Classical Gravitational Bound States with Amplitudes

Riccardo Gonzo
based on work with with T.Adamo (2212.13269), C.Shi (2304.06066)
and work in progress with T.Adamo and A.Ilderton

THE UNIVERSITY of EDINBURGH

New Frontiers in Theoretical Physics - XXXVII Convegno Nazionale di Fisica Teorica

Cortona, 27 September 2023

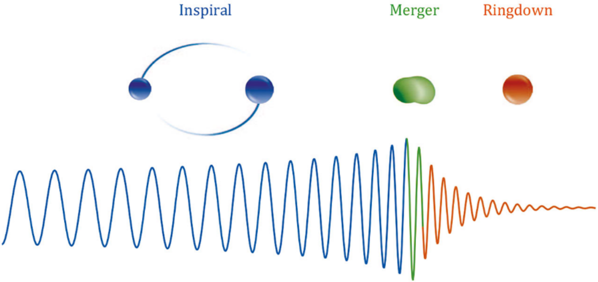
Content

Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.

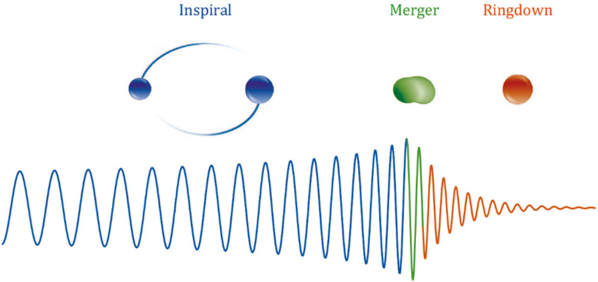
Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.
- We need waveform templates to extract the signal: the effective one-body (EOB) [Buonanno, Damour] allows to combine analytical and numerical techniques valid for different stages of the evolution of compact binaries



Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.
- We need waveform templates to extract the signal: the effective one-body (EOB) [Buonanno, Damour] allows to combine analytical and numerical techniques valid for different stages of the evolution of compact binaries

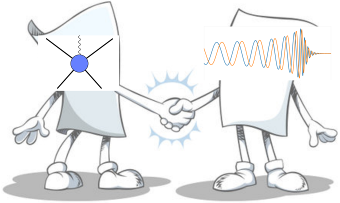


- Today: focus on the inspiral phase, where we can model compact objects as point particles in the spirit of effective field theory [Goldberger,Rothstein]

Motivation and introduction (II)

- Why amplitudes?

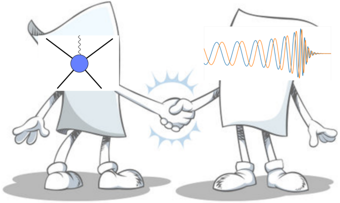
Amplitudes are gauge-invariant, universal objects which encode in a compact way the perturbative scattering dynamics for point particles in a QFT. New perspective on GR!



Motivation and introduction (II)

- Why amplitudes?

Amplitudes are gauge-invariant, universal objects which encode in a compact way the perturbative scattering dynamics for point particles in a QFT. New perspective on GR!



- Question: How can we study classical bound states from amplitude techniques?

This seems an hard question . . .
'It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in an entirely satisfactory shape.'
(Weinberg, QFT I, page 560)

The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the (non-relativistic) hydrogen atom problem with a fixed potential.

The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the (non-relativistic) hydrogen atom problem with a fixed potential.
- We are familiar to the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle\,, \qquad H = \frac{p^2}{2m} + V, \qquad V(r) \propto -\frac{g}{r}\,,$$

which can be solved exactly (at all orders in the coupling g)

$E > 0 \leftrightarrow$ scattering plane wave $\psi \propto e^{i\vec{k}^{>}\cdot\vec{x}}$ $\leftrightarrow$ continuous spectrum $E_{\vec{k}}$

$E < 0 \leftrightarrow$ normalizable wavefunction $\psi \propto e^{-\vec{k}^{<}\cdot\vec{x}}$ $\leftrightarrow$ discrete spectrum E_n

where $>$ (resp. $<$) stands for scattering orbits (resp. bound orbits).

The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the (non-relativistic) hydrogen atom problem with a fixed potential.
- We are familiar to the eigenvalue problem

$$H\,|\psi\rangle = E\,|\psi\rangle\ , \qquad H = \frac{p^2}{2m} + V, \qquad V(r) \propto -\frac{g}{r}\ ,$$

which can be solved exactly (at all orders in the coupling g)

$$E > 0 \leftrightarrow \text{scattering plane wave}\ \psi \propto e^{i\vec{k}^{>}\cdot\vec{x}} \leftrightarrow \text{continuous spectrum}\ E_{\vec{k}}$$

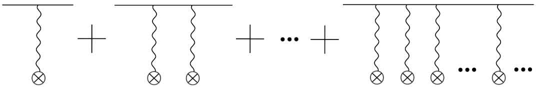
$$E < 0 \leftrightarrow \text{normalizable wavefunction}\ \psi \propto e^{-\vec{k}^{<}\cdot\vec{x}} \leftrightarrow \text{discrete spectrum}\ E_n$$

where $>$ (resp. $<$) stands for scattering orbits (resp. bound orbits).

- Note: In partial wave basis there is an analytic continuation $k^{>} \to i\,k^{<}$ ($\sqrt{y^2-1} \to i\sqrt{1-y^2}$ with $y = E/m$) from scattering to bound solutions in the semiclassical limit of large quantum numbers $\hbar n \to L$ [Adamo, RG].

