Non-perturbative RG-running of the tensor operator for N_f=3 QCD in a χSF setup

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A bit of context

- Lattice QCD → non-perturbative calculations
- Tensor operator $T^{f_1 f_2}_{\mu\nu}(x) = i\,\bar{\psi}_{f_1}(x)\sigma_{\mu\nu}\frac{1}{2}\psi_{f_2}(x)$, with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$

  Phenomenological interest: [arXiv:1606.00916, arXiv:1110.6448] → not discussed here!

- What interests here more is that this work belongs in a **line of** ALPHA Collaboration **projects** of non-perturbative renormalisation and RG-running

RG-running of T in SF → new variant of SF (χSF) → RG-running of P (quark mass) in χSF → WE ARE HERE → ...

arXiv:1706.06674
arXiv:2309.04314

arXiv:1008.4857
arXiv:1603.00046
...

arXiv:2112.10606

non-perturbative RG-running of bilinear operators

Why is non-perturbative **RG**-running useful?

Case of a mass-independent renormalisation scheme

we can define the RGE for a renormalised operator $\bar{O}(\mu) = Z_O(\mu) O$ and the coupling $\bar{g}(\mu)$

$$\mu \frac{\partial \bar{O}}{\partial \mu} = \gamma_O(\bar{g}(\mu)) \bar{O}(\mu) \qquad \mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}(\mu))$$

with

$$\gamma_O(\bar{g}) \underset{\bar{g} \to 0}{\sim} -\bar{g}^2(\gamma_0 + \gamma_1 \bar{g}^2 + \gamma_2 \bar{g}^4 + O(\bar{g}^6))$$

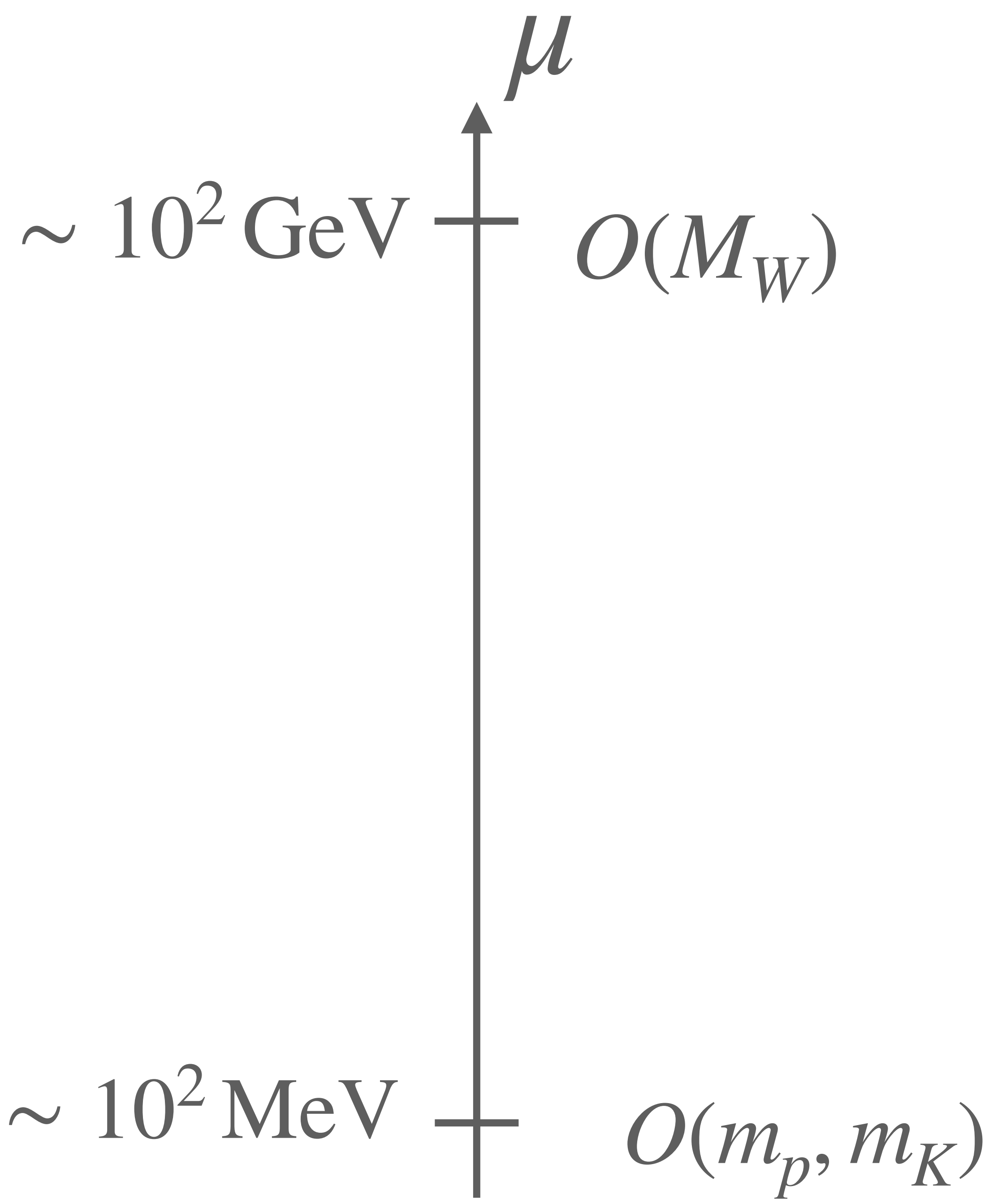
$$\beta(\bar{g}) \underset{\bar{g} \to 0}{\sim} -\bar{g}^3(b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + O(\bar{g}^6))$$

from which we obtain the RG-invariants

$$\Lambda_{QCD} = \mu[b_0\,\bar{g}^2(\mu)]^{-\frac{b_1}{2b_0^2}}\, e^{-\frac{1}{2b_0\bar{g}(\mu)^2}}\, e^{-\int_0^{\bar{g}(\mu)} dg\left(\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}\right)}$$

$$O^{RGI} = \bar{O}(\mu)\left[\frac{\bar{g}^2(\mu)}{4\pi}\right]^{-\frac{\gamma_0}{2b_0}}\, e^{-\int_0^{\bar{g}(\mu)} dg\left(\frac{\gamma_O(g)}{\beta(g)}-\frac{\gamma_0}{b_0 g}\right)}$$

However we need to deal with very different energy scales!



$$\mu\frac{\partial\bar{O}(\mu)}{\partial\mu} = \gamma_O(\bar{g})\bar{O}(\mu) \rightarrow \frac{O^{RGI}}{\bar{O}(\mu_{pt})}$$

perturbation theory

?

