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# Modular Invariance and the Strong CP Problem

**Arsenii Titov**

*Dipartimento di Fisica “Enrico Fermi”, Università di Pisa, Italy  
INFN, Sezione di Pisa, Italy*



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**Modular invariance and the QCD angle**

Ferruccio Feruglio (INFN, Padua), Alessandro Strumia (Pisa U.), Arsenii Titov (Pisa U.) (May 15, 2023)

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# Outline

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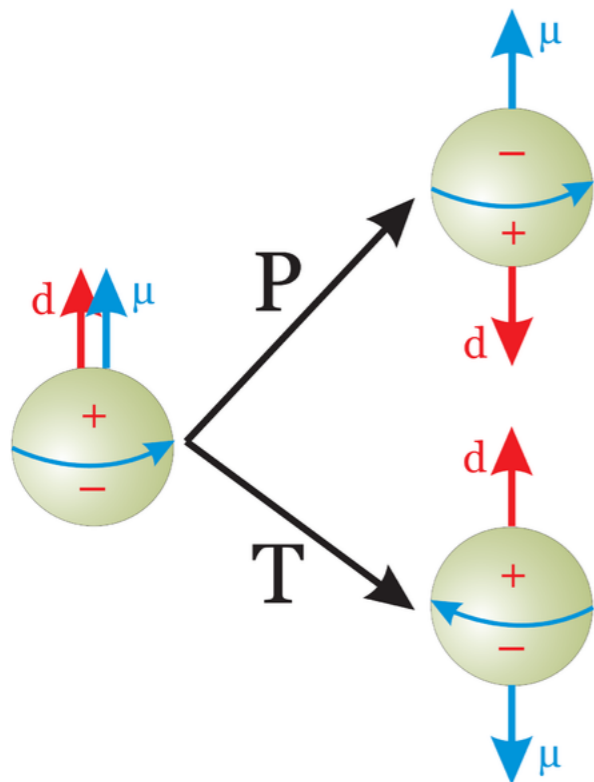
1. Strong CP problem
2. Existing solutions
3. **Modular invariance** and global supersymmetry
4. Modular anomalies and their cancellation
5. Corrections to  $\bar{\theta} = 0$
6. Conclusions

# The strong CP problem

$$\mathcal{L}_{\text{QCD}} = \bar{q} \left( i\not{D} - M_q \right) q - \frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu} + \theta_{\text{QCD}} \frac{g_3^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a,\mu\nu}$$

$$\bar{\theta} = \theta_{\text{QCD}} + \arg \det M_q \quad \text{CPV parameter}$$

Neutron EDM  $d$



$$d = 2.4 \times 10^{-16} \bar{\theta} e \cdot \text{cm} \quad \text{Pospelov, Ritz, hep-ph/9908508v4}$$

$$|d| \leq 1.8 \times 10^{-26} e \cdot \text{cm} \quad (90\% \text{ C.L.}) \quad \text{Abel et al., 2001.11966}$$

$$|\bar{\theta}| \lesssim 10^{-10}$$

Why so small???

... and the CPV phase in the CKM matrix  $\delta_{\text{CKM}} \approx 1.2$

# Solution 1: the Axion

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Promote  $\bar{\theta}$  to a dynamical scalar field  $a$ , the **axion**, which washes out CP violation in QCD

$$\mathcal{L}_a = \frac{1}{2}(\partial_\mu a)^2 + \frac{g_3^2}{32\pi^2} \frac{a}{f_a} G\tilde{G} + \dots$$

$$\bar{\theta} = \frac{\langle a \rangle}{f_a} \quad \text{with} \quad \langle a \rangle = 0$$

New global  $U(1)_{\text{PQ}}$

Peccei, Quinn, PRL **38** (1977) 1440; PRD **16** (1977) 1791

- ▶ spontaneously broken  $\Rightarrow$  the axion is a NGB
- ▶ anomalous under QCD  $(\partial_\mu J_{\text{PQ}}^\mu \propto G\tilde{G}) \Rightarrow$  the **axion is a pNGB**

Quality problem

- ▶ Corrections of order  $(f_a/M_{\text{Pl}})^\#$  from higher-dimensional operators
- ▶  $U(1)_{\text{PQ}}$  should be an accidental symmetry in a complete model

# Solution 2: CP (P) is symmetry of UV

- ▶ CP (P) is a symmetry of the UV
- ▶ It is broken spontaneously in such a way that  $\bar{\theta} = 0$  and  $\delta_{\text{CKM}} = \mathcal{O}(1)$

Nelson—Barr models

Nelson, PLB **136** (1984) 387; Barr, PRL **53** (1984) 329

New heavy vector-like quarks  $Q$  and scalars  $\eta$  with CPV complex VEVs  $\langle \eta \rangle$

$$(q_R \ Q_R) M_q \begin{pmatrix} q_L \\ Q_L \end{pmatrix} = (q_R \ Q_R) \begin{pmatrix} y v_H & y' \langle \eta \rangle \\ 0 & \mu \end{pmatrix} \begin{pmatrix} q_L \\ Q_L \end{pmatrix}$$

- ▶ CP is a symmetry  $\Rightarrow \theta_{\text{QCD}} = 0$  and the couplings  $(y, y', \mu)$  are real
- ▶  $\det M_q = y v_H \mu$  is real (and positive)  $\Rightarrow \arg \det M_q = 0$
- ▶ Effective light quark mass matrix depends on  $\langle \eta \rangle \Rightarrow \delta_{\text{CKM}} \neq 0$

Additional matter, tuning, loop corrections...