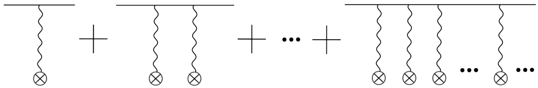
The bound state equation in quantum mechanics (II)

- Using perturbation theory this would correspond to an infinite sum $V + VGV + \cdots + V(GV)^n$: bound states are intrinsically non-perturbative!

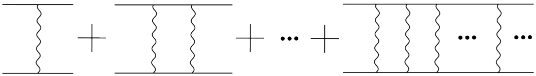


The bound state equation in quantum mechanics (II)

- Using perturbation theory this would correspond to an infinite sum $V + VGV + \cdots + V(GV)^n$: bound states are intrinsically non-perturbative!



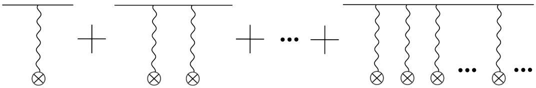
- The natural generalization of the previous picture to the non-relativistic two-body problem is given by the "ladder approximation"



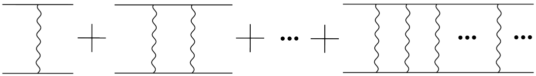
which is actually used to study QED bound systems like positronium e^+e^-!

The bound state equation in quantum mechanics (II)

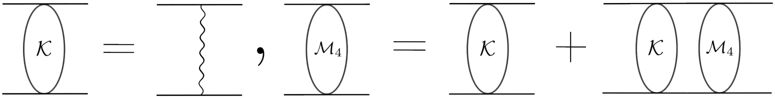
- Using perturbation theory this would correspond to an infinite sum $V + VGV + \cdots + V(GV)^n$: bound states are intrinsically non-perturbative!



- The natural generalization of the previous picture to the non-relativistic two-body problem is given by the "ladder approximation"



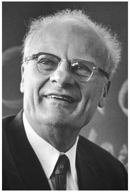
which is actually used to study QED bound systems like positronium e^+e^-! We can write it as an amplitude recursion relation

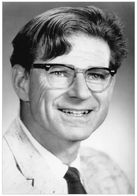


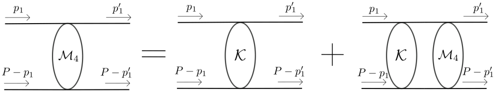
which is nothing else that the (quantum) Bethe-Salpeter equation!

The bound state equation in quantum field theory

- The Bethe-Salpeter equation is a non-perturbative recursion relation for 4-pt amplitudes, which generate the bound state energy poles via the iteration of a two-massive particle irreducible kernel $\mathcal{K}$







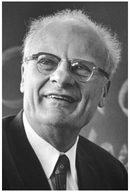
Bethe- Salpeter equation

$$\mathcal{M}_4(p_1, p_1'; P) = \mathcal{K}(p_1, p_1'; P) + \int \hat{\mathrm{d}}^4 l\, \mathcal{K}(p_1, l; P) G(l, P) \mathcal{M}_4(l, p_1'; P)\,,$$

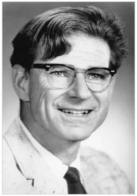
where $G(l, P)$ is the two-body propagator.

The bound state equation in quantum field theory

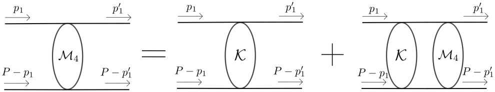
- The Bethe-Salpeter equation is a non-perturbative recursion relation for 4-pt amplitudes, which generate the bound state energy poles via the iteration of a two-massive particle irreducible kernel $\mathcal{K}$



Bethe-



Salpeter



equation

$$\mathcal{M}_4(p_1, p_1'; P) = \mathcal{K}(p_1, p_1'; P) + \int \hat{\mathrm{d}}^4 l\, \mathcal{K}(p_1, l; P) G(l, P) \mathcal{M}_4(l, p_1'; P)\,,$$

where $G(l, P)$ is the two-body propagator.

- How can we take the classical limit?

What is the classical expansion?

- Framework: QFT scattering amplitudes techniques for the classical gravitational interaction of two massive (spinless or spinning) point particles

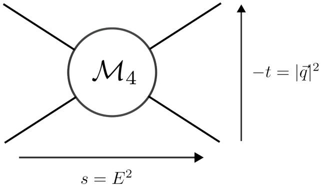
What is the classical expansion?

- Framework: QFT scattering amplitudes techniques for the classical gravitational interaction of two massive (spinless or spinning) point particles
- Consider the conservative 4-pt amplitude: the classical expansion $\hbar \to 0$ is equivalent to consider the Heavy Particle Effective Theory (HEFT) scheme [Damgaard,Aoude,Haddad,Helset;Brandhuber,Chen,Travaglini,Wen]

$$p_1^\mu := p_A^\mu + \hbar\frac{\bar{q}^\mu}{2}\,, \qquad (p_1')^\mu := p_A^\mu - \hbar\frac{\bar{q}^\mu}{2}\,, \qquad s = (p_A + p_B)^2\,,$$

$$p_2^\mu := p_B^\mu - \hbar\frac{\bar{q}^\mu}{2}\,, \qquad (p_2')^\mu := p_B^\mu + \hbar\frac{\bar{q}^\mu}{2}\,, \qquad t = -\hbar^2|\vec{\bar{q}}|^2\,,$$

where p_A, p_B are the classical momenta and q is the momentum transfer.