$\downarrow$

$\bar{O}(\mu_{had})$

renormalisation through hadronic scheme

Idea: we can factorise the running in many evolutions between two scales:

μ

$\sim 10^2\,\mathrm{GeV}$ — $O(M_W)$

$$\mu \frac{\partial \bar{O}(\mu)}{\partial \mu} = \gamma_O(\bar{g})\bar{O}(\mu) \;\rightarrow\; \frac{O^{RGI}}{\bar{O}(\mu_{pt})}$$

perturbation theory

$$\downarrow$$

$$\frac{O^{RGI}}{\bar{O}(\mu_{pt})} \; \frac{\bar{O}(\mu_{pt})}{\bar{O}(\mu_{pt}/2)}$$

$$\downarrow$$

$$\frac{O^{RGI}}{\bar{O}(\mu_{pt})} \; \frac{\bar{O}(\mu_{pt})}{\bar{O}(\mu_{pt}/2)} \; \frac{\bar{O}(\mu_{pt}/2)}{\bar{O}(\mu_{pt}/2^2)}$$

$$\downarrow$$

$$\cdots$$

$$\downarrow$$

$$\frac{O^{RGI}}{\bar{O}(\mu_{pt})} \cdots \frac{\bar{O}(\mu_{pt}/2^k)}{\bar{O}(\mu_{had})} \qquad \bar{O}(\mu_{had})$$

$\sim 10^2\,\mathrm{MeV}$ — $O(m_p, m_K)$

renormalisation through hadronic scheme

from that it is natural to define

$$\sigma_O(s,u) = \frac{\bar{O}(\mu_2)}{\bar{O}(\mu_1)} \qquad \text{with} \quad s = \frac{\mu_1}{\mu_2}$$

while for the coupling

$$u = \bar{g}^2(\mu) \qquad\qquad \sigma(s,u) = \bar{g}^2(\mu/s)$$

Typically s=2. From the RGE

$$2 = \exp - \int_{\sqrt{u}}^{\sqrt{\sigma(u)}} dg \, \frac{1}{\beta(g)}$$

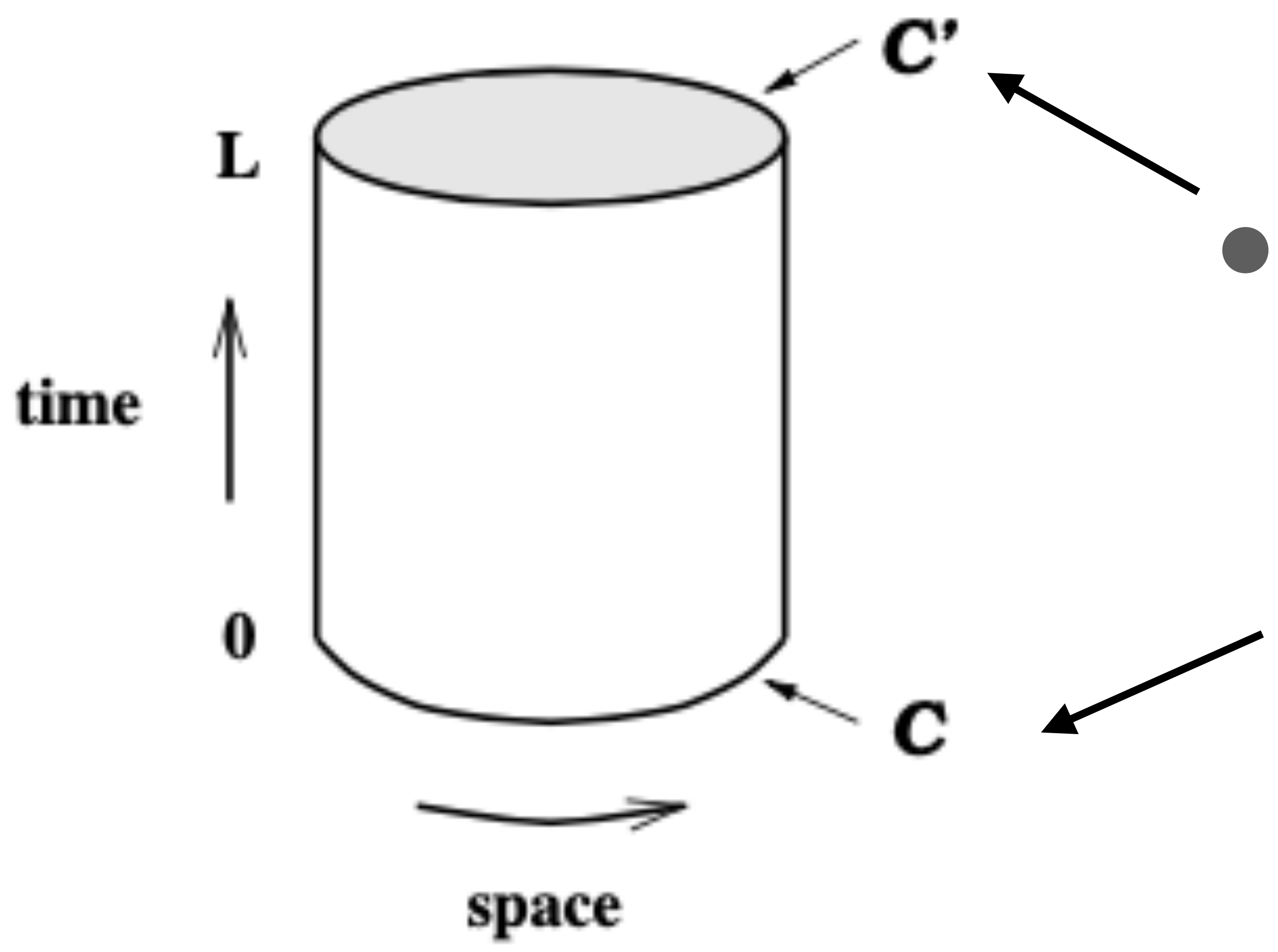
$$\sigma_O(u) = \exp \int_{\sqrt{u}}^{\sqrt{\sigma(u)}} dg \, \frac{\gamma_O(g)}{\beta(g)}$$

On the lattice

Renormalisation scheme:

SF (Schrödinger functional)

- periodic boundary conditions on the space directions



$L^3 x T$ lattice

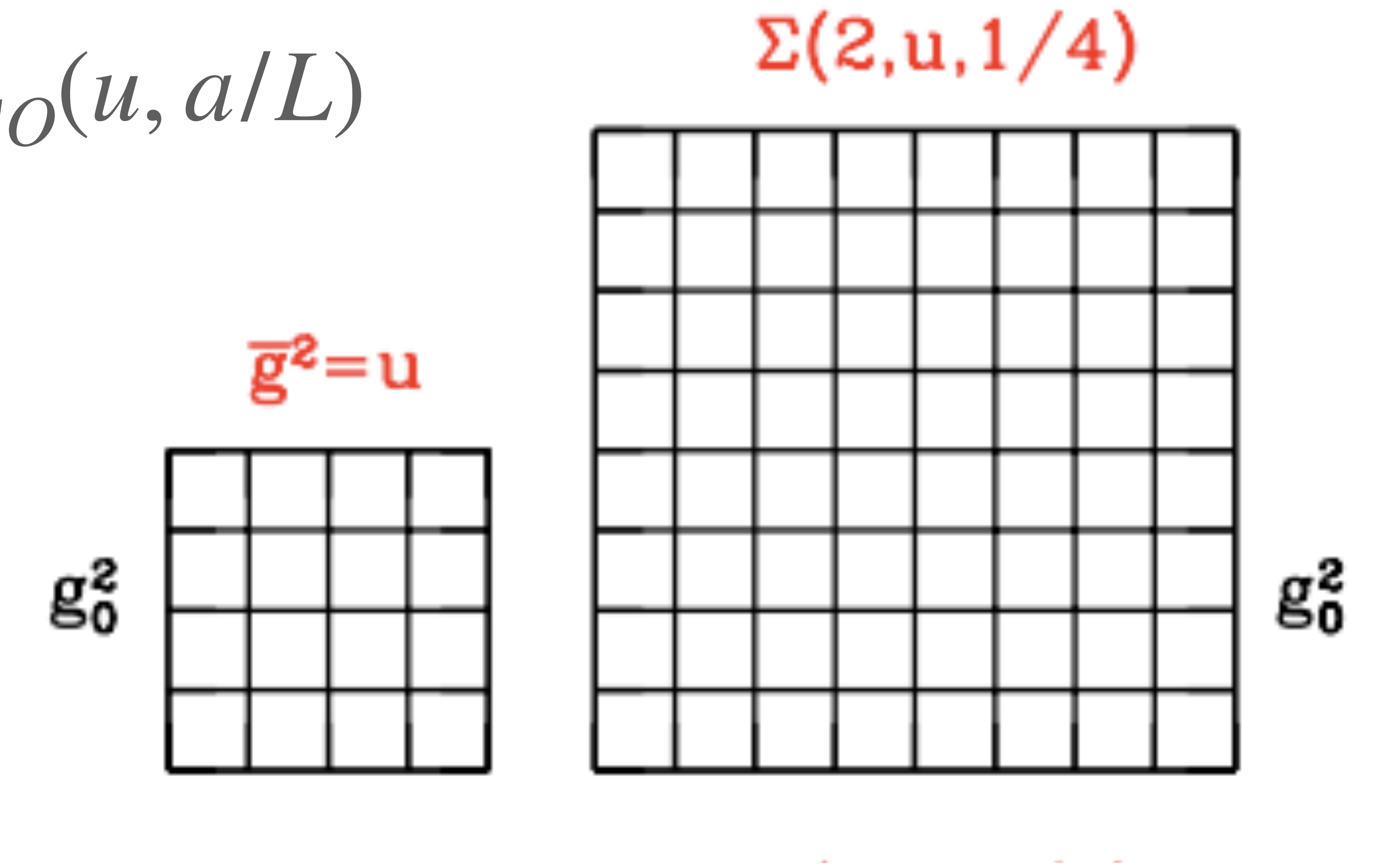
- Dirichlet boundary conditions on the time direction

