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On the ultraviolet asymptotics of the glueball effective action in large-N Yang-Mills theory

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Based on:

[1] M. Bochicchio, M. Papinutto and F. Scardino, JHEP 08 (2021) 142, [2104.13163].

[2] M. Bochicchio, Eur. Phys. J. C 81 (2021) 749, [2103.15527]

[3] M. Bochicchio, M. Papinutto and F. Scardino, Phys. Rev. D 108 (2023), [2208.14382].

[4] M. Bochicchio, M. Papinutto and F. Scardino, to appear.

In the first part of the talk we briefly explain our computation of the renormalization-group (RG) improved generating functional of twist-2 operators with maximal spin projection on the p_+ light-cone direction in pure SU(N) Yang-Mills theory [3].

3 Steps:

1) Computation of the lowest-order conformal generating functional as the logarithm of a funcional determinant [1][3]

2) Construction of a renormalization scheme where the operators become multiplicatively renormalizable [2][3]

3) RG improvement of the Euclidean conformal generating functional [3]

In the second part we discuss the non-perturbative interpretation of our results in the large-N expansion of Yang-Mills theory [4].

Yang-Mills theory is conformal to order g^0 (leading order) and also g^2 (next-to-leading) of perturbation theory (as the beta function enters the solution of the Callan-Symanzik equation to order g^4).

Twist-2 operators transform to the leading order as primary operators with respect to the conformal group [5]

$$\mathbb{O}^{\mathcal{T}=2}_{\rho_1\ldots\rho_s} = \mathrm{Tr}\; F^{\mu}_{(\rho_1} \overleftrightarrow{D}_{\rho_2} \ldots \overleftrightarrow{D}_{\rho_{s-1}} F_{\rho_s)\mu} - \text{ traces}$$ Generalization of the stress-energy tensor

$$\tilde{\mathbb{O}}^{\mathcal{T}=2}_{\rho_1\ldots\rho_s} = \mathrm{Tr}\; \tilde{F}^{\mu}_{(\rho_1} \overleftrightarrow{D}_{\rho_2} \ldots \overleftrightarrow{D}_{\rho_{s-1}} F_{\rho_s)\mu} - \text{ traces}$$

$$\mathbb{S}^{\mathcal{T}=2}_{\mu\nu\rho_1\ldots\rho_{s-2}\lambda\sigma} = \mathrm{Tr}\,(F_{\mu(\nu} + i\tilde{F}_{\mu(\nu)}) \overleftrightarrow{D}_{\rho_1} \ldots \overleftrightarrow{D}_{\rho_{s-2}} (F_{\lambda)\sigma} + i\tilde{F}_{\lambda)\sigma}) - \text{ traces}$$

$\mathbb{O}^{\mathcal{T}=2}_{\rho_1\ldots\rho_s}$, $\tilde{\mathbb{O}}^{\mathcal{T}=2}_{\rho_1\ldots\rho_s}$ are the "balanced" operators that appear as leading terms in the OPE of tensor currents in Minkowskian space-time near the light-cone. They are balanced as the number of dotted and undotted indices in the spinor representation coincides.

$\mathbb{S}^{\mathcal{T}=2}_{\mu\nu\rho_1\ldots\rho_{s-2}\lambda\sigma}$ are the "unbalanced" operators as they are chiral, i.e. have a different number of dotted and undotted indices in the spinor representation

[5] N. Beisert, G. Ferretti, R. Heise and K. Zarembo, One-loop QCD spin chain and its spectrum, Nucl. Phys. B 717 (2005) 137–189, [hep-th/0412029].

These operators can be projected onto the + light-cone direction and have nice transformation properties with respect to the collinear conformal subgroup.

In the light-cone gauge they are exactly bilinear and take the form [6][3]

$$\mathbb{O}_s = \frac{1}{2}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x)$$

$$\tilde{\mathbb{O}}_s = \frac{1}{2}\bar{A}^a(x)\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x)$$

$$\mathbb{S}_s = \frac{1}{2\sqrt{2}}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)\bar{A}^a(x)$$

$$\bar{\mathbb{S}}_s = \frac{1}{2\sqrt{2}}A^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x)$$

Even spin

$$\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) = \overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+$$
$$= \frac{\Gamma(3)\Gamma(s+3)}{\Gamma(5)\Gamma(s+1)} i^{s-2}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-1)^{s-k}\overleftarrow{\partial}_+^{s-k-1}\overrightarrow{\partial}_+^{k+1}$$