Equivalently, we can use the eikonal formulation (see Carlo's talk!)

The classical Bethe-Salpeter equation

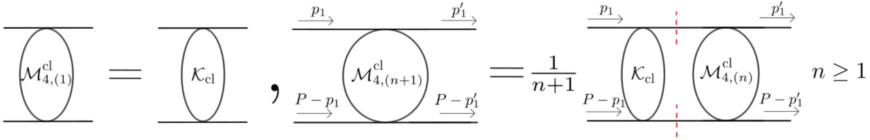
- We obtain the classical Bethe-Salpeter equation from quotienting diagrams by symmetrization over internal graviton exchanges: [Adamo, RG]

$$\mathcal{M}^{\text{cl}}_{4,(n+1)}(p_A, p_B, q) = \begin{cases} \mathcal{K}_{\text{cl}}(p_A, p_B, q) & \text{if } n = 0 \\ \frac{1}{n+1} \int \hat{\mathrm{d}}^4 l \, \mathcal{K}_{\text{cl}}(p_A, p_B, l) G_{\text{cl}}(p_A, p_B, l) \mathcal{M}^{\text{cl}}_{4,(n)}(p_A, p_B, q - l) & \text{if } n \geq 1 \end{cases} .$$

where the two-body propagator is replaced by its on-shell version

$$G_{\text{cl}}(p_A, p_B, l) = \hat{\delta}(2l \cdot p_A) \hat{\delta}(2l \cdot p_B) \, ,$$

and (n) is the number of classical two-massive particle irreducible diagrams.



Exponentiation of the classical kernel: an exact solution

- Going to impact parameter space (i.e. to the partial wave basis)

$$\widetilde{f}\,(x_q) \equiv \int \hat{\mathrm{d}}^4 q \hat{\delta}\,(2p_A \cdot q)\,\hat{\delta}\,(2p_B \cdot q)\, e^{i(q\cdot x_q)/\hbar} f(q)\,,$$

the classical BSE becomes

$$\widetilde{\mathcal{M}}^{\mathrm{cl}}_{4,(n+1)}(p_A, p_B, x_\perp) = \begin{cases} \widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, x_\perp) & \text{if } n = 0 \\ \frac{1}{n+1}\widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, x_\perp)\widetilde{\mathcal{M}}^{\mathrm{cl}}_{4,(n)}(p_A, p_B, x_\perp) & \text{if } n \geq 1 \end{cases},$$

which means that the final solution exponentiates exactly [Adamo,RG]

$$\widetilde{\mathcal{M}}^{\mathrm{cl}}_4(p_A, p_B, x_\perp) = e^{\widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, x_\perp)}\,.$$

Natural generalization for spinning particles!

Exponentiation of the classical kernel: an exact solution

- Going to impact parameter space (i.e. to the partial wave basis)

$$\widetilde{f}\,(x_q) \equiv \int \hat{\mathrm{d}}^4 q \hat{\delta}\,(2p_A \cdot q)\,\hat{\delta}\,(2p_B \cdot q)\, e^{i(q\cdot x_q)/\hbar} f(q)\,,$$

the classical BSE becomes

$$\widetilde{\mathcal{M}}^{\mathrm{cl}}_{4,(n+1)}(p_A, p_B, x_\perp) = \begin{cases} \widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, x_\perp) & \text{if } n = 0 \\ \frac{1}{n+1}\widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, x_\perp)\widetilde{\mathcal{M}}^{\mathrm{cl}}_{4,(n)}(p_A, p_B, x_\perp) & \text{if } n \geq 1 \end{cases},$$

which means that the final solution exponentiates exactly [Adamo,RG]

$$\widetilde{\mathcal{M}}^{\mathrm{cl}}_4(p_A, p_B, x_\perp) = e^{\widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, x_\perp)}\,.$$

Natural generalization for spinning particles!

- The analytic structure (poles, etc.) in momentum space arise completely from

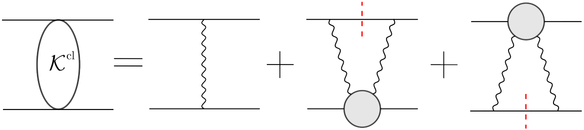
$$\mathrm{i}\mathcal{M}^{\mathrm{cl}}_4(p_A, p_B; q_\perp) = \frac{4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}}{\hbar^2} \int \mathrm{d}^2 x_\perp \mathrm{e}^{-\mathrm{i}\bar{q}_\perp \cdot x_\perp} \left(\mathrm{e}^{\widetilde{\mathcal{K}}_{\mathrm{cl}}(p_A, p_B, x_\perp)} - 1\right).$$

An example: classical kernel for spinless particles at 2PM

- We can consider for example the classical kernel up to 2 PM

$$\widetilde{\mathcal{K}}^{\mathrm{cl},>}(p_A,p_B,x_\perp) = \frac{i}{\hbar}\Bigg[-2G_N \log(\mu_{\mathrm{IR}}|x_\perp|) m_A m_B \frac{2y^2-1}{\sqrt{y^2-1}} + \frac{3\pi}{4} G_N^2 m_A m_B (m_A+m_B) \frac{5y^2-1}{\sqrt{y^2-1}} \frac{1}{|x_\perp|} \Bigg],$$

which encodes the conservative dynamics of two spinless particles.