- it allows us to define a renormalised coupling $u = g_R^2(\mu)$ that scales only with the scale $1/L = \mu$
- Dirichlet boundary conditions make the inversion of the Dirac operator well defined even for $m = 0 \rightarrow$ mass-independent renormalisation scheme

- Here we define the lattice step scaling functions

$$u = \bar{g}^2(L) \rightarrow \quad \Sigma(u, a/L) = \bar{g}^2(2L)$$

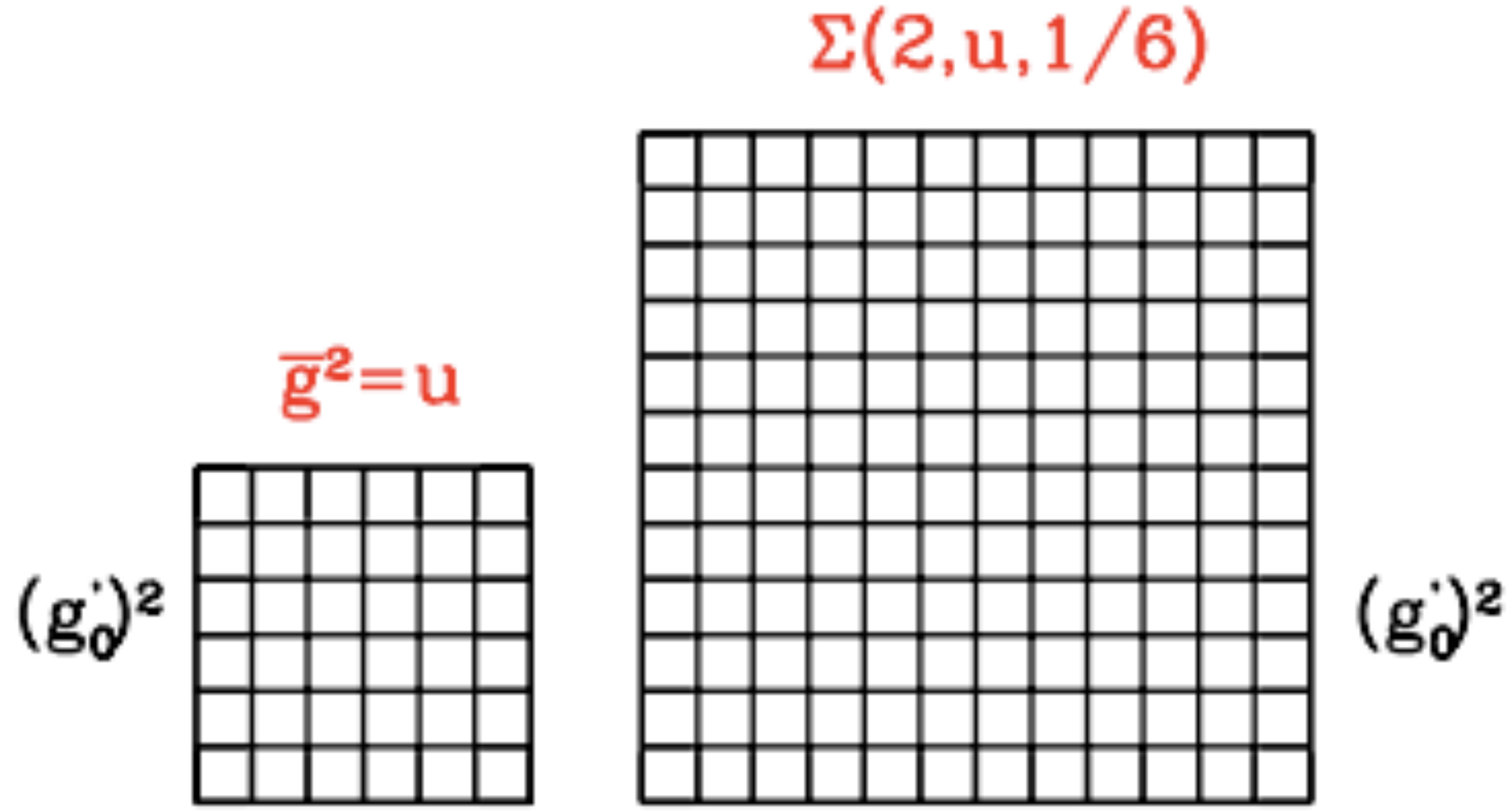
$$\sigma(u) = \lim_{a\to 0} \Sigma(u, a/L) \qquad \sigma_O(u) = \lim_{a\to 0} \Sigma_O(u, a/L)$$



In practice, at a given energy scale μ

- At a given cut-off a, we tune g_0 so to fix $\bar{g}(L)$
- We compute Z_O at $(g_0, a/L)$
- We then compute Z_O on the doubled-lattice$(g_0, a/(2\,L))$
- $$\frac{O_R(\mu)}{O_R(\mu/2)} = \frac{Z_O(g_0^2, a/2L)\cancel{O_{bare}}}{Z_O(g_0^2, a/L)\cancel{O_{bare}}} := \Sigma_O(u, a/L)$$

  - We do the same for different values of $(g_0, a/L)$ to do the continuum limit



Why from χSF to SF?

- SF: $O(a)$ lattice artifacts (without Symanzik improvement)
- e.g for the tensor $< T^I_{\mu\nu} > = < T_{\mu\nu} > + a\, c_T(g_0^2) < (\tilde{\partial}_\mu V_\nu - \tilde{\partial}_\nu V_\mu) >$
- χSF: $O(a^2)$ lattice artifacts (authomatically)
- $< T^I_{\mu\nu} > = < T_{\mu\nu} > + O(a^2)$