Odd spin

$$\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) = \overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+$$
$$= \frac{\Gamma(3)\Gamma(s+3)}{\Gamma(5)\Gamma(s+1)} i^{s-2}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-1)^{s-k}\overleftarrow{\partial}_+^{s-k-1}\overrightarrow{\partial}_+^{k+1}$$

where **s** is the collinear spin, i.e. the eigenvalue of the spin operator along the light-cone direction and $C^{\frac{5}{2}}_{s-2}$ are the Gegenbauer polynomials, which are a special case of the Jacobi polynomials.

[6] V. M. Braun, G. P. Korchemsky and D. Mueller, The Uses of conformal symmetry in QCD, Prog. Part. Nucl. Phys. 51 (2003) 311–398, [hep-ph/0306057].

Step 1

We have pointed out [3] that the above operators are quadratic to all orders of perturbation theory in the light-cone gauge. Hence, we have computed their generating functional as a Gaussian functional integral from the definition of the Yang-Mills theory in the light-cone gauge to the lowest order [3]

$$\mathcal{Z}_{conf}[J_{\mathbb{O}}, J_{\tilde{\mathbb{O}}}, J_{\mathbb{S}}, J_{\bar{\mathbb{S}}}] = \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}\, e^{-i\int d^4x\, \bar{A}^a \Box A^a} \exp\left(\int d^4x \sum_s J_{\mathbb{O}_s}\mathbb{O}_s + J_{\tilde{\mathbb{O}}_s}\tilde{\mathbb{O}}_s + J_{\mathbb{S}_s}\mathbb{S}_s + J_{\bar{\mathbb{S}}_s}\bar{\mathbb{S}}_s\right)$$

For simplicty we only report in the present talk the generating functional of the even-spin balanced operators [3]. The conformal result is

$$\mathcal{W}_{conf}[J_{\mathbb{O}}, 0, 0, 0] = \log \mathcal{Z}_{conf}[J_{\mathbb{O}}, 0, 0, 0] = -(N^2-1)\log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2} i\Box^{-1} J_{\mathbb{O}_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

Step 2

Defining $\langle \mathcal{O}_{k_1}(x_1)\ldots\mathcal{O}_{k_n}(x_n)\rangle = G^{(n)}_{k_1\ldots k_n}(x_1,\ldots,x_n;\mu,g(\mu))$

in general, the above operators mix with derivatives of lower spin operators of twist-2.
As a consequence we get from the Callan-Symanzik equation the corresponding UV asymptotics for $\lambda \to 0$

$$G^{(n)}_{k_1\ldots k_n}(\lambda x_1,\ldots,\lambda x_n;\mu,g(\mu)) \sim \sum_{j_1\ldots j_n} Z_{k_1 j_1}(\lambda)\ldots Z_{k_n j_n}(\lambda)\,\lambda^{-\sum_{i=1}^{n} D_{\mathcal{O}_i}} G^{(n)}_{conf\; j_1\ldots j_n}(x_1,\ldots,x_n)$$

provided that $G^{(n)}_{conf\; j_1\ldots j_n}(x_1,\ldots,x_n)$ (which can be computed at the lowest order of perturbation theory) is not zero.

The idea in [2] is to find a (formal) holomorphic transformation depending on the coupling that defines a finite change of renormalization scheme

$$\mathcal{O}'(x) = S(g)\mathcal{O}(x)$$

In [2] it demonstrated that under the so called "non-resonant" condition $Z(\lambda)$ is diagonalizable and it is one loop exact to all orders of perturbation theory and its eigenvalues take the form

$$Z_{\mathcal{O}_i}(\lambda) = \left(\frac{g(\mu)}{g(\frac{\mu}{\lambda})}\right)^{\frac{\gamma_0 \mathcal{O}_i}{\beta_0}}$$