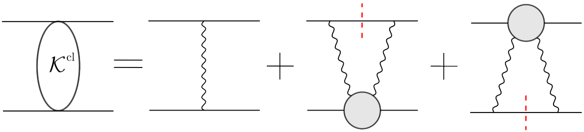


An example: classical kernel for spinless particles at 2PM

- We can consider for example the classical kernel up to 2 PM

$$\widetilde{\mathcal{K}}^{\mathrm{cl},>}(p_A,p_B,x_\perp)=\frac{i}{\hbar}\Bigg[-2G_N\log(\mu_{\mathrm{IR}}|x_\perp|)m_Am_B\frac{2y^2-1}{\sqrt{y^2-1}} + \frac{3\pi}{4}G_N^2m_Am_B(m_A+m_B)\frac{5y^2-1}{\sqrt{y^2-1}}\frac{1}{|x_\perp|}\Bigg],$$

which encodes the conservative dynamics of two spinless particles.

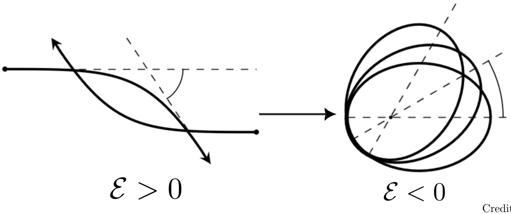


- Note that the motion is restricted to a plane and completely determined by the conserved quantities $(\mathcal{E}, L)$!

$$\mathcal{E}:=\frac{E-m_A-m_B}{\mu}\,,\qquad L=p_\infty(E,m_A,m_B)|x_\perp|\,,\qquad y=\frac{E^2-m_A^2-m_B^2}{2m_Am_B}\,,$$

The Hamilton-Jacobi action from amplitudes (I)

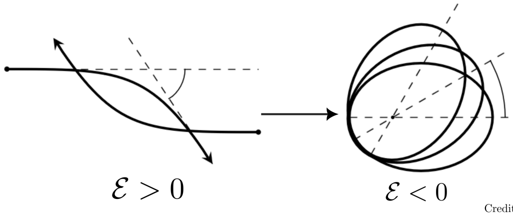
- Since $\mathcal{E} > 0$ for scattering orbits and $\mathcal{E} < 0$ for bound orbits, how do we perform an analytic continuation?



Credit: Kälin,Porto

The Hamilton-Jacobi action from amplitudes (I)

- Since $\mathcal{E} > 0$ for scattering orbits and $\mathcal{E} < 0$ for bound orbits, how do we perform an analytic continuation?



- Natural connection of the kernel with the scattering Hamilton-Jacobi action

$$\widetilde{\mathcal{K}}_{\text{cl}}^{>}(p_A, p_B; x_\perp) = \frac{i}{\hbar} I^{>}(\mathcal{E}, L) \ , \quad I_r^{>}(\mathcal{E}, L) = \oint_{\mathcal{C}^{>}} dr\, p_r(r, \mathcal{E}, L) + L\pi \ ,$$

where p_r is the radial momentum and $\mathcal{C}^{>}$ is the contour of integration for scattering orbits. This is the "amplitude-action" relation! [Bern et al.; Damgaard,Plante,Vanhove; Kol,O'Connell,Telem; Adamo,RG]

The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between the boundary conditions for scattering and bound planar orbits [Kälin,Porto]

$$\int_{\mathcal{C}_r^>} = 2\int_{r_m(\mathcal{E},L)}^{\infty}\,, \qquad \int_{\mathcal{C}_r^<} = 2\int_{r_-(\mathcal{E},L)}^{r_+(\mathcal{E},L)}\,,$$

$$r_-(\mathcal{E},L) \overset{\mathcal{E}\leq 0}{=} r_m(\mathcal{E},L)\,, \qquad r_+(\mathcal{E},L) \overset{\mathcal{E}\leq 0}{=} r_m(\mathcal{E},-L)\,,$$

thanks to which (p_r is invariant under $L \to -L$)

$$I_r^<(\mathcal{E}<0,L) = I_r^>(\mathcal{E}<0,L) - I_r^>(\mathcal{E}<0,-L)\,.$$

Alternatively, analytically continue in the rapidity y at fixed L [Adamo, RG; Di Vecchia, Heissenberg, Russo, Veneziano]

$$I_r^<\left(\sqrt{y^2-1},L\right) = I_r^>\left(i\sqrt{1-y^2},L\right) + I_r^>\left(-i\sqrt{1-y^2},L\right)\,.$$

The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between the boundary conditions for scattering and bound planar orbits [Kälin,Porto]

$$\int_{\mathcal{C}_r^>} = 2\int_{r_m(\mathcal{E},L)}^{\infty}\,, \qquad \int_{\mathcal{C}_r^<} = 2\int_{r_-(\mathcal{E},L)}^{r_+(\mathcal{E},L)}\,,$$

$$r_-(\mathcal{E},L) \overset{\mathcal{E}\leq 0}{=} r_m(\mathcal{E},L)\,, \qquad r_+(\mathcal{E},L) \overset{\mathcal{E}\leq 0}{=} r_m(\mathcal{E},-L)\,,$$

thanks to which (p_r is invariant under $L \to -L$)

$$I_r^<(\mathcal{E}<0,L) = I_r^>(\mathcal{E}<0,L) - I_r^>(\mathcal{E}<0,-L)\,.$$

Alternatively, analytically continue in the rapidity y at fixed L [Adamo, RG; Di Vecchia, Heissenberg, Russo, Veneziano]

$$I_r^<\left(\sqrt{y^2-1},L\right) = I_r^>\left(i\sqrt{1-y^2},L\right) + I_r^>\left(-i\sqrt{1-y^2},L\right)\,.$$