- same continuum limit

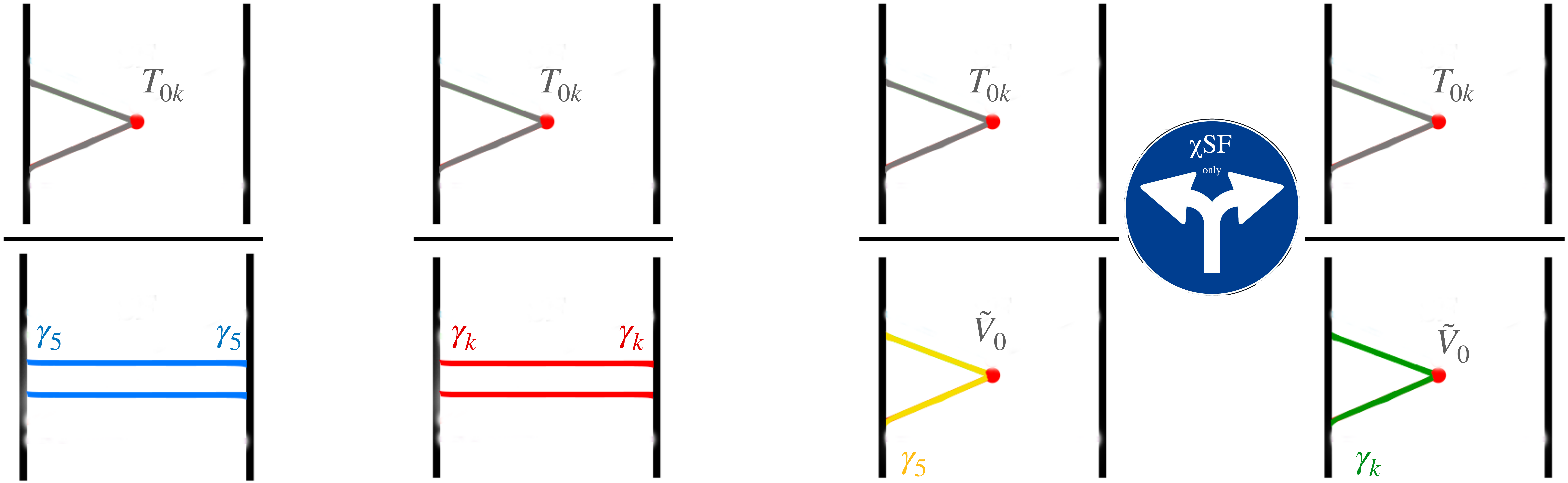
$$\chi\text{SF} \xleftarrow[\text{with } R(\alpha) = exp(i\frac{\alpha}{2}\gamma_5\tau^3)]{\psi = R(\frac{\pi}{2})\psi' \qquad \bar{\psi}=\bar{\psi}'R(\frac{\pi}{2})} \text{SF}$$

Let's start to play!

... with 4 renormalisation conditions (**4 schemes**)

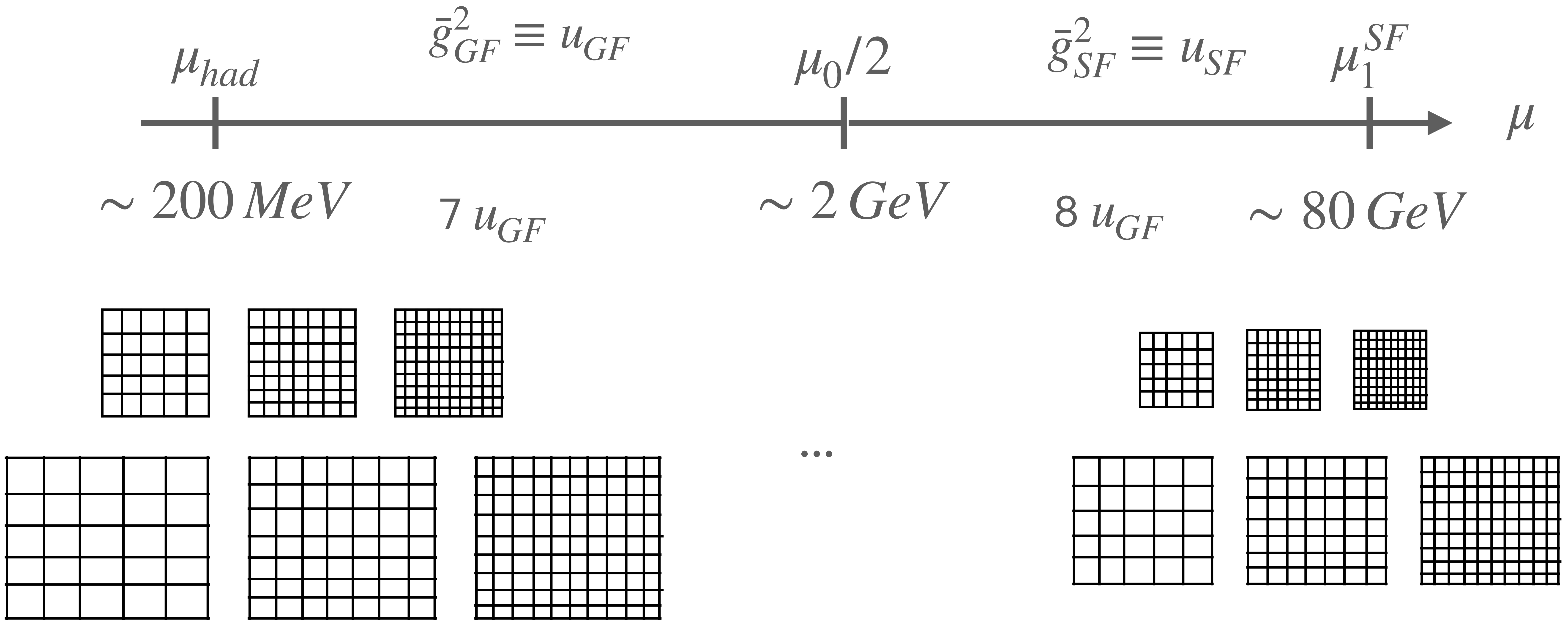
$$Z_T^{ud}(g_0^2, L/a)\ l_T^{ud}(T/2)\frac{1}{(g_1^{ud})^{\alpha}\,(l_1^{ud})^{\beta}\,(-ig_{\tilde{V}}^{ud})^{\gamma}\,(l_{\tilde{V}}^{uu'})^{\delta}} = \left[l_T^{ud}(T/2)\frac{1}{(g_1^{ud})^{\alpha}\,(l_1^{ud})^{\beta}\,(-ig_{\tilde{V}}^{ud})^{\gamma}\,(l_{\tilde{V}}^{uu'})^{\delta}} \right]^{(0,a/L)}$$

scheme-α scheme-β scheme-γ scheme-δ



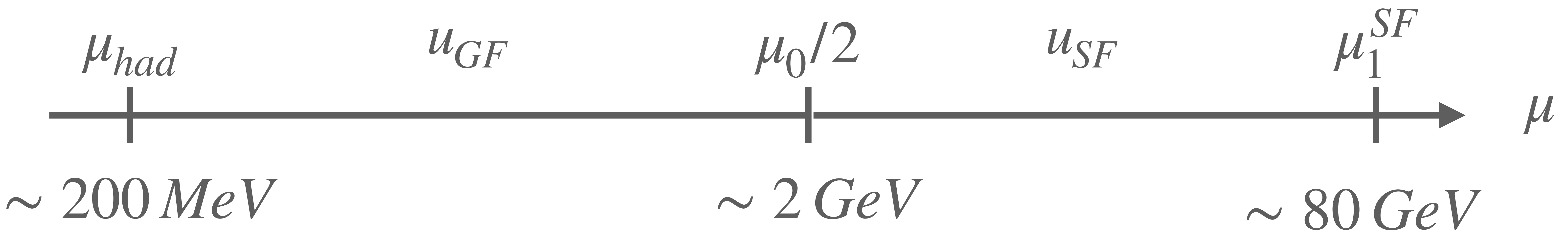
... in a range of energies that span from $\sim [\Lambda_{QCD}, M_W]$ (at least 3 a/L for each μ)

optimization issues: different coupling definitions but same renormalisation conditions on both sides



$\Sigma_T(\bar{g}^2_{GF,1}, a/L)$ $\Sigma_T(\bar{g}^2_{GF,1}, (a/L)')$ $\Sigma_T(\bar{g}^2_{GF,1}, (a/L)'')$ ... $\Sigma_T(\bar{g}^2_{SF,1}, a/L)$ $\Sigma_T(\bar{g}^2_{SF,1}, (a/L)')$ $\Sigma_T(\bar{g}^2_{SF,1}, (a/L)'')$