Dine, Draper, 1506.05433

# Our solution: CP + modular invariance

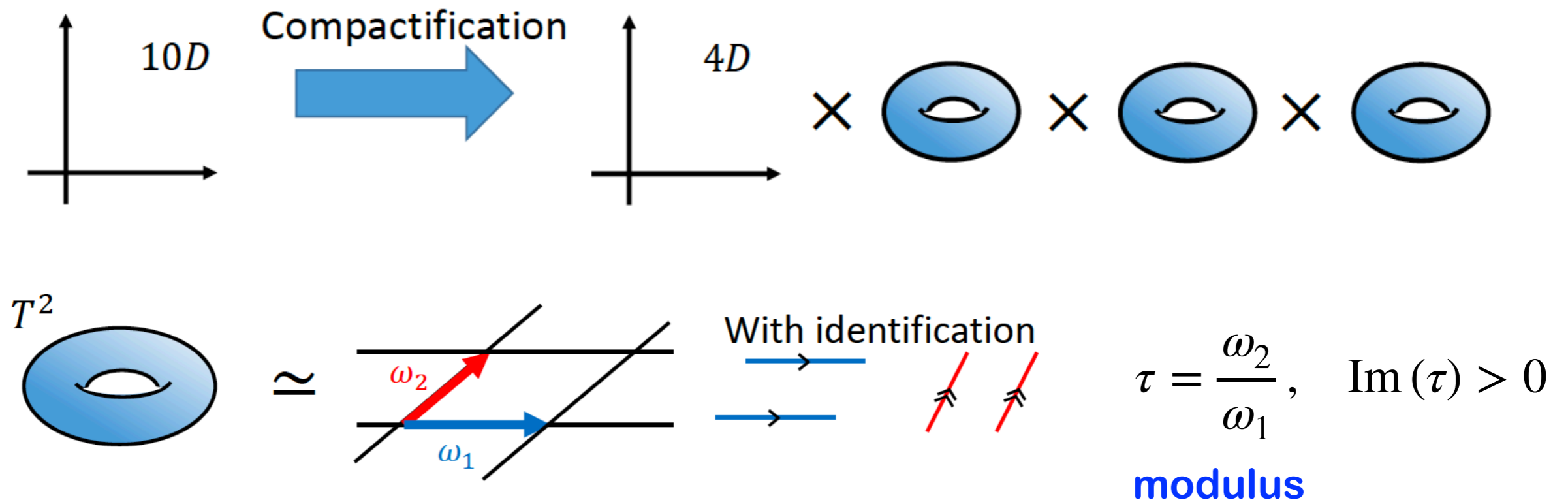
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1. CP is a symmetry  $\Rightarrow \theta_{\text{QCD}} = 0$  (and real Lagrangian couplings)
2. **Modular invariance**/anomaly cancellation  $\Rightarrow \arg \det M_q = 0$
3. CP is broken spontaneously by the VEV of a single complex field, the modulus  $\tau \Rightarrow \delta_{\text{CKM}} = \mathcal{O}(1)$
4. Quark mass hierarchies and mixing angles are reproduced by  $\mathcal{O}(1)$  parameters
5. Corrections to  $\bar{\theta} = 0$  are small under **certain assumptions on SUSY breaking**

# Modular invariance

String theory requires extra dimensions

Images: [Takuya H. Tatsuishi](#)



Lattice left invariant by modular transformations

$$\tau \rightarrow \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} \quad a, b, c, d \in \mathbb{Z} \quad ad - bc = 1$$

These transformations form an infinite discrete group

# Modular group

## Homogeneous modular group

$$\Gamma = \langle S, T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2, \mathbb{Z})$$

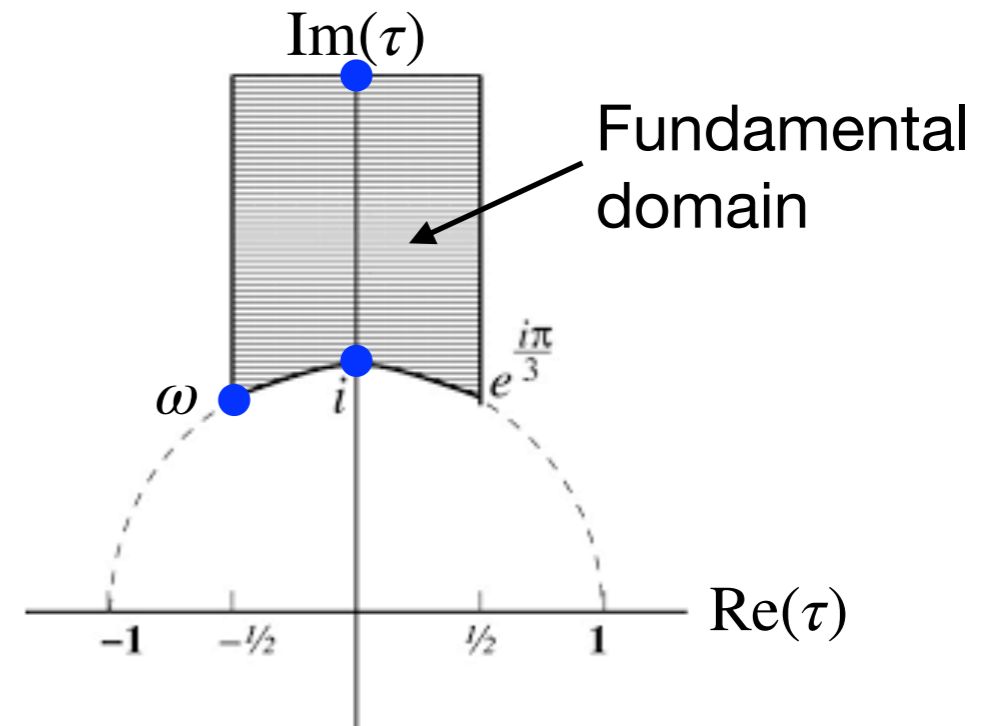
$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau}$$

duality

$$\tau \xrightarrow{T} \tau + 1$$

discrete shift symmetry



## Special points

$$\blacktriangleright \tau = i: \quad i \xrightarrow{S} -\frac{1}{i} = i \quad \Rightarrow \quad \mathbb{Z}_4^S$$

$$\blacktriangleright \tau = \omega \equiv e^{\frac{2\pi i}{3}}: \quad \omega \xrightarrow{ST} -\frac{1}{\omega + 1} = \omega \quad \Rightarrow \quad \mathbb{Z}_3^{ST} \times \mathbb{Z}_2^{S^2}$$

$$\blacktriangleright \tau = i\infty: \quad i\infty \xrightarrow{T} i\infty + 1 = i\infty \quad \Rightarrow \quad \mathbb{Z}^T$$



# Modular forms

Holomorphic functions on  $\mathcal{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$  transforming under  $\Gamma$  as

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma$$

$k$  is **weight**, a non-negative even integer

Normalised Eisenstein series

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(m,n) \neq (0,0)} \frac{1}{(m + n\tau)^k}$$

Each modular form can be written as a polynomial in  $E_4$  and  $E_6$

$$f(\tau) = \sum_{a,b \geq 0} c_{ab} E_4^a(\tau) E_6^b(\tau) \quad \text{with} \quad 4a + 6b = k$$