- This picture generalize to the case of aligned-spin particles $\vec{L}//\vec{a}_1,\vec{a}_2$, since the motion still remains on the equatorial plane. [Kälin,Porto]

The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between the boundary conditions for scattering and bound planar orbits [Kälin,Porto]

$$\int_{\mathcal{C}_r^>} = 2\int_{r_m(\mathcal{E},L)}^{\infty} , \qquad \int_{\mathcal{C}_r^<} = 2\int_{r_-(\mathcal{E},L)}^{r_+(\mathcal{E},L)} ,$$

$$r_-(\mathcal{E},L) \overset{\mathcal{E}\leq 0}{=} r_m(\mathcal{E},L) , \qquad r_+(\mathcal{E},L) \overset{\mathcal{E}\leq 0}{=} r_m(\mathcal{E},-L) ,$$

  thanks to which (p_r is invariant under $L \to -L$)

$$I_r^< (\mathcal{E} < 0, L) = I_r^> (\mathcal{E} < 0, L) - I_r^> (\mathcal{E} < 0, -L) \; .$$

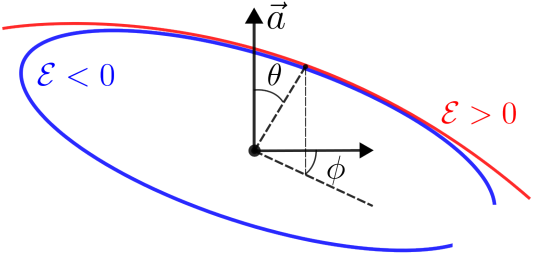
  Alternatively, analytically continue in the rapidity y at fixed L [Adamo, RG; Di Vecchia, Heissenberg, Russo, Veneziano]

$$I_r^< \left(\sqrt{y^2-1}, L\right) = I_r^> \left(i\sqrt{1-y^2}, L\right) + I_r^> \left(-i\sqrt{1-y^2}, L\right) \; .$$

- This picture generalize to the case of aligned-spin particles $\vec{L}//\vec{a}_1, \vec{a}_2$, since the motion still remains on the equatorial plane. [Kälin,Porto]
- What happens when $\vec{L}$ and $\vec{a}$ are oriented in generic directions?

Hamiltonian for geodesics in Kerr

- Let's consider the generic orbit in a Kerr black hole of spin $\vec{a}$ for a massive point particle with orbital angular momentum $\vec{L} \cdot \vec{a} \neq |\vec{L}||\vec{a}|$



which are described by the geodesic Hamiltonian $H(x, p) = 1/2\, g^{\mu\nu}_{\text{Kerr}} p_\mu p_\nu$ in some convenient Boyer-Lindquist coordinates (t, r, θ, ϕ).

The Hamilton-Jacobi action for generic Kerr orbits

- Using the integrability of Kerr, we can generalize the H-J action for generic orbits of a massive spinless probe in a Kerr black hole [Carter]

$$\boxed{I := I_r + I_\theta}\,, \qquad I_r = \int_{\mathcal{C}_r} p_r \, \mathrm{d}r\,, \qquad I_\theta = \int_{\mathcal{C}_\theta} p_\theta \, \mathrm{d}\theta\,,$$

with the radial (resp.polar) momentum p_r (resp.p_θ) and contour $\mathcal{C}_r$ (resp. $\mathcal{C}_\theta$)

The Hamilton-Jacobi action for generic Kerr orbits

- Using the integrability of Kerr, we can generalize the H-J action for generic orbits of a massive spinless probe in a Kerr black hole [Carter]

$$\boxed{I := I_r + I_\theta}\,, \qquad I_r = \int_{\mathcal{C}_r} p_r \, \mathrm{d}r\,, \qquad I_\theta = \int_{\mathcal{C}_\theta} p_\theta \, \mathrm{d}\theta\,,$$

with the radial (resp.polar) momentum p_r (resp.p_θ) and contour $\mathcal{C}_r$ (resp. $\mathcal{C}_\theta$)

- The radial and the polar action obey the analytic continuation [RG, Shi]

$$I_r^<(\mathcal{E}, L, a, L_Q) \overset{\mathcal{E}\leq 0}{=} I_r^>(\mathcal{E}, L, a, L_Q) - I_r^>(\mathcal{E}, -L, -a, -L_Q)\,,$$

$$I_\theta^<(\mathcal{E}, L, a, L_Q; n^>, \eta^>_{\text{out}}) \overset{\mathcal{E}\leq 0}{=} I_\theta^>(\mathcal{E}, L, a, L_Q; n^<, \eta^<_{\text{out}})\,.$$

How do we compute scattering and bound observables?