... with 4 different classes of fits



$\Sigma_T(\bar{g}^2_{GF,1}, a/L)$ $\Sigma_T(\bar{g}^2_{GF,1}, (a/L)')$ $\Sigma_T(\bar{g}^2_{GF,1}, (a/L)'')$ ... $\Sigma_T(\bar{g}^2_{SF,1}, a/L)$ $\Sigma_T(\bar{g}^2_{SF,1}, (a/L)')$ $\Sigma_T(\bar{g}^2_{SF,1}, (a/L)'')$

fits to extract $\sigma_T(u)$ and $\gamma_T(u)$

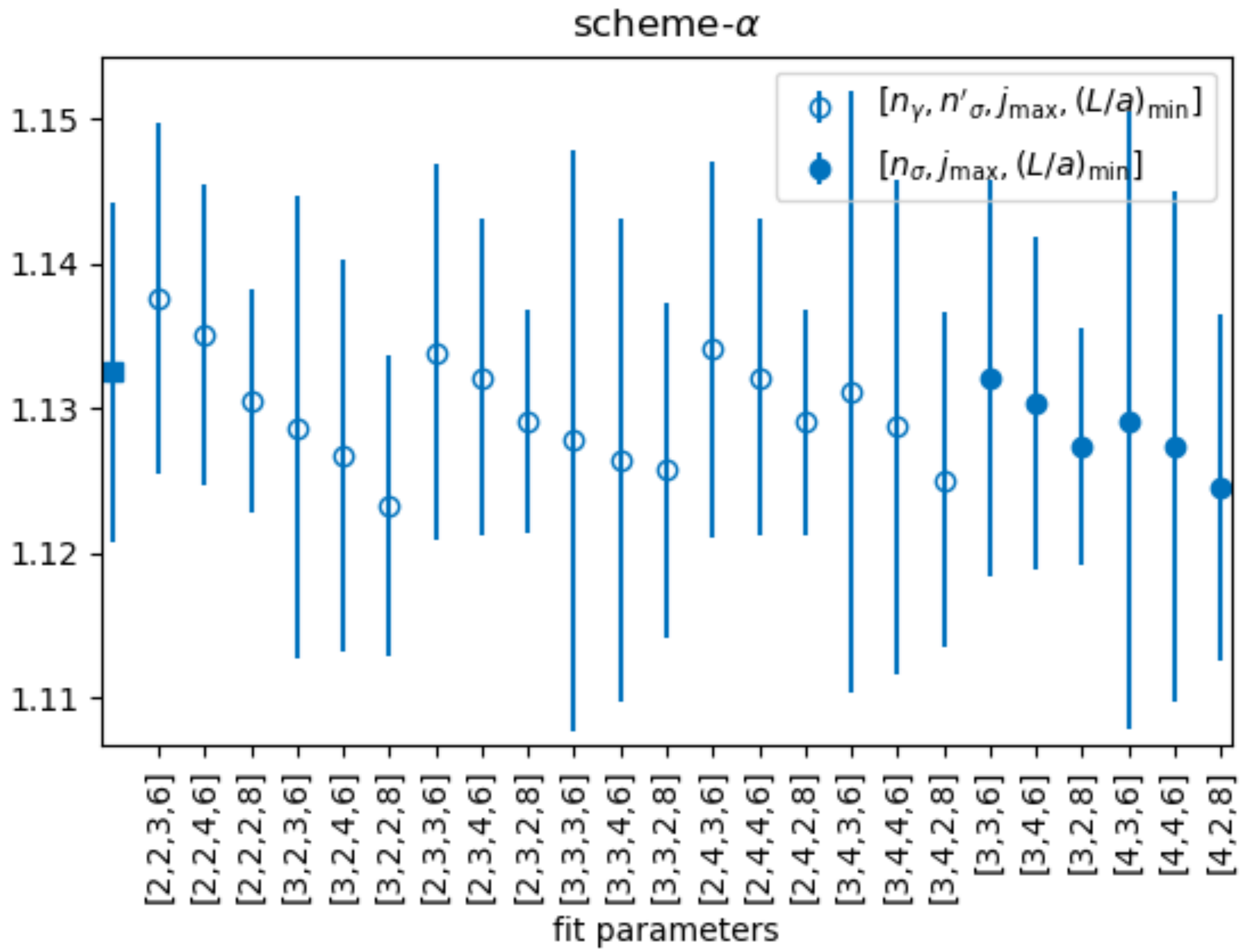
$$\frac{\bar{T}(\mu_0/2)}{\bar{T}(\mu_{had})}$$

$$\frac{T^{RGI}}{\bar{T}(\mu_0/2)}$$

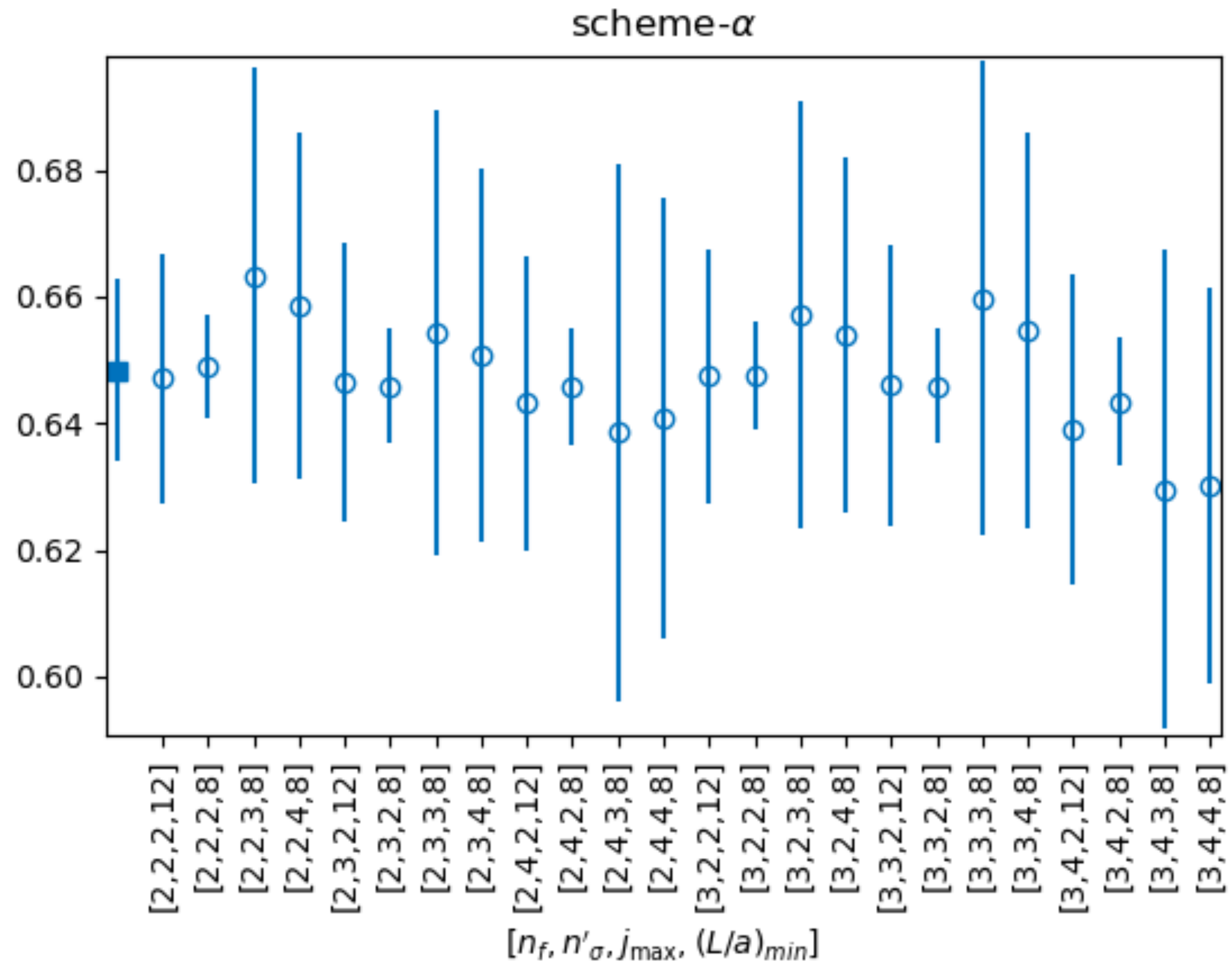
u-by-u fits, global fits...