Modular weight $k$	0	2	4	6	8	10	12	14
Modular forms	1	–	$E_4$	$E_6$	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	$E_4^3, E_6^2$	$E_{14} = E_4^2 E_6$

# Modular-invariant SUSY theories

$\mathcal{N} = 1$  global SUSY action

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\tau, e^{2V}\Phi, \tau^\dagger, \Phi^\dagger) + \left[ \int d^2\theta W(\tau, \Phi) + \frac{1}{16} \int d^2\theta f(\tau) \mathcal{G} \mathcal{G} + \text{h.c.} \right]$$

Kähler potential  $K$   
(kinetic terms,  
gauge interactions)

Superpotential  $W$   
(Yukawa interactions)

Gauge kinetic function  $f$   
 $f_3 = \frac{1}{g_3^2} - i \frac{\theta_{\text{QCD}}}{8\pi^2}$

Under modular transformations  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\begin{cases} \tau \rightarrow \frac{a\tau + b}{c\tau + d} & \tau \text{ is promoted to a (dimensionless) superfield} \\ \Phi \rightarrow (c\tau + d)^{-k_\Phi} \Phi & \text{matter supermultiplets} \\ V \rightarrow V & \text{vector supermultiplets} \end{cases}$$

Modular symmetry acts **non-linearly**

Ferrara et al., PLB **225** (1989) 363; PLB **233** (1989) 147; Feruglio, 1706.08749

# Modular-invariant SUSY theories

Minimal Kähler potential

$$K = -h^2 \log(-i\tau + i\tau^\dagger) + \sum_{\Phi} \frac{\Phi^\dagger e^{2V} \Phi}{(-i\tau + i\tau^\dagger)^{k_\Phi}}$$

Superpotential

$$W = Y_{ij}^u(\tau) u_{Ri} Q_j H_u + Y_{ij}^d(\tau) d_{Ri} Q_j H_d$$

$\tau$ -dependent Yukawa couplings

$$Y_{ij}^q(\tau) \rightarrow (c\tau + d)^{k_{ij}^q} Y_{ij}^q(\tau) \quad \text{with} \quad k_{ij}^q = k_{q_{Ri}} + k_{Q_j} + k_{H_q}$$

are modular forms!

$$Y_{ij}^q(\tau) = c_{ij}^q F_{k_{ij}^q}(\tau) \quad \text{with} \quad c_{ij}^q \in \mathbb{R} \quad \text{because of CP}$$

Gauge kinetic function

$$f = \frac{1}{g_3^2} \quad \theta_{\text{QCD}} = 0 \quad \text{because of CP}$$

# Modular invariance and CP

## Fields

$$\tau \xrightarrow{\text{CP}} -\tau^\dagger \quad \text{and} \quad \Phi \xrightarrow{\text{CP}} \Phi^\dagger$$

## Modular forms

$$F(\tau) \xrightarrow{\text{CP}} F(-\tau^*) = F(\tau)^*$$

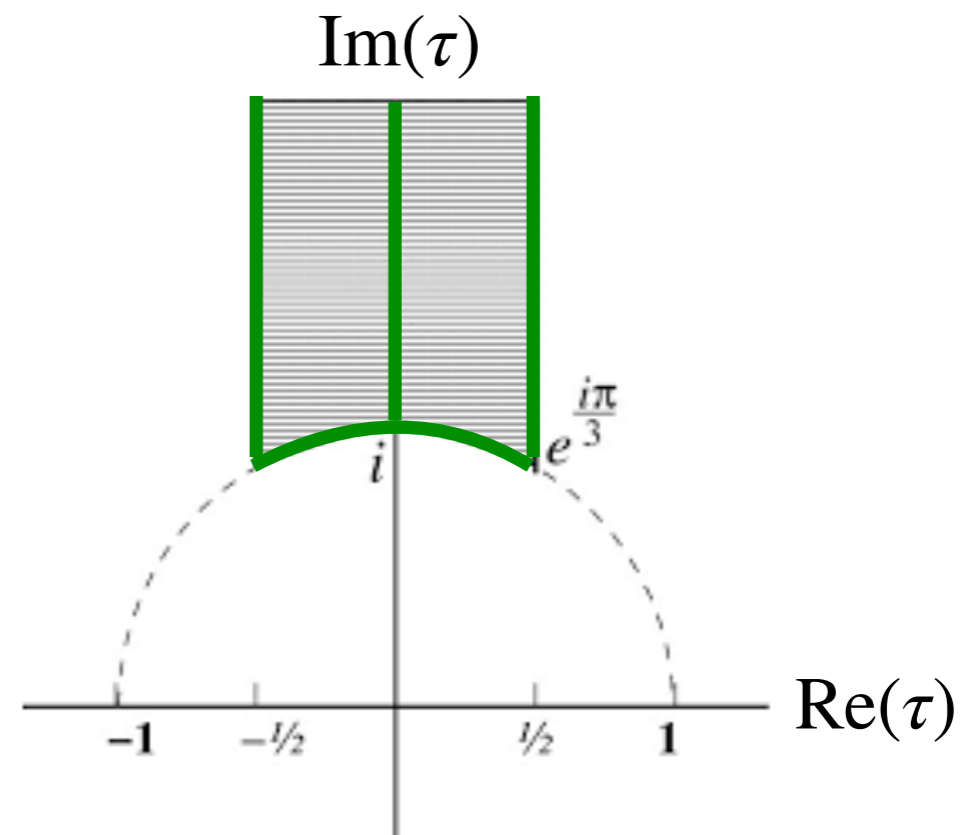
## CP-conserving values of $\tau$

$$\tau \xrightarrow{\text{CP}} -\tau^* = \gamma\tau \quad (\text{goes to itself up to } \gamma)$$