Azimuthal deflection angle for generic Kerr orbits

- Once we have the H-J action, we can obtain the azimuthal deflection angle by differentiation over L

$$\Delta\phi + \pi = -\frac{\partial I}{\partial L} = -\frac{\partial I_r}{\partial L} - \frac{\partial I_\theta}{\partial L}$$

which gives in the perturbative expansion in $G_N M$ ($a \ll G_N M$)

$$\Delta\phi = \frac{2G_N M m L(2\mathcal{E}+1)}{\sqrt{\mathcal{E}} L_Q^2} + \frac{3\pi G_N^2 M^2 m^2 L(5\mathcal{E}+4)}{4L_Q^3} + a\sqrt{\mathcal{E}+1}\left(\frac{4G_N M m^2 \sqrt{\mathcal{E}}(L_Q^2 - 2L^2)}{L_Q^4}\right) + \dots$$

Azimuthal deflection angle for generic Kerr orbits

- Once we have the H-J action, we can obtain the azimuthal deflection angle by differentiation over L

$$\Delta\phi + \pi = -\frac{\partial I}{\partial L} = -\frac{\partial I_r}{\partial L} - \frac{\partial I_\theta}{\partial L}$$

which gives in the perturbative expansion in $G_N M$ ($a \ll G_N M$)

$$\Delta\phi = \frac{2G_N MmL(2\mathcal{E}+1)}{\sqrt{\mathcal{E}}L_Q^2} + \frac{3\pi G_N^2 M^2 m^2 L(5\mathcal{E}+4)}{4L_Q^3} + a\sqrt{\mathcal{E}+1}\left(\frac{4G_N Mm^2\sqrt{\mathcal{E}}(L_Q^2 - 2L^2)}{L_Q^4}\right) + \dots$$

- Matches known result in the equatorial limit $Q \to 0$ [Damgaard, Hoogeveen, Luna, Vines], but the expansion is new for generic orbits! [RG,Shi]

Time delay for generic null Kerr orbits

- The time delay is obtained by differentiating over the energy E in the HJ action. For generic null geodesics with fixed b relative to an observer with $b' \gg b$ but at the same energy $E' = E$ [Camanho, Edelstein, Maldacena, Zhiboedov; Accettulli Huber, Brandhuber, De Angelis, Travaglini]

$$
\begin{aligned}
\Delta T &= \left.\frac{\partial I}{\partial E}\right|_{L_Q, E} - \left.\frac{\partial I}{\partial E}\right|_{L'_Q \gg L_Q, E'=E} \\
&= 4 G_N M \log\left(\frac{L'_Q}{L_Q}\right) + \frac{15\pi G_N^2 M^2 E}{2 L_Q} + \frac{64 G_N^3 M^3 E^2}{L_Q^2} \\
&\quad - \frac{aLE}{L_Q^4}\left(8 G_N M L_Q^2 + 15\pi G_N^2 M^2 E L_Q + 256 G_N^3 M^3 E^2\right) + \dots
\end{aligned}
$$

which in the equatorial limit matches [Bautista, Guevara, Kavanagh, Vines].

Fundamental frequencies for Kerr orbits

- The H-J action provides an intrinsic definition of the fundamental frequencies for the bound motion $\omega_r, \omega_\phi, \omega_\theta$ via the action-angle representation [Schmidt]:

$$\omega_r = -\frac{1}{\Omega}\frac{\partial J_\theta}{\partial Q}\,, \quad \omega_\theta = \frac{1}{\Omega}\frac{\partial J_r}{\partial Q}\,, \quad \omega_\phi = \frac{1}{\Omega}\left(\frac{\partial J_r}{\partial L}\frac{\partial J_\theta}{\partial Q} - \frac{\partial J_r}{\partial Q}\frac{\partial J_\theta}{\partial L}\right),$$

with $\Omega := \frac{\partial J_r}{\partial H}\frac{J_\theta}{\partial Q} - \frac{\partial J_r}{\partial Q}\frac{\partial J_\theta}{\partial H}$, $J_\phi = L$, $J_r = \oint p_r \, \mathrm{d}r = I_r^{<}$, $J_\theta = \oint p_\theta \mathrm{d}\theta = I_\theta^{<(1)}$.

Fundamental frequencies for Kerr orbits

- The H-J action provides an intrinsic definition of the fundamental frequencies for the bound motion $\omega_r, \omega_\phi, \omega_\theta$ via the action-angle representation [Schmidt]:

$$\omega_r = -\frac{1}{\Omega}\frac{\partial J_\theta}{\partial Q}\,, \quad \omega_\theta = \frac{1}{\Omega}\frac{\partial J_r}{\partial Q}\,, \quad \omega_\phi = \frac{1}{\Omega}\left(\frac{\partial J_r}{\partial L}\frac{\partial J_\theta}{\partial Q} - \frac{\partial J_r}{\partial Q}\frac{\partial J_\theta}{\partial L}\right),$$

with $\Omega := \frac{\partial J_r}{\partial H}\frac{J_\theta}{\partial Q} - \frac{\partial J_r}{\partial Q}\frac{\partial J_\theta}{\partial H}$, $J_\phi = L$, $J_r = \oint p_r \, \mathrm{d}r = I_r^<$, $J_\theta = \oint p_\theta \mathrm{d}\theta = I_\theta^{<(1)}$.