$$\frac{T^{RGI}}{\bar{T}(\mu_{had})}$$

High-energies fits



Low-energies fits



compatible with previous SF estimates

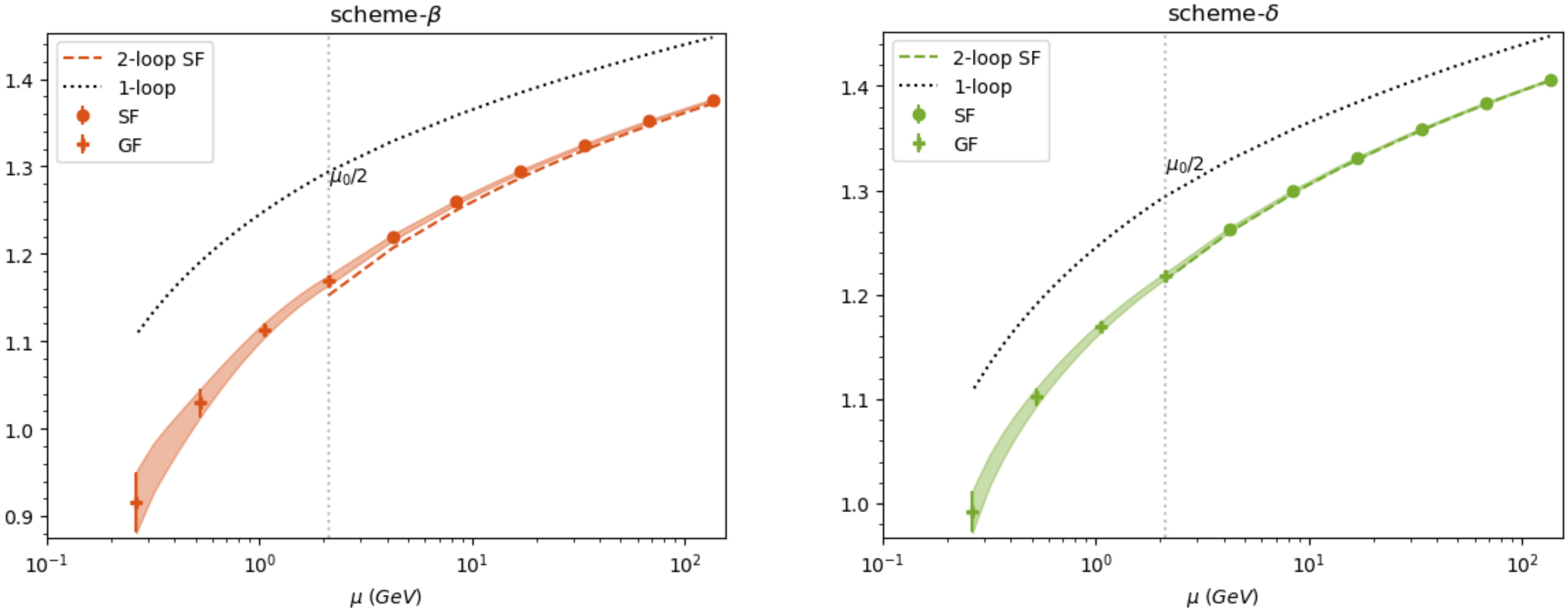
$$\frac{T^{RGI}}{\bar{T}(\mu_0/2)} =$$

α	β	γ	δ
1.133(12)	1.167(10)	1.2952(90)	1.2194(65)

$$\frac{\bar{T}(\mu_0/2)}{\bar{T}(\mu_{had})} =$$

α	β	γ	δ
0.648(14)	0.752(11)	0.754(16)	0.7825(96)

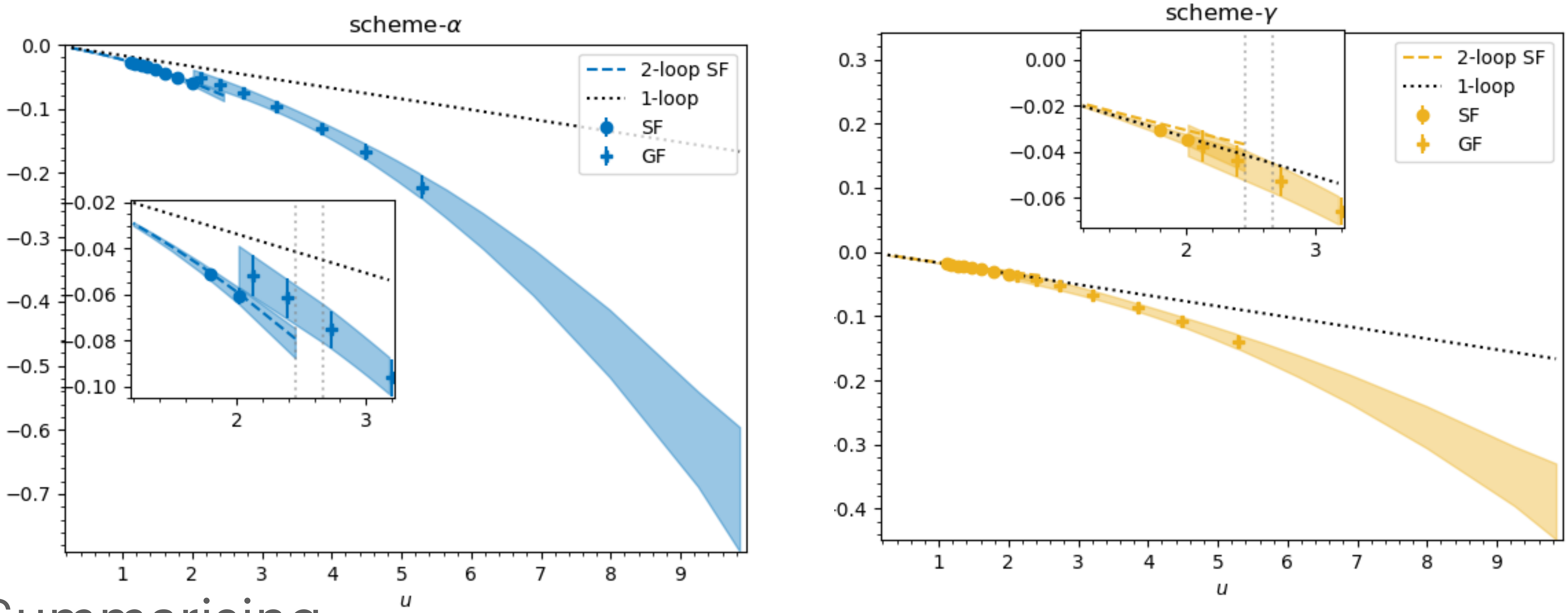
Running of the tensor operator
$(T^{RGI}/\bar{T}(\mu))$



$$\frac{T^{RGI}}{\bar{T}(\mu_{had})} =$$

α	β	γ	δ
0.734(17)	0.882(14)	0.964(21)	0.963(12)