1.  $\tau = iy \xrightarrow{\text{CP}} iy$

2.  $\tau = -\frac{1}{2} + iy \xrightarrow{\text{CP}} \frac{1}{2} + iy = T\tau$

3.  $\tau = e^{i\varphi} \xrightarrow{\text{CP}} -e^{-i\varphi} = S\tau$



Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

# Determinant of quark mass matrix

$$M_u = v_u Y^u \quad M_d = v_d Y^d$$

$$\det M_q = \det M_u \det M_d \propto \det Y^u \det Y^d$$

$$Y^q(\tau) = \begin{pmatrix} F_{k_{11}^q} & F_{k_{12}^q} & F_{k_{13}^q} \\ F_{k_{21}^q} & F_{k_{22}^q} & F_{k_{23}^q} \\ F_{k_{31}^q} & F_{k_{32}^q} & F_{k_{33}^q} \end{pmatrix} \Rightarrow \det Y^q(\tau) \text{ is a modular form of weight } k_{\det}^q$$

$$k_{\det}^q = k_{11}^q + k_{22}^q + k_{33}^q = \dots = \sum_{i=1}^3 (k_{q_{Ri}} + k_{Q_i}) + 3k_{H_q}$$

And  $\det Y^u(\tau) \det Y^d(\tau)$  is a modular form of weight  $k_{\det}$

$$k_{\det} = k_{\det}^u + k_{\det}^d = \sum_{i=1}^3 (2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}}) + 3(k_{H_u} + k_{H_d})$$

$$k_{\det} = 0 \Rightarrow \det Y^u(\tau) \det Y^d(\tau) = (\text{real}) \text{ constant}$$

# Matter fields and canonical normalisation

Gauge quantum numbers

	$Q$	$u_R$	$d_R$	$L$	$e_R$	$H_u$	$H_d$
$SU(3)_C$	<b>3</b>	$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}$	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>
$SU(2)_L$	<b>2</b>	<b>1</b>	<b>1</b>	<b>2</b>	<b>1</b>	<b>2</b>	<b>2</b>
$U(1)_Y$	$\frac{1}{6}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$

Canonical normalisation

$$K \supset \frac{\Phi^\dagger \Phi}{(-i\tau + i\tau^\dagger)^{k_\Phi}} = \Phi_{\text{can}}^\dagger \Phi_{\text{can}} \quad \Phi_{\text{can}} = \{\phi_{\text{can}}, \psi_{\text{can}}\}$$

$$\psi_{\text{can}} \rightarrow \left( \frac{c\tau + d}{c\tau^\dagger + d} \right)^{-\frac{k_\Phi}{2}} \psi_{\text{can}} = e^{-ik_\Phi \alpha(\tau)} \psi_{\text{can}} \quad \alpha(\tau) = \arg(c\tau + d)$$

Modular symmetry acts on canonically normalised fields  
as a  $\tau$ -dependent phase rotation

# Cancellation of modular anomalies

Conditions for modular-gauge anomaly cancellation

$$\text{SU}(3)_C : A \equiv \sum_{i=1}^3 \left( 2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}} \right) = 0$$

$$\text{SU}(2)_L : \sum_{i=1}^3 \left( 3k_{Q_i} + k_{L_i} \right) + k_{H_u} + k_{H_d} = 0$$

$$\text{U}(1)_Y : \sum_{i=1}^3 \left( k_{Q_i} + 8k_{u_{Ri}} + 2k_{d_{Ri}} + 3k_{L_i} + 6k_{e_{Ri}} \right) + 3 \left( k_{H_u} + k_{H_d} \right) = 0$$

Simplest solution

$$k_Q = k_{u_R} = k_{d_R} = k_L = k_{e_R} = (-k, 0, k) \quad \text{and} \quad k_{H_u} + k_{H_d} = 0$$

Cancellation of modular-QCD anomaly along with  $k_{H_u} + k_{H_d} = 0$  implies

$$k_{\text{det}} = \sum_{i=1}^3 \left( 2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}} \right) + 3 \left( k_{H_u} + k_{H_d} \right) = 0$$

# Simplest example: quarks

Simplest non-trivial example giving  $k_{\det} = 0$  and  $A = 0$

$$k_Q = k_{u_R} = k_{d_R} = (-6, 0, 6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Yukawa matrices

$$Y^q = \begin{pmatrix} 0 & 0 & c_{13}^q \\ 0 & c_{22}^q & c_{23}^q E_6 \\ c_{31}^q & c_{32}^q E_6 & c_{33}^q E_4^3 + c_{33}'^q E_6^2 \end{pmatrix} \Rightarrow Y^q|_{\text{can}} = \begin{pmatrix} 0 & 0 & c_{13}^q \\ 0 & c_{22}^q & c_{23}^q (2\text{Im}\tau)^3 E_6 \\ c_{31}^q & c_{32}^q (2\text{Im}\tau)^3 E_6 & (2\text{Im}\tau)^6 [c_{33}^q E_4^3 + c_{33}'^q E_6^2] \end{pmatrix}$$

$$\det Y^q|_{\text{can}} = -c_{13}^q c_{22}^q c_{31}^q \in \mathbb{R}$$

Fixing  $\tau = 1/8 + i$  and  $\tan \beta = 10$

$$c_{ij}^u \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.56 \\ 0 & -1.86 & 0.87 \\ 1.29 & 4.14 & 3.51, 1.40 \end{pmatrix} \quad c_{ij}^d \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.55 \\ 0 & -2.59 & 4.59 \\ 0.378 & 0.710 & 0.734, 1.76 \end{pmatrix}$$

reproduce the quark masses, mixing angles and  $\delta_{\text{CKM}}$  at the GUT scale



# Simplest example: leptons

$$k_L = k_{e_R} = (-6, 0, 6)$$

Weinberg operator  $\mathcal{C}_{ij}^\nu (L_i H_u)(L_j H_u)$  for neutrino masses

Charged lepton Yukawa matrix and coefficient of the Weinberg operator

$$Y^e = \begin{pmatrix} 0 & 0 & c_{13}^e \\ 0 & c_{22}^e & c_{23}^e E_6 \\ c_{31}^e & c_{32}^e E_6 & c_{33}^e E_4^3 + c_{33}'^e E_6^2 \end{pmatrix} \quad \mathcal{C}^\nu = \begin{pmatrix} 0 & 0 & c_{13}^\nu \\ 0 & c_{22}^\nu & c_{23}^\nu E_6 \\ c_{31}^\nu & c_{32}^\nu E_6 & c_{33}^\nu E_4^3 + c_{33}'^\nu E_6^2 \end{pmatrix}$$