Since, for twist-2 operators in SU(N) Yang-Mills theory the non-resonant condition is verified [8][3] then the UV asymptotics is greatly simplified

$$G^{(n)}_{j_1\ldots j_n}(\lambda x_1,\ldots,\lambda x_n;\mu,g(\mu)) \sim \frac{Z_{\mathcal{O}_{j_1}}(\lambda)\ldots Z_{\mathcal{O}_{j_n}}(\lambda)}{\lambda^{D_{\mathcal{O}_1}+\cdots+D_{\mathcal{O}_n}}} G^{(n)}_{conf\, j_1\ldots j_n}(x_1,\ldots,x_n)$$

with $\quad Z_{\mathcal{O}_i}(\lambda) = \left(\frac{g(\mu)}{g(\frac{\mu}{\lambda})}\right)^{\frac{\gamma_{0\mathcal{O}_i}}{\beta_0}}$ $\quad$ and $\quad g^2(\frac{\mu}{\lambda}) \sim \frac{1}{\beta_0 \log(\frac{1}{\lambda^2})}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{1}{\lambda^2})}{\log(\frac{1}{\lambda^2})}\right)$

[8] U. Aglietti, M. Becchetti, M. Bochicchio, M. Papinutto and F. Scardino, Operator mixing, UV asymptotics of nonplanar/planar 2-point correlators, and nonperturbative large-N expansion of QCD-like theories, [2105.11262].

Step 3

It follows the RG-improved Euclidean generating functional of the even-spin balanced (rescaled so that the 2-point correlators are of order 1 for large N) operators in the non-resonant diagonal renormalization scheme [3]

$$\mathcal{W}^{E}_{asym}[J_{\mathbb{O}^E},0,0,0] = -(N^2-1)\log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{1}{N}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}J_{\mathbb{O}^E_s}\otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

Large-N generating functional

In the ‘t Hooft large-N expansion, Feynmann diagrams -in the double line representation- are topologically equivalent to closed Riemann surfaces (with punctures).

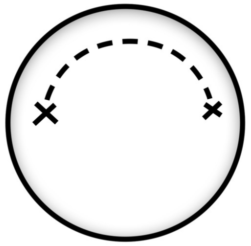
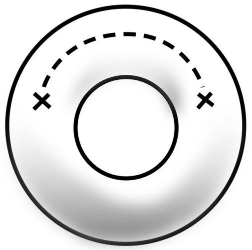
Then, the perturbative series can be rearranged so that the topologically equivalent contributions are resummed together.

Therefore, our generating functional should be interpreted as a sum of topologically distinct terms

$$\mathcal{W} = \sum_n N^{2-n} \mathcal{W}_{\text{sphere}}(g, n) + N^{-n} \mathcal{W}_{\text{torus}}(g, n) + \ldots$$

where n stands for the number of operator insertions and the ellipses for higher-genus contributions.

Graphically,

$$\mathcal{W} = \sum_n N^{2-n}$$  $$+ N^{-n}$$  $$+ \ldots$$

Large-N generating functional

Explictly, asymptotically

$$\mathcal{W}^{E}_{asym}[J_{\mathbb{O}^E},0,0,0] = -\,N^2 \log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{1}{N}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} J_{\mathbb{O}^E_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

$$+ \log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{1}{N}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} J_{\mathbb{O}^E_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

Non-perturbative interpretation of the large-N generating functional

Yang-Mills theory in the large-N expansion should be interpreted as a theory of an infinite number of weakly interacting glueballs, with interaction of order 1/N.
The generating functional reads schematically

$$\mathcal{Z}_{\text{glueball}}[j] = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int \phi*(-\Delta+M^2)\phi - \frac{1}{N}\frac{1}{3!}\int \phi*\phi*\phi+\dots} e^{\int j\phi}$$

where * stands for a presently unkown algebraic structure on the glueball fields.

Hence, the connected generating functional reads

$$\begin{aligned}\mathcal{W}_{\text{glueball}}[j] = \log \mathcal{Z}_{\text{glueball}}[j] &= -\frac{1}{2}\int \phi_j * (-\Delta + M^2)\phi_j - \frac{1}{N}\frac{1}{3!}\int \phi_j * \phi_j * \phi_j + \int j\phi_j \\ &-\frac{1}{2}\log \mathrm{Det}\left(*(-\Delta + M^2) + \frac{1}{N} * \phi_j * + \dots\right) + \dots\end{aligned}$$

With $\left.\frac{\delta S}{\delta \phi}\right|_{\phi_j} = j$ and $\phi_j = (*(-\Delta + M^2))^{-1} j - \frac{1}{2N}(*(-\Delta + M^2))^{-1} * \phi_j * \phi_j + \dots$