Anomalous dimension $\gamma_T(u)$



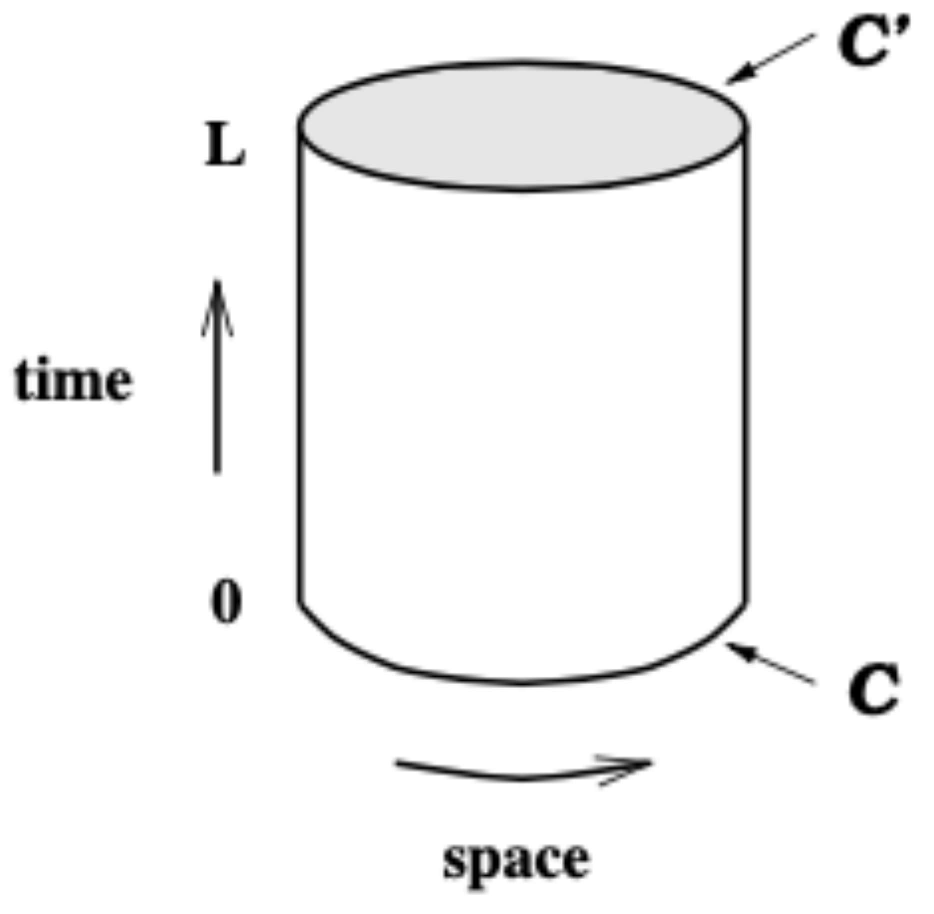
Summarising

- Full non-perturbative RG running of T operator in χSF: done
- The results are compatible with the previous SF ones, but with improvements in the errors and in the evaluation of the systematics

Backup slides

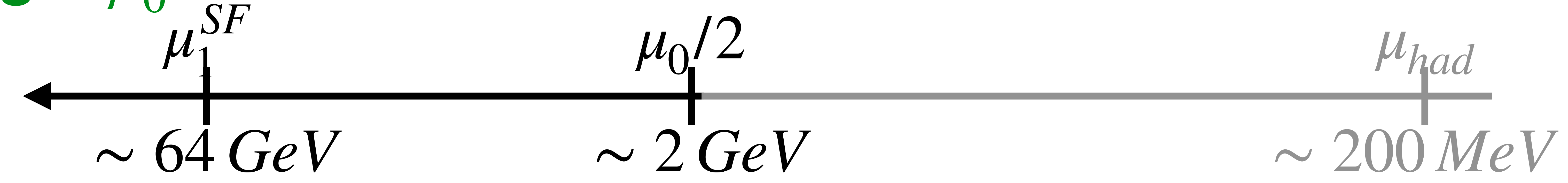
Simulation details

- N_f = 3 massless QCD, Wilson-clover fermions
- Gauge configurations of arXiv:1802.05243, arXiv:2112.10606
- Schrödinger Functional (SF) boundary conditions. $\theta = 0.5$
- χSF valence fermions



SF boundary conditions

- Scheme switching at $\mu_0/2 \sim$ 2 GeV. Same renormalisation conditions for O on both sides!

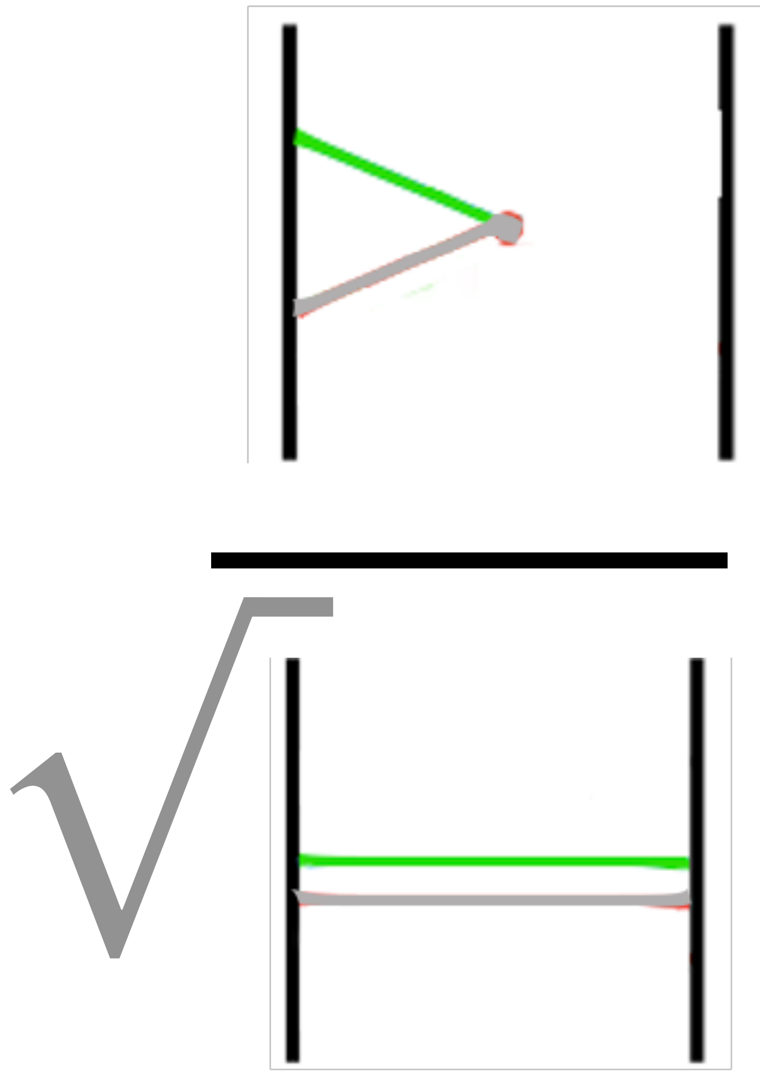


$$T^{RGI} = \frac{T^{RGI}}{\bar{T}(\mu_1^{SF})} \frac{\bar{T}(\mu_1^{SF})}{\bar{T}(\mu_2^{SF})} \cdots \frac{\bar{T}(\mu_k^{SF})}{\bar{T}(\mu_0/2)} \frac{\bar{T}(\mu_0/2)}{\bar{T}(\mu_1^{GF})} \frac{\bar{T}(\mu_2^{GF})}{\bar{T}(\mu_3^{GF})} \cdots \frac{\bar{T}(\mu_n^{GF})}{\bar{T}(\mu_{had}^{GF})} \bar{T}(\mu_{had})$$