Fixing  $\tau = 1/8 + i$  and  $\tan \beta = 10$

$$c_{ij}^e = 10^{-3} \begin{pmatrix} 0 & 0 & 1.29 \\ 0 & 5.95 & 0.35 \\ -2.56 & 1.47 & 1.01, 1.32 \end{pmatrix} \quad c_{ij}^\nu = \frac{1}{10^{16} \text{ GeV}} \begin{pmatrix} 0 & 0 & 3.4 \\ 0 & 7.1 & 1.2 \\ 3.4 & 1.2 & 0.19, 0.95 \end{pmatrix}$$

reproduce the lepton masses and mixings, including  $\delta_{\text{PMNS}}$

# Models with larger modular charges

Yukawa matrices $Y_{u,d}$	Modular weights			Alternative bigger weights		
	$(u_L, d_L)_{1,2,3}$	$u_{R1,2,3}$	$d_{R1,2,3}$	$(u_L, d_L)_{1,2,3}$	$u_{R1,2,3}$	$d_{R1,2,3}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_6 \\ 1 & E_6 & E_4^3 + E_6^2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -2 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -2 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^2 \\ 1 & E_4 & E_4^3 + E_6^2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^2 \\ 1 & E_4^2 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4 E_6 \\ 1 & E_6 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ -2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 2 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^3 + E_6^2 \\ 1 & E_4 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ -4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$

# Corrections to $\bar{\theta} = 0$

$M_{\text{Pl}}$	<p><b>SUSY unbroken</b></p> <ul style="list-style-type: none"> <li>▶ Modular invariance determines completely (up to real couplings) the functional dependence <math>W(\tau)</math></li> <li>▶ It is not the case for <math>K</math>, but <math>\bar{\theta}</math> is insensitive to <math>K</math></li> <li>▶ No-renormalisation theorems <a href="#">Ellis, Ferrara, Nanopoulos, PLB 114 (1982) 231</a></li> </ul>
$\Lambda_{\text{flavour/CP}}$	
$\Lambda_{\text{SUSY}}$	<p><b>SUSY breaking corrections</b></p> <ul style="list-style-type: none"> <li>▶ In general, can be large</li> </ul>
$m_{\text{SUSY}}$	<ul style="list-style-type: none"> <li>▶ Small if <math>\Lambda_{\text{flavour/CP}} \gg \Lambda_{\text{SUSY}}</math> (as e.g. in gauge mediation) and soft breaking terms respect the flavour structure of the SM</li> </ul>
$v$	$\bar{\theta} \lesssim \frac{M_t^4 M_b^4 M_c^2 M_s^2}{v^{12}} J_{\text{CP}} \tan^6 \beta \sim 10^{-28} \tan^6 \beta$ <p>SM corrections are negligible</p> <ul style="list-style-type: none"> <li>▶ <math>\bar{\theta} \sim 10^{-18}</math> at four loops <a href="#">Khriplovich, PLB 173 (1986) 193</a> <a href="#">Ellis, Gaillard, NPB 150 (1979) 141</a></li> </ul>

# Conclusions

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- ▶ **Modular invariance** is inherent to **toroidal compactifications** in string theory
- ▶ It can be consistently implemented in a **supersymmetric QFT**
- ▶ The VEV of the modulus  $\tau$  is the only source of **spontaneous CP violation**

$$\bar{\theta} = \theta_{\text{QCD}} + \arg \det M_q$$

- ▶  $\theta_{\text{QCD}} = 0$  because the UV theory is CP-conserving
- ▶  $\arg \det M_q = 0$  because of **anomaly-free modular symmetry**
- ▶ Corrections to  $\bar{\theta} = 0$  are small under certain assumptions on SUSY breaking

# Back-up slides

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# Phenomenology and cosmology

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- ▶ Couplings to matter are suppressed by  $1/h$  ( $1/M_{\text{Pl}}$  in SUGRA)
- ▶ No couplings to gauge bosons in the exact SUSY limit
  
- ▶  $m_\tau \gtrsim 10$  TeV not to spoil BBN
  
- ▶ Fermionic component of  $\tau$  could be LSP and maybe DM
  
- ▶ Scalar potential  $V(\tau) = V(-\tau^*) \Rightarrow$  CP-conjugated minima  
(domain walls are inflated away if CP breaking occurs before inflation)

# Modular group

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Homogeneous modular group

$$\Gamma = \langle S, T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2, \mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau}$$

duality

$$\tau \xrightarrow{T} \tau + 1$$

discrete shift symmetry

Inhomogeneous modular group

$$\bar{\Gamma} = \langle S, T \mid S^2 = (ST)^3 = I \rangle \cong \mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z}) / \{I, -I\}$$

In other words,  $\mathrm{SL}(2, \mathbb{Z})$  matrices  $\gamma$  and  $-\gamma$  are identified

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau + b}{c\tau + d} \quad \tau \xrightarrow{-\gamma} (-\gamma)\tau = \frac{-a\tau - b}{-c\tau - d} = \gamma\tau$$

# Modular forms

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Holomorphic functions on  $\mathcal{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$  transforming under  $\Gamma$  as

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma$$

$k$  is **weight**, a non-negative even integer

$$\gamma = -I \Rightarrow f(\tau) = (-1)^k f(\tau) \Rightarrow k \text{ is even}$$

Modular forms are periodic and admit  **$q$ -expansions**

$$\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow f(\tau + 1) = f(\tau) \Rightarrow f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}$$

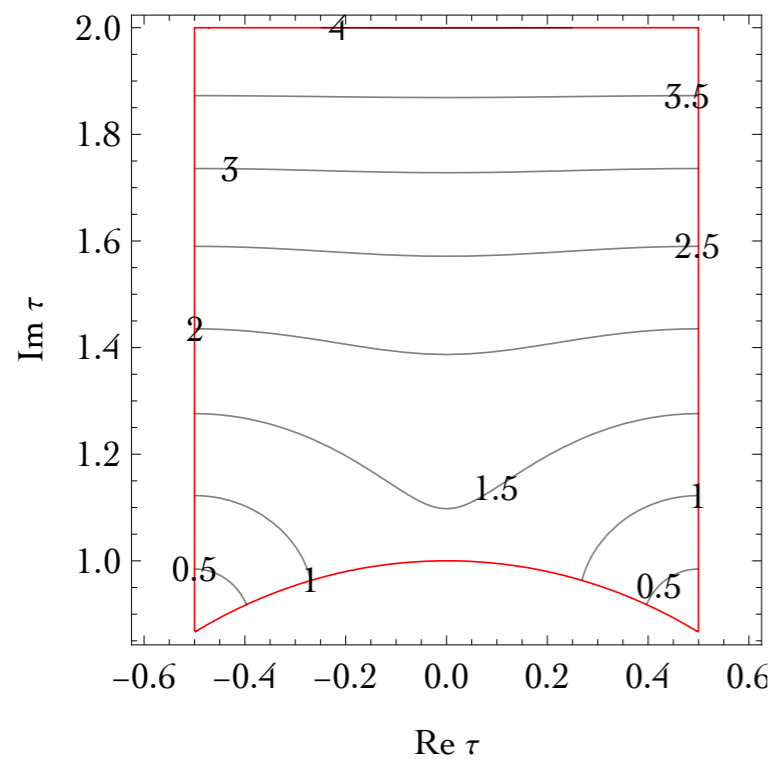
Modular forms of weight  $k$  form a **linear space**  $\mathcal{M}_k$  of finite dimension

$$\dim \mathcal{M}_k = \begin{cases} 0 & \text{if } k \text{ is negative or odd} \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12} \end{cases}$$

