alpha}\xi^*\right) - e^{i\alpha}\mathbf{U}(\xi, \xi^*) = 0$

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$$\begin{array}{ccc} \mathbf{z} & \xrightarrow{\mathbf{F}} & \mathbf{z}' \\ \Phi\uparrow & & \uparrow\Phi \\ \xi & \xrightarrow{\mathbf{U}} & \xi' \end{array} \Rightarrow \begin{cases} z = \Phi(\xi) = \underbrace{\xi}_{\text{Start with Identity}} + [\Phi(\xi)]_{n\geq 2} \\ \xi' = \underbrace{U(\xi, \xi^*)}_{\mathbf{U}\text{ same linear motion of }\mathbf{F}} = \begin{cases} \xi_1' = e^{i\omega_1}\xi_1 + [U_1(\xi)]_{n\geq 2} \\ \xi_2' = e^{i\omega_2}\xi_2 + [U_2(\xi)]_{n\geq 2} \end{cases} \end{cases} \tag{5}$$

A. Bazzani, G. Servizi, E. Todesco and G. Turchetti, *CERN Yellow Report*, 94-02, 1994.

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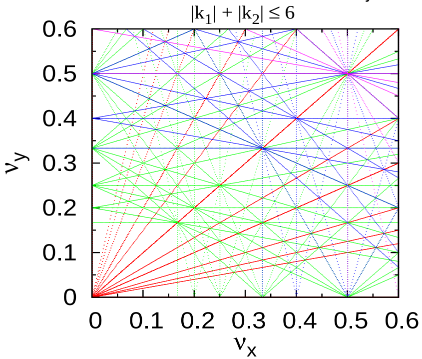

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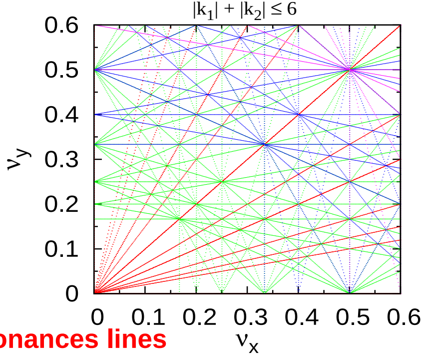
$\omega \cdot \mathbf{k} = 0 \leftrightarrow \mathbf{k} = l\mathbf{e}$

$l \in \mathbb{N}, \mathbf{e} \in \mathbb{N}^3$

DOUBLE-RESONANCE CASE

$\omega \cdot \mathbf{k} = 0 \leftrightarrow \mathbf{k} = l_1\mathbf{e}_1 + l_2\mathbf{e}_2$

$l_j \in \mathbb{N}, \mathbf{e}_j \in \mathbb{N}^3$



Intricate net of resonances lines

4D resonances and Henón map 2/2

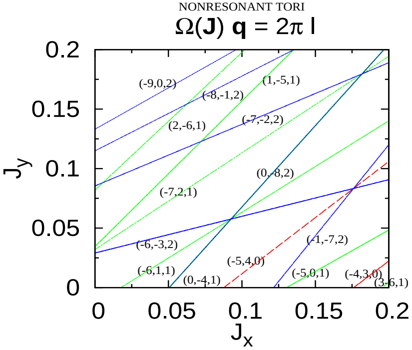
If we care about **non linear stuff**...

4D resonances and Henón map 2/2

If we care about **non linear stuff**...$\Omega = \Omega(j)$

4D resonances and Henón map 2/2

If we care about **non linear stuff**...$\Omega = \Omega(j) \Rightarrow$ **amplitude detuning**
We can write in the action space the resonance lines...