- SF coupling
- plaquette gauge action

- Gradient Flow coupling
- Lüscher-Weisz gauge action

Renormalisation conditions



χSF

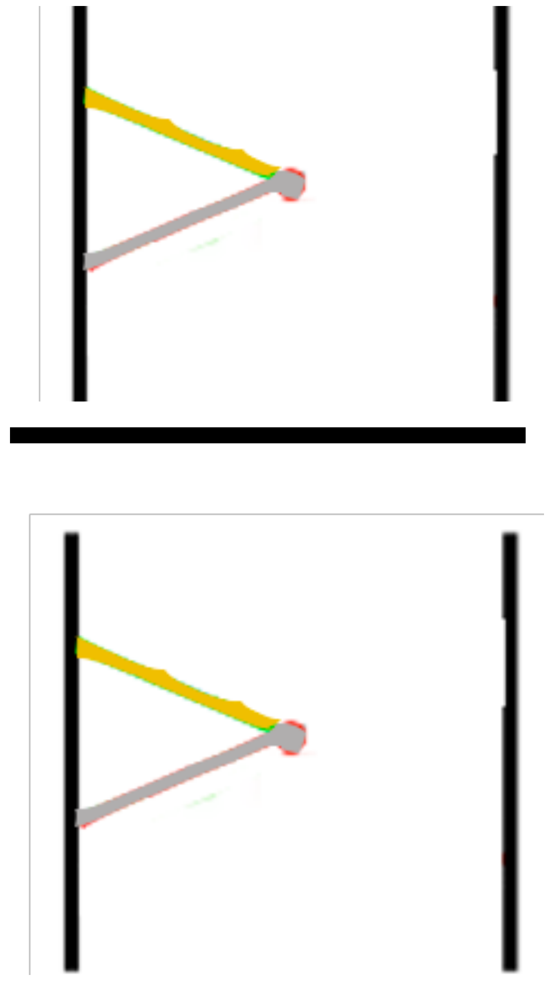
1. $Z_T^{ud}(g_0^2, L/a)\dfrac{l_T^{ud}(T/2)}{\sqrt{g_1^{ud}}} = \left(\dfrac{l_T^{ud}(T/2)}{\sqrt{g_1^{ud}}}\right)_{g_0^2=0,\, a/L\neq 0}$

2. $Z_T^{ud}(g_0^2, L/a)\dfrac{l_T^{ud}(T/2)}{\sqrt{l_1^{ud}}} = \left(\dfrac{l_T^{ud}(T/2)}{\sqrt{l_1^{ud}}}\right)_{g_0^2=0,\, a/L\neq 0}$

SF

1. $Z_T(g_0^2, L/a)\dfrac{k_T(T/2)}{\sqrt{f_1}} = \left(\dfrac{k_T(T/2)}{\sqrt{f_1}}\right)_{g_0^2=0,\, a/L\neq 0}$

2. $Z_T(g_0^2, L/a)\dfrac{k_T(T/2)}{\sqrt{k_1}} = \left(\dfrac{k_T(T/2)}{\sqrt{k_1}}\right)_{g_0^2=0,\, a/L\neq 0}$



3. $Z_T^{ud}(g_0^2, L/a)\dfrac{l_T^{ud}(T/2)}{-ig_{\tilde{V}}^{ud}} = \left(\dfrac{l_T^{ud}(T/2)}{-ig_{\tilde{V}}^{ud}}\right)_{g_0^2=0,\, a/L\neq 0}$

4. $Z_T^{ud}(g_0^2, L/a)\dfrac{l_T^{ud}(T/2)}{l_{\tilde{V}}^{uu'}} = \left(\dfrac{l_T^{ud}(T/2)}{l_{\tilde{V}}^{uu'}}\right)_{g_0^2=0,\, a/L\neq 0}$

since

$\tilde{V}$ is the conserved lattice vector current

and

$Z_{\tilde{V}}^{f_1 f_2} = 1$

... with 4 different classes of fits

AIC model average arXiv:2008.01069

$$< O > = \sum_{n=1}^{N_{fits}} w_n^{AIC} < O >_n$$

$$w_n^{AIC} \sim N \exp\Big[-\frac{1}{2}(\chi_n^2 + 2N_n^{param} + 2N_n^{cut})\Big]$$

u-by-u fits

$$\Sigma_T(u, a/L) = \sigma_T(u) + \rho(u)(\frac{a}{L})^2$$

$$\sigma_T(u) = \exp\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} dx \frac{\gamma_T(x)}{\beta(x)}$$

$$\sigma_T(u) = \left(1 + \rho_1 u + \rho_2 u^2 + \ldots + \rho_{n_s} u^{n_s}\right)$$

global fits

$$\Sigma_T^{SF} = \exp\Big[\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} dx \frac{\gamma(x)}{\beta(x)}\Big] + \sum_{i=1}^{i_{max}} \sum_{j=2}^{j_{max}} c_{ij} u^i (\frac{a}{L})^j$$

$$\Sigma_T(u, a/L) = 1 + \sum_{i=1, j=0}^{n_s, j_{max}} b_{ij} u^i (a/L)^j$$

γ-fits

$$\gamma(g) = -g^2 \sum_{n=0}^{n_t} \gamma^{(n)} g^{2n}$$

$$\frac{T^{RGI}}{\bar{T}(\mu_0/2)} = (\frac{\bar{g}^2(\mu_0/2)}{4\pi})^{-\gamma^{(0)}/2b_0} exp\Big[-\int_0^{\bar{g}(\mu_0/2)} dx(\frac{\gamma(x)}{\beta(x)} - \frac{\gamma^{(0)}}{b_0 x})\Big]$$

σ-fits

$$\frac{T^{RGI}}{\bar{T}(\mu_0/2)} = \frac{T^{RGI}}{\bar{T}(2^k\mu_0)} \frac{\bar{T}(2^k\mu_0)}{\bar{T}(\mu_0/2)}$$

$$\frac{\bar{T}(2^k\mu_0)}{\bar{T}(\mu_0/2)} = \prod_{n=0}^{k} \sigma_T^{-1}(u_n)$$

Improving lattice artefacts

- Nice, but

$$\Sigma(u, a/L) = \sigma(u) + O(a)$$

- Symanzik's theory gives us that

$$< O > = < O >^{cont} - a < S_1 O >^{cont} + a < \delta O >^{cont} + O(a^2)$$

- So, for a chirally-even observable

$$< O_{even} > = < O_{even} >^{cont} + O(a^2)$$

- but SF's boundary conditions break chiral symmetry (even if $m = 0$)

- We still need the introduction of counterterms in the action and in the operators to reduce lattice artefacts

Chirally-rotated Schrödinger functional

- Trick: we rotate fermion fields with a chiral-non singlet transformation

$$\psi = R(\frac{\pi}{2})\psi' \qquad \bar{\psi} = \bar{\psi}' R(\frac{\pi}{2}) \quad \text{with } R(\alpha) = exp(i\frac{\alpha}{2}\gamma_5\tau^3)$$

- obtaining new boundary conditions

$$\tilde{Q}_+\psi'(x)\big|_{x_0=0} = 0 \qquad \tilde{Q}_-\psi'(x)\big|_{x_0=T} = 0$$

$$\bar{\psi}'(x)\tilde{Q}_{+|x_0=0} = 0 \qquad \bar{\psi}'(x)\tilde{Q}_{-|x_0=T} = 0$$

for which Symanzik's improvement works automatically.

- Price: a counterterm to remove artefacts from boundary (needed already in SF), a counterterm to restore parity/flavour symmetry in the new lattice regularisation (z_f)