- From the amplitude perspective we want bound observables invariant under the choice of the time coordinate, i.e. the frequency ratios [RG, Shi]

$$K^{\phi r} := \frac{\omega_\phi}{\omega_r} = \frac{\partial J_r/\partial Q}{\partial J_\theta/\partial Q}\frac{\partial J_\theta}{\partial L} - \frac{\partial J_r}{\partial L}, \qquad K^{\theta r} := \frac{\omega_\theta}{\omega_r} = -\frac{\partial J_r/\partial Q}{\partial J_\theta/\partial Q}\,.$$

corresponding to the precession of the periastron and of the orbital plane.

Frequency ratios for Kerr orbits

- The periastron advance is [RG,Shi]

$$K^{\phi r} = 1 + \frac{3G_N^2 M^2 m^2 (5\mathcal{E}+4)}{4L_Q^2} + \frac{a G_N^2 M^2 m^3 \sqrt{\mathcal{E}+1}(L_Q - 3L)(5\mathcal{E}+2)}{L_Q^4}$$
$$+ \frac{3a^2 G_N^2 M^2 m^4}{32 L_Q^6}\left[L^2(445\mathcal{E}^2 + 416\mathcal{E} + 40) - L_Q(L_Q + 2L)(85\mathcal{E}^2 + 80\mathcal{E} + 8)\right]$$

which matches [Kälin,Porto] in the equatorial limit.

Frequency ratios for Kerr orbits

- The periastron advance is [RG,Shi]

$$K^{\phi r} = 1 + \frac{3G_N^2 M^2 m^2 (5\mathcal{E}+4)}{4L_Q^2} + \frac{a G_N^2 M^2 m^3 \sqrt{\mathcal{E}+1}(L_Q - 3L)(5\mathcal{E}+2)}{L_Q^4} + \frac{3a^2 G_N^2 M^2 m^4}{32 L_Q^6}\left[L^2(445\mathcal{E}^2 + 416\mathcal{E} + 40) - L_Q(L_Q + 2L)(85\mathcal{E}^2 + 80\mathcal{E} + 8)\right]$$

  which matches [Kälin,Porto] in the equatorial limit.

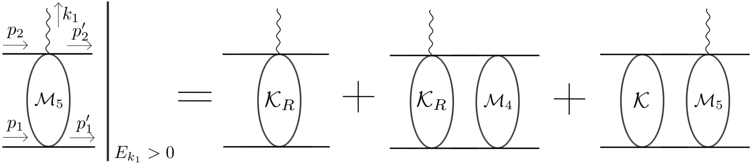
- The precession of the orbital plane is

$$K^{\theta r} = 1 + \frac{3G_N^2 M^2 m^2 (5\mathcal{E}+4)}{4L_Q^2} - \frac{3a G_N^2 M^2 m^3 L \sqrt{\mathcal{E}+1}(5\mathcal{E}+2)}{L_Q^4} + \frac{3a^2 G_N^2 M^2 m^4}{32 L_Q^6}\left[L^2(445\mathcal{E}^2 + 416\mathcal{E} + 40) - L_Q^2(85\mathcal{E}^2 + 80\mathcal{E} + 8)\right].$$

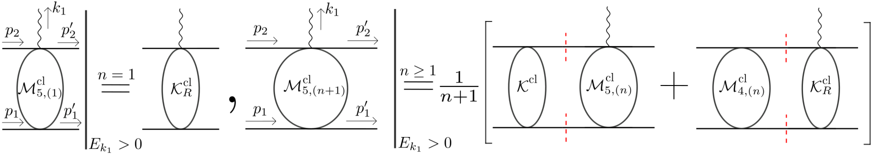
As expected frequencies become degenerate in the spinless case $\omega_\theta \overset{a\to 0}{\to} \omega_\phi$

Radiative observables in the S-matrix formalism (I)

- How can the previous picture be generalized in the presence of radiation? Consider the 5-pt recursion with the emission of a positive energy graviton



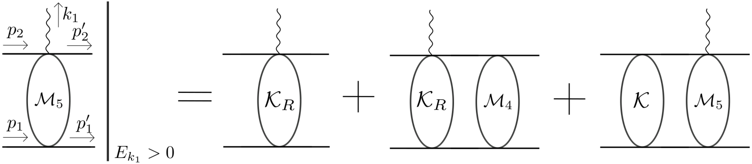
and apply the symmetrization procedure [Adamo, RG, Ilderton]



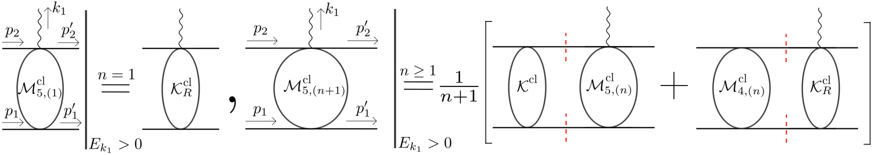
A similar recursion holds for the emission of n gravitons.

Radiative observables in the S-matrix formalism (I)

- How can the previous picture be generalized in the presence of radiation? Consider the 5-pt recursion with the emission of a positive energy graviton



and apply the symmetrization procedure [Adamo, RG, Ilderton]



A similar recursion holds for the emission of n gravitons.

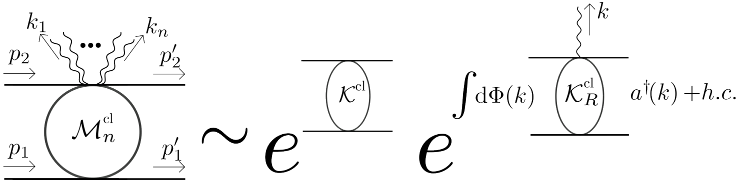
- Can we find an exact solution from the resummation?