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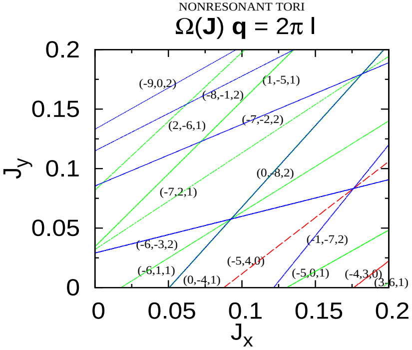

lowest order

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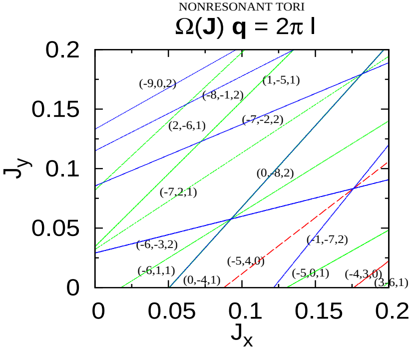

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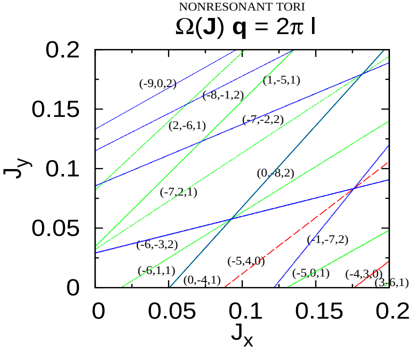

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$$\begin{cases} \Omega_x = \omega_x + 2\,\alpha_{20}\,j_1 + \alpha_{11}\,j_2 \\ \Omega_y = \omega_y + \alpha_{11}\,j_2 + 2\,\alpha_{02}\,j_2 \end{cases} \quad (9)$$

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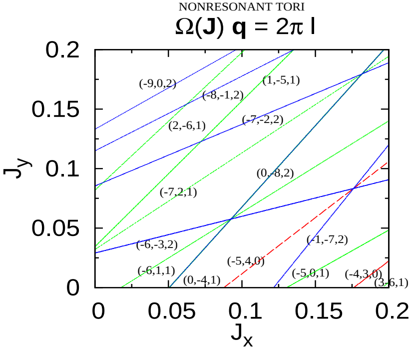

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Resonance conditions as function of emittances $\rightarrow \epsilon_i = 2\,j_i$

Details: A. Bazzani, L. Bongini, G. Turchetti "Analysis of resonances in action space for symplectic maps" Phys. Rev. E 57, 1178

2D Henón map and Normal Forms application 1/3

$x-$plane **sextupolar** dynamics from Eq. (3) if we let $\beta \to 0$

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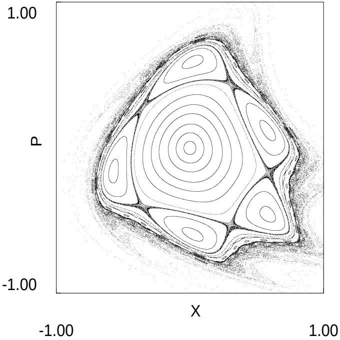

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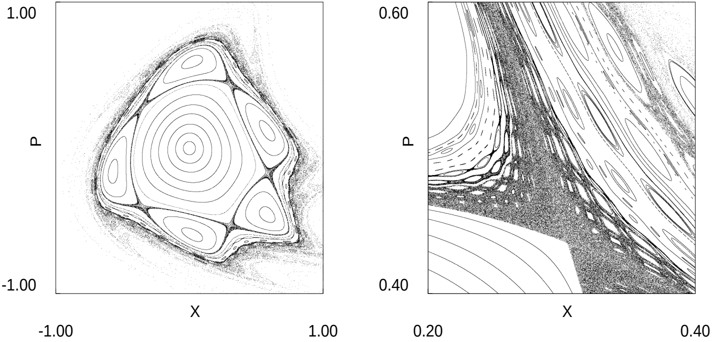

..phase portrait with $\nu = \omega/2\pi = 0.212$...

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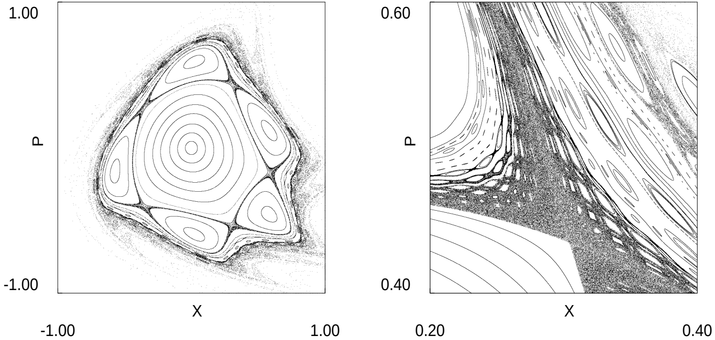

..phase portrait with $\nu = \omega/2\pi = 0.212$...**1/5 resonance!**

2D Henón map and Normal Forms application 2/3

Analytical **nonlinear tune** value...