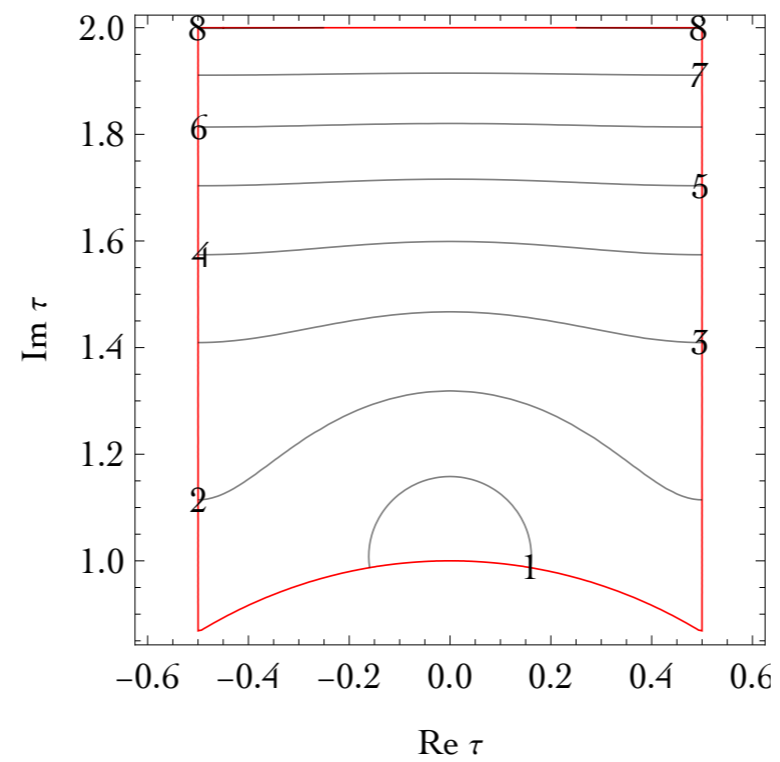


# E4 and E6

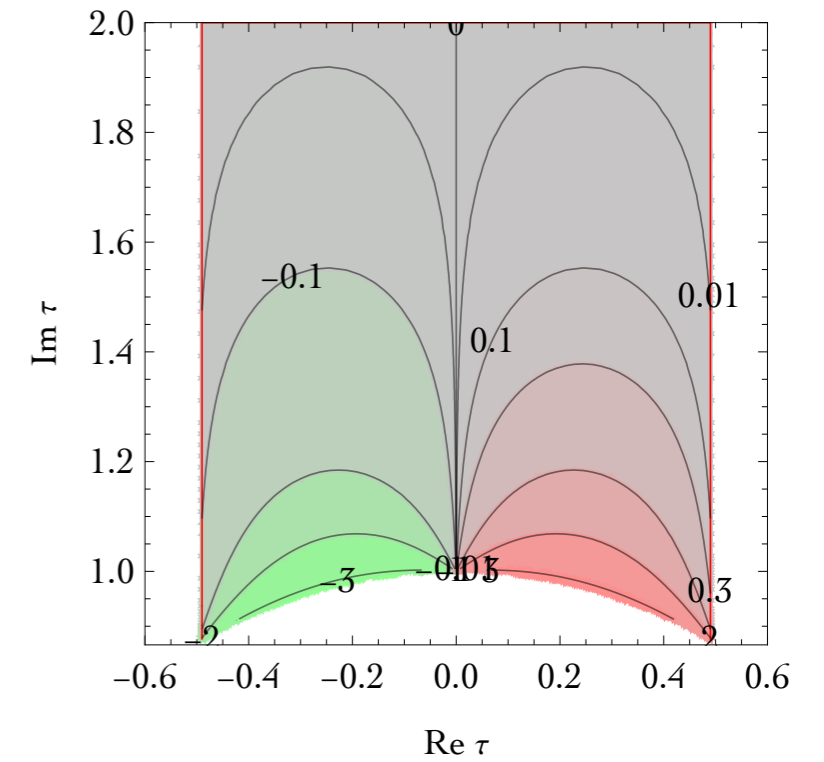
$$|(\operatorname{Im} \tau)^2 E_4(\tau)|$$



$$|(\operatorname{Im} \tau)^3 E_6(\tau)|$$



$$\arg E_4^3/E_6^2$$



# Modular-invariant SUSY theories

$\mathcal{N} = 1$  global SUSY action

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\tau, e^{2V}\Phi, \tau^\dagger, \Phi^\dagger) + \left[ \int d^2\theta W(\tau, \Phi) + \frac{1}{16} \int d^2\theta f(\tau) \mathcal{G} \mathcal{G} + \text{h.c.} \right]$$

Kähler potential  $K$   
(kinetic terms,  
gauge interactions)

Superpotential  $W$   
(Yukawa interactions)

Gauge kinetic function  $f$   
$$f_3 = \frac{1}{g_3^2} - i \frac{\theta_{\text{QCD}}}{8\pi^2}$$

Modular invariance of the action requires

$$\begin{cases} K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) \rightarrow K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) + f_K(\tau, \Phi) + \bar{f}_K(\tau^\dagger, \Phi^\dagger) \\ W(\tau, \Phi) \rightarrow W(\tau, \Phi) \\ f(\tau) \rightarrow f(\tau) \end{cases}$$

Ferrara et al., PLB **225** (1989) 363; PLB **233** (1989) 147; Feruglio, 1706.08749