Radiative observables in the S-matrix formalism (II)

- The classical S-matrix is [Cristofoli, RG, Moynihan, O'Connell, Ross, Sergola, White; Britto, RG, Jehu; Di Vecchia, Heissenberg, Russo, Veneziano]

$$\widetilde{\mathcal{S}}^{\mathrm{cl}}\Big|_{E_{k_1},\ldots,E_{k_N}>0} \sim e^{\widetilde{\mathcal{K}}^{\mathrm{cl}}(p_A,p_B;x_{q_1},x_{q_2})} e^{\sum_\sigma \int \mathrm{d}\Phi(k)\widetilde{\mathcal{K}}^{\mathrm{cl}}_{\mathcal{R}}(p_A,p_B;x_{q_1},x_{q_2},k^\sigma)a^\dagger_\sigma(k)+\mathrm{h.c.}},$$

with a coherent state of gravitons representing the gravitational wave

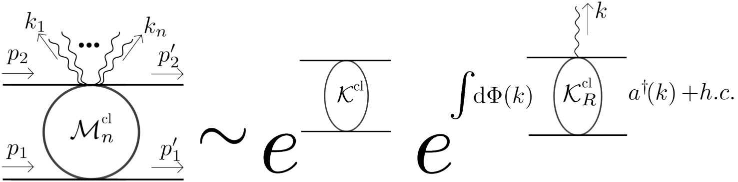


Radiative observables in the S-matrix formalism (II)

- The classical S-matrix is [Cristofoli, RG, Moynihan, O'Connell, Ross, Sergola, White; Britto, RG, Jehu; Di Vecchia, Heissenberg, Russo, Veneziano]

$$\widetilde{\mathcal{S}}^{\rm cl}\Big|_{E_{k_1},\ldots,E_{k_N}>0} \sim e^{\widetilde{\mathcal{K}}^{\rm cl}(p_A,p_B;x_{q_1},x_{q_2})} e^{\sum_\sigma \int \mathrm{d}\Phi(k)\widetilde{\mathcal{K}}^{\rm cl}_{\mathcal{R}}(p_A,p_B;x_{q_1},x_{q_2},k^\sigma)a^\dagger_\sigma(k)+\text{h.c.}},$$

with a coherent state of gravitons representing the gravitational wave



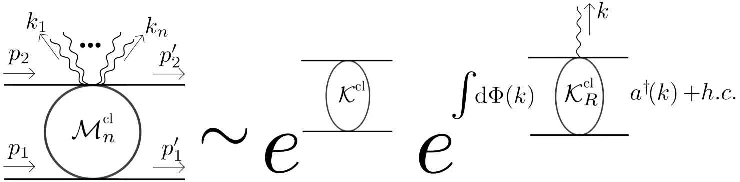
- Comments: 1) All amplitude observables for the two-body problem can derived from such gauge-invariant representation; 2) Compact expression which unifies the treatment of potential and radiative modes.

Radiative observables in the S-matrix formalism (II)

- The classical S-matrix is [Cristofoli, RG, Moynihan, O'Connell, Ross, Sergola, White; Britto, RG, Jehu; Di Vecchia, Heissenberg, Russo, Veneziano]

$$\widetilde{\mathcal{S}}^{\rm cl}\Big|_{E_{k_1},\dots,E_{k_N}>0} \sim e^{\widetilde{\mathcal{K}}^{\rm cl}(p_A,p_B;x_{q_1},x_{q_2})} e^{\sum_\sigma \int \mathrm{d}\Phi(k)\widetilde{\mathcal{K}}^{\rm cl}_{\mathcal{R}}(p_A,p_B;x_{q_1},x_{q_2},k^\sigma)a^\dagger_\sigma(k)+\text{h.c.}},$$

with a coherent state of gravitons representing the gravitational wave



- Comments: 1) All amplitude observables for the two-body problem can derived from such gauge-invariant representation; 2) Compact expression which unifies the treatment of potential and radiative modes.
- Open problem: can we understand the analytic continuation of the waveform?

Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes

Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes
- We solved the BS equation in impact parameter space, making a connection with the Hamilton-Jacobi action via the amplitude-action relation

Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes
- We solved the BS equation in impact parameter space, making a connection with the Hamilton-Jacobi action via the amplitude-action relation
- We discussed a generalization of the boundary to bound dictionary for generic angular momentum orientations (i.e. including both radial and polar action), focusing on a massive probe particle moving in a Kerr background and its scattering (azimuthal and polar deflection angles, time delay) and bound (periastron advance, precession of the orbital plane) exact observables

Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes
- We solved the BS equation in impact parameter space, making a connection with the Hamilton-Jacobi action via the amplitude-action relation
- We discussed a generalization of the boundary to bound dictionary for generic angular momentum orientations (i.e. including both radial and polar action), focusing on a massive probe particle moving in a Kerr background and its scattering (azimuthal and polar deflection angles, time delay) and bound (periastron advance, precession of the orbital plane) exact observables
- Future directions: understand the amplitude-action relation for generic spin orientations, generalized Carter constant for spinning particles, analytic continuation for radiative observables, self-force from amplitudes, ...

Summary and future directions