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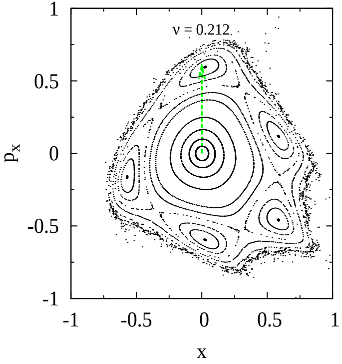

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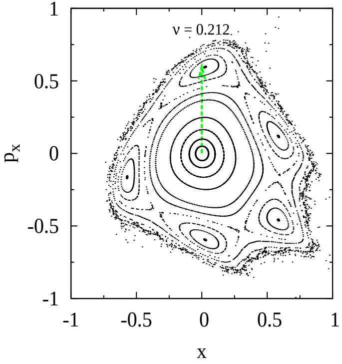

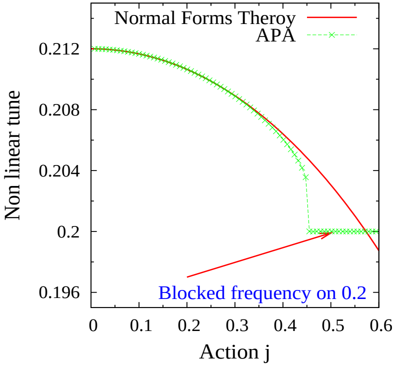

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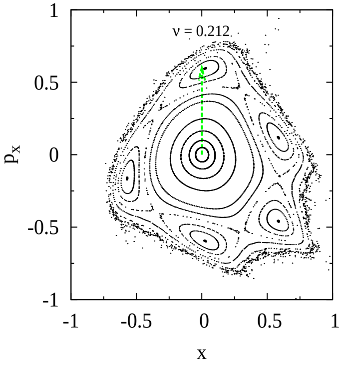
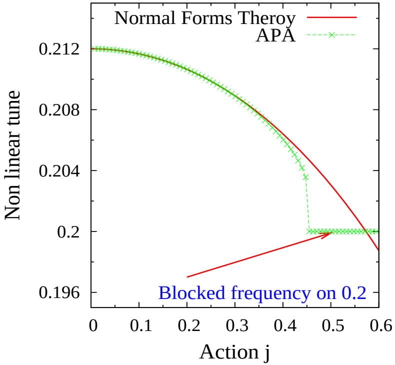
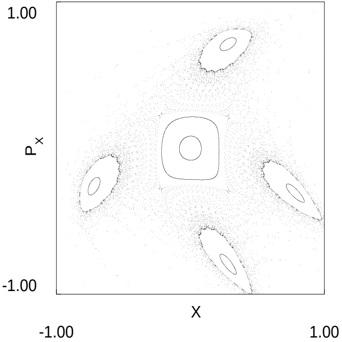




Good agreement between numerical and theoretical values!!!

2D Henón map and Normal Forms application 3/3

We can also build up the **interpolating Hamiltonian** to control the system stability



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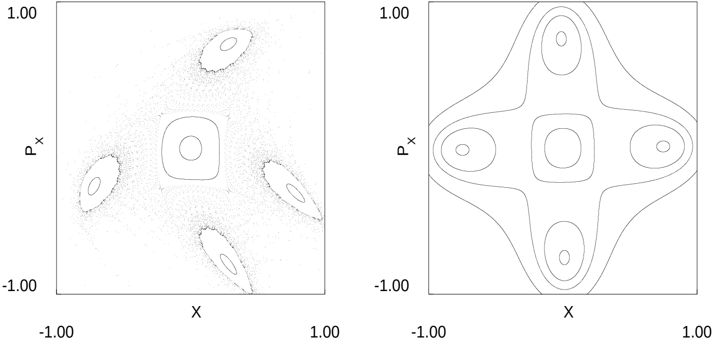