# Heavy quarks and singularities

- ▶ Heavy quarks are not needed for the mechanism to work, but assume they exist

$$k_q = (-6, -2, 0, +2, +6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Light chiral quarks    Heavy vector-like quarks

- ▶ In the full theory  $f_{UV} \in \mathbb{R}$  and  $\det M_{\text{all}} \in \mathbb{R} \Rightarrow \bar{\theta} = 0$

$$M_{\text{all}} = \begin{pmatrix} M_{LL} & M_{LH} \\ M_{HL} & M_{HH} \end{pmatrix}$$

$$M_{\text{light}} \approx M_{LL} - M_{LH} M_{HH}^{-1} M_{HL} \quad M_{\text{heavy}} \approx M_{HH}$$

Singularities where  $\det M_{\text{heavy}}(\tau) = 0$   
(breakdown of EFT)

$$\det M_{\text{all}} = \det M_{\text{light}} \det M_{\text{heavy}}$$

$$\det M_{\text{light}} \rightarrow (c\tau + d)^{k_{\text{light}}} \det M_{\text{light}} \quad \det M_{\text{heavy}} \rightarrow (c\tau + d)^{k_{\text{heavy}}} \det M_{\text{heavy}}$$

$$k_{\text{all}} = k_{\text{light}} + k_{\text{heavy}} = 0$$

# EFT of light quarks

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In the EFT of light quarks

$$\bar{\theta} = \theta_{\text{QCD}} + \arg \det M_{\text{light}} = -8\pi^2 \text{Im} f_{\text{IR}} + \arg \det M_{\text{light}}$$

The EFT has anomalous field content with  $k_q = (-6, -2, 0)$

Anomaly is cancelled by a new contribution to the gauge kinetic function arising from the integration over the heavy quarks

$$f_{\text{IR}} = f_{\text{UV}} - \frac{1}{8\pi^2} \log \det M_{\text{heavy}}$$

Thus

$$\bar{\theta} = \arg \det M_{\text{heavy}} + \arg \det M_{\text{light}} = \arg \det M_{\text{all}} = 0$$

# Modular invariance and SUGRA

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$\mathcal{N} = 1$  SUGRA action depends on

$$G = \frac{K}{M_{\text{Pl}}^2} + \log \left| \frac{W}{M_{\text{Pl}}^3} \right|^2$$

For  $G$  to be invariant, both  $K$  and  $W$  have to transform

$$K \rightarrow K + M_{\text{Pl}}^2 (F + F^\dagger) \quad \text{and} \quad W \rightarrow e^{-F} W$$

In the case of modular transformations

$$F = \frac{h^2}{M_{\text{Pl}}^2} \log(c\tau + d)$$

$$W \rightarrow (c\tau + d)^{-k_W} W \quad \text{with} \quad k_W = \frac{h^2}{M_{\text{Pl}}^2} > 0$$

The superpotential is a **modular function**, having singularities at some values of  $\tau$

$$k_W \rightarrow 0 \quad \text{rigid SUSY limit}$$

# Modular invariance and SUGRA

$$W = Y_{ij}^u(\tau) u_{Ri} Q_j H_u + Y_{ij}^d(\tau) d_{Ri} Q_j H_d$$

$$Y_{ij}^q(\tau) \rightarrow (c\tau + d)^{k_{ij}^q} Y_{ij}^q(\tau) \quad \text{with} \quad k_{ij}^q = k_{q_{Ri}} + k_{Q_j} + k_{H_q} - k_W$$

Furthermore, the Kähler transformation must be accompanied by a U(1) rotation

$$\psi \rightarrow e^{\frac{F-F^\dagger}{4}} \psi \quad \lambda \rightarrow e^{-\frac{F-F^\dagger}{4}} \lambda \quad \text{how gaugino enters the game}$$

$$\psi_{\text{can}} \rightarrow \left( \frac{c\tau + d}{c\tau^\dagger + d} \right)^{\frac{k_W}{4} - \frac{k_\Phi}{2}} \psi_{\text{can}} \quad \lambda \rightarrow \left( \frac{c\tau + d}{c\tau^\dagger + d} \right)^{-\frac{k_W}{4}} \lambda$$

Modular-QCD anomaly modifies as

$$A = \sum_{i=1}^3 \left( 2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}} - 2k_W \right) + Ck_W$$

$C = 3$  is quadratic Casimir of  $\mathbf{8}$  of  $SU(3)_C$

# Glino mass

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$$\bar{\theta} = \theta_{\text{QCD}} + \arg \det M_q + C \arg M_3$$

Assume  $k_{\text{det}} = 0$  and the quark contribution to  $A$  vanishes. Then

$$\bar{\theta} = \theta_{\text{QCD}} + C \arg M_3$$

Glino mass requires SUSY breaking

$$M_3 = \frac{g_3^2}{2} e^{K/2M_{\text{Pl}}^2} K^{i\bar{j}} D_{\bar{j}} W^\dagger f_i$$

Assuming  $D_\tau W = 0$  and no additional phases from SUSY breaking

$$\arg M_3 = - \arg W$$

$$W = \dots + \frac{c_0 M_{\text{Pl}}^3}{\eta(\tau)^{2k_W}} \quad \text{and} \quad f = \dots + \frac{C k_W}{4\pi^2} \log \eta(\tau)$$

$$\bar{\theta} = - 8\pi^2 \text{Im} f - C \arg W = 0$$

# More on modular invariance in SUGRA

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$$\det M_q \rightarrow \left( \frac{c\tau + d}{c\tau^\dagger + d} \right)^{\frac{k_{\text{det}}}{2}} \det M_q \quad k_{\text{det}} = \sum_{i=1}^3 \left( 2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}} - 2k_W \right) + 3 \left( k_{H_u} + k_{H_d} \right)$$

$$M_3 \rightarrow \left( \frac{c\tau + d}{c\tau^\dagger + d} \right)^{\frac{k_W}{2}} M_3 \quad (\text{gluino mass arises only if SUSY is broken})$$



# Quark masses and mixings

At the GUT scale of  $2 \times 10^{16}$  GeV,  
assuming MSSM with  $\tan \beta = 10$  and SUSY breaking scale of 10 TeV