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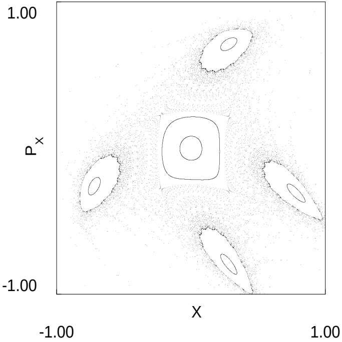

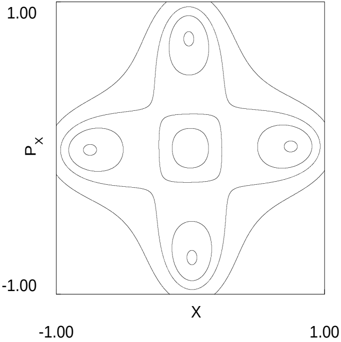

Map iteration (left) flow of $H(\xi, \xi^*)$ (right) ν being close to 1/4 resonance

Outline

Dynamic aperture 1/2

Studies on slow extraction dynamic aperture... exciting the 1/3 resonance

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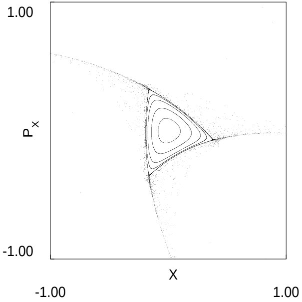

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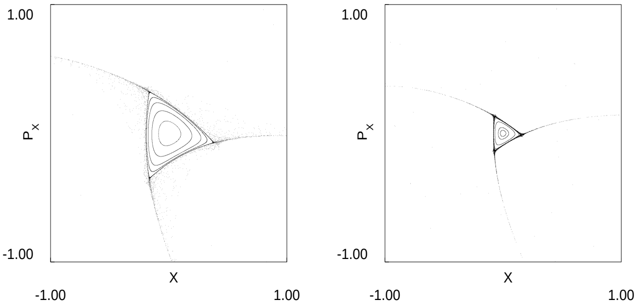

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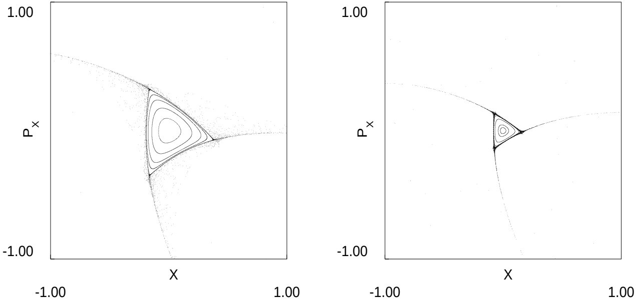


Approaching the (unstable!) $\nu = 1/3$ sextupolar resonance...the dynamic aperture is shrinking... particles get extracted. Studies on hyperbolic NF

L. Bongini, A. Bazzani, G. Turchetti, I. Hofmann *Analysis of a model of resonant extraction of intense beams by normal forms and frequency map* Phys. Rev. Special Topics - Accelerators and Beams **4**, 114201 (2001)

Dynamic aperture 2/2

Nekhoroshev like estimates on the time stability.

Dynamic aperture 2/2

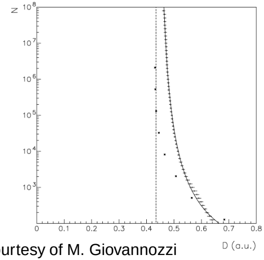
Nekhoroshev like estimates on the time stability.
A particle in $\mathcal{B}(0; r/2)$ remains bounded in $\mathcal{B}'(0; r)$ for a time

$$\tau(r) \geq \tau_0 \exp\left[\left(\frac{r^*}{r}\right)^{\frac{2}{1+d}}\right] \quad \mathcal{D}(N) = \mathcal{D}_\infty\left(1 + \frac{b}{\log^k N}\right) \qquad (11)$$

A. Bazzani, S. Marmi, G. Turchetti, *Nekhoroshev estimates for non resonant symplectic maps* Celestial Mechanics **47**, 333 (1990)