$m_u/m_c$	$(1.93 \pm 0.60) \times 10^{-3}$
$m_c/m_t$	$(2.82 \pm 0.12) \times 10^{-3}$
$m_d/m_s$	$(5.05 \pm 0.62) \times 10^{-2}$
$m_s/m_b$	$(1.82 \pm 0.10) \times 10^{-2}$
$\sin^2 \theta_{12}$	$(5.08 \pm 0.03) \times 10^{-2}$
$\sin^2 \theta_{13}$	$(1.22 \pm 0.09) \times 10^{-5}$
$\sin^2 \theta_{23}$	$(1.61 \pm 0.05) \times 10^{-3}$
$\delta/\pi$	$0.385 \pm 0.017$

$$m_t = 87.46 \text{ GeV}$$

$$m_b = 0.9682 \text{ GeV}$$

Antusch, Maurer, 1306.6879

Yao, Lu, Ding, 2012.13390

# Lepton masses and mixings

NuFIT 5.2 (2022)

	Normal Ordering (best fit)		Inverted Ordering ( $\Delta\chi^2 = 6.4$ )		
	bfp $\pm 1\sigma$	$3\sigma$ range	bfp $\pm 1\sigma$	$3\sigma$ range	
with SK atmospheric data	$\sin^2 \theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \rightarrow 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \rightarrow 0.341$
	$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \rightarrow 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \rightarrow 35.74$
	$\sin^2 \theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \rightarrow 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \rightarrow 0.613$
	$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \rightarrow 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \rightarrow 51.5$
	$\sin^2 \theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \rightarrow 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \rightarrow 0.02416$
	$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \rightarrow 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \rightarrow 8.94$
	$\delta_{\text{CP}}/^\circ$	$232^{+36}_{-26}$	$144 \rightarrow 350$	$276^{+22}_{-29}$	$194 \rightarrow 344$
	$\frac{\Delta m_{21}^2}{10^{-5} \text{ eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \rightarrow 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \rightarrow 8.03$
	$\frac{\Delta m_{3\ell}^2}{10^{-3} \text{ eV}^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \rightarrow +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \rightarrow -2.406$

$$m_e/m_\mu = 0.0048 \pm 0.0002$$

$$m_\mu/m_\tau = 0.0565 \pm 0.0045$$

Esteban et al., 2007.14792 and [www.nu-fit.org](http://www.nu-fit.org)

# Finite modular groups

Infinite normal subgroups of  $SL(2, \mathbb{Z})$ ,  $N = 2, 3, 4, \dots$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Principal congruence subgroups of the modular group

$$\bar{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\} \quad \bar{\Gamma}(N) \equiv \Gamma(N), \quad N > 2$$

Finite modular groups

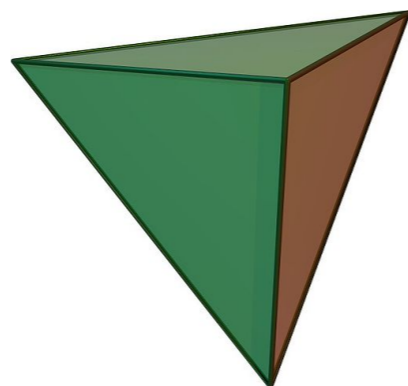
$$\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$$

$$\Gamma_N = \langle S, T \mid S^2 = (ST)^3 = T^N = I \rangle, \quad N = 2, 3, 4, 5$$

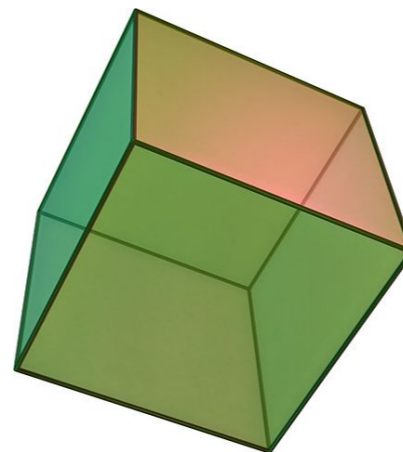
$$\Gamma_2 \cong S_3$$



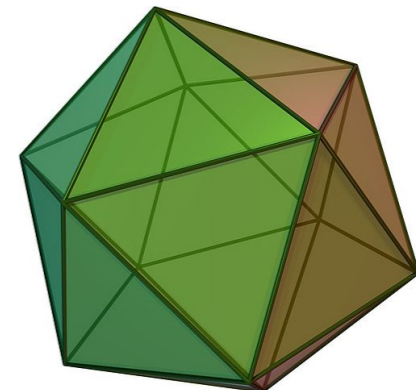
$$\Gamma_3 \cong A_4$$



$$\Gamma_4 \cong S_4$$



$$\Gamma_5 \cong A_5$$



# Theories based on finite modular groups

$\mathcal{N} = 1$  rigid SUSY matter action

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} K(\tau, \bar{\tau}, \psi, \bar{\psi}) + \int d^4x d^2\theta W(\tau, \psi) + \int d^4x d^2\bar{\theta} \bar{W}(\bar{\tau}, \bar{\psi})$$

Ferrara, Lust, Shapere, Theisen, PLB **225** (1989) 363

Ferrara, Lust, Theisen, PLB **233** (1989) 147

$$\begin{cases} \tau \rightarrow \frac{a\tau + b}{c\tau + d} \\ \psi_i \rightarrow (c\tau + d)^{-k_i} \rho_i(\tilde{\gamma}) \psi_i \end{cases} \Rightarrow \begin{cases} W(\tau, \psi) \rightarrow W(\tau, \psi) \\ K(\tau, \bar{\tau}, \psi, \bar{\psi}) \rightarrow K(\tau, \bar{\tau}, \psi, \bar{\psi}) + f_K(\tau, \psi) + \bar{f}_K(\bar{\tau}, \bar{\psi}) \end{cases}$$

Feruglio, 1706.08749

unitary representation of  $\Gamma_N$

$$W(\tau, \psi) = \sum_n \sum_{\{i_1, \dots, i_n\}} \sum_s g_{i_1 \dots i_n, s} \left( Y_{i_1 \dots i_n, s}(\tau) \psi_{i_1} \dots \psi_{i_n} \right)_{\mathbf{1}, s}$$

$$Y(\tau) \xrightarrow{\gamma} (c\tau + d)^{k_Y} \rho_Y(\tilde{\gamma}) Y(\tau)$$

$$k_Y = k_{i_1} + \dots + k_{i_n}$$

$$\rho_Y \otimes \rho_{i_1} \otimes \dots \otimes \rho_{i_n} \supset \mathbf{1}$$

Yukawa couplings are modular forms!