Numerical evidences of this scaling for the Hénon map and for a realistic 4-6D LHC model are given in

M. Giovannozzi, W. Scandale, E. Todesco *Dynamic aperture extrapolation in presence of tune modulation*, Phys. Rev. E **57**, 3432 (1998)



Picture is a courtesy of M. Giovannozzi

Multi Turn Extraction

Splitting the beam in phase space by means of **nonlinear elements**

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As the linear tune changes the particles get trapped in resonance islands

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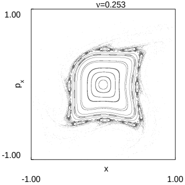

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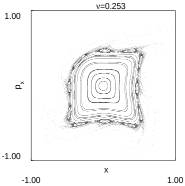

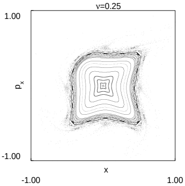

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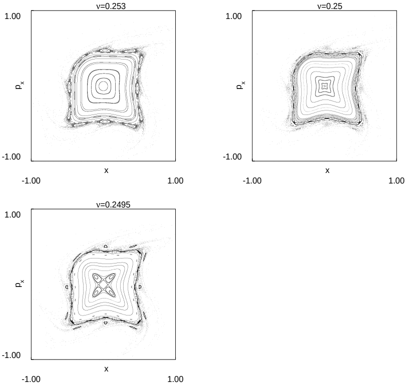

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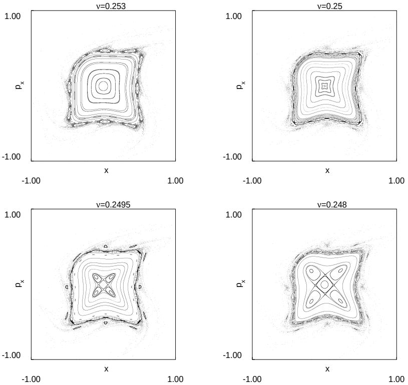

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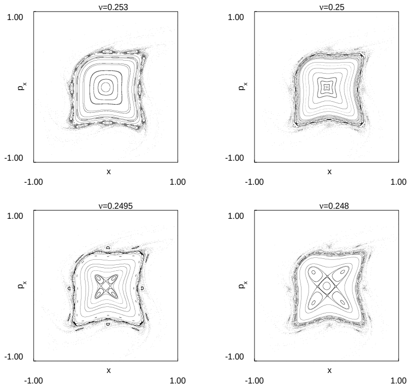

@ CERN using NF theory and improving the number of particles in the islands...

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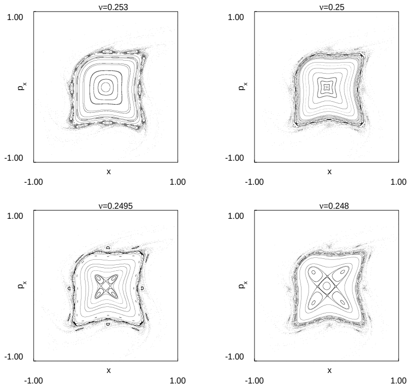

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Experimental data

Movie is a courtesy of A. Franchi

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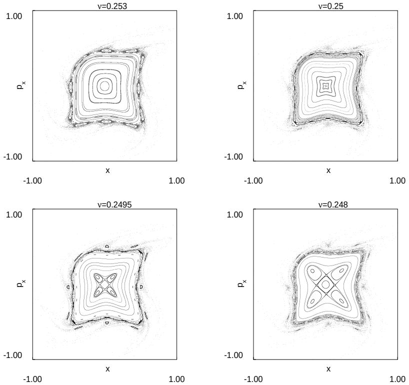

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Nonlinear parameters from NF analytical calculations!

D. Quatraro, Master Degree Thesis, Bologna University

Conclusion and future perspectives

- NF is an efficient tool to deal with nonlinear problems
- Analytical & numerical methods to get informations concerning the stability and the dynamic aperture
- Several experiment @ CERN (MTE) agreed with NF theory
- Extensions also to space charge applications

  C. Benedetti, G. Turchetti *An analytic map for space charge in a nonlinear lattice* Physics Letters **A340**, 461-465 (2006)
- MTE studies still ongoing @ CERN