The minus sign in front of the logDet in $\mathcal{W}_{\text{glueball}}[j]$ arises from the spin-statistics theorem, since all the gauge-invariant glueball interpolating fields have integer spin, and thus the glueballs should be bosons

$$-\frac{1}{2}\log\operatorname{Det}\left(*(-\Delta+M^2)+\frac{1}{N}*\phi_j*+\dots\right)$$

Remarkably, our asymptotic result reproduces the logDet structure of the glueball one-loop generating functional Yet, surprisingly, the sign is the opposite of what would follow from the spin-statistics theorem

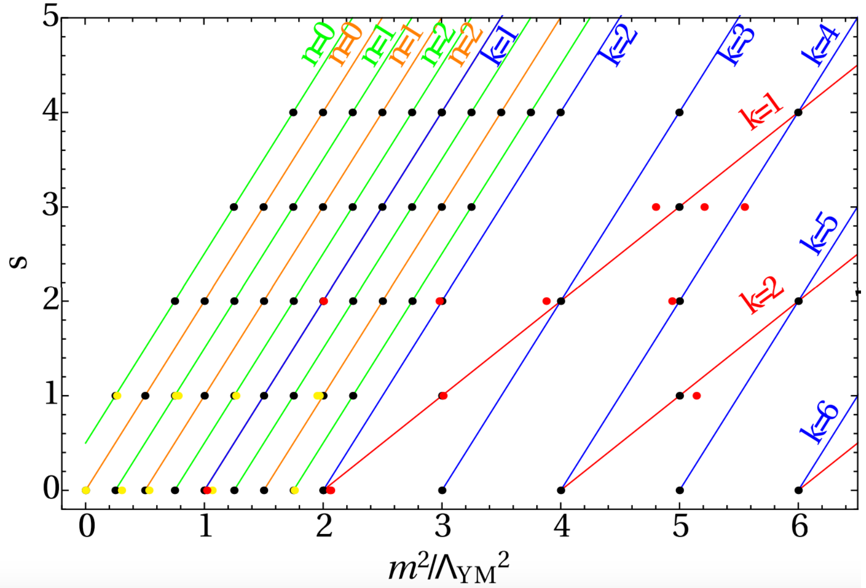
$$\mathcal{W}^{E}_{\text{Torus asym}}[J_{\mathbb{O}^E},0,0,0]=+\log\operatorname{Det}\left(\mathbb{I}+\frac{1}{2}\frac{1}{N}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}J_{\mathbb{O}^E_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

The aim of this talk is to solve the above sign puzzle.

One of the hypotheses of the spin-statistics theorem is that there must be a finite number of fields. Indeed, in theories where there is an infinite number of fields several counterexamples to the theorem are known.
These counterexamples are based on massive infinite dimensional representations of the Lorentz group. For instance, in [9] examples are constructed of theories with an infinite number of integer-spin fields where fermionic statistics must be imposed in order to ensure positivity of the energy.

Conversely, there are examples of fields with half-integer spins that require bosonic quantization [9][10][11][12].

A significant issue with this hypothetical way out of the sign puzzle is that the aforementioned infinite-dimensional representations of the Lorentz group have infinite mass degeneracy. Namely, the fields constructed by means of these representations decompose –according to the Wigner theorem- into the sum of irreducible representations of the Poincaré group corresponding to an infinite number of particles of any spin, all having the same mass.
This would correspond to vertical Regge trajectories that is not acceptable in large-N Yang-Mills theory, as evidence from lattice calculations shows [13]



[9] Harish-Chandra. "Infinite Irreducible Representations of the Lorentz Group." Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 189, no. 1018 (1947)

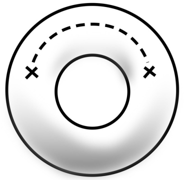
[10] Feldman, G. and Matthews, P. T., Unitarity, Causality, and Fermi Statistics, Phys. Rev. 151 , 4, (1966)

[11] Streater, R.F. Local fields with the wrong connection between spin and statistics. Commun.Math. Phys. **5**, 88–96 (1967).

[12] R. Casalbuoni, Majorana and the Infinite Component Wave Equations, PoS EMC2006 (2006), [hep-th/0610252].

[13] M. Bochicchio, Glueball and Meson Spectrum in Large-N QCD, Few Body Syst. 57 (2016) no.6, 455-459

We have found a different way out of the sign puzzle: The ‘t Hooft topological expansion must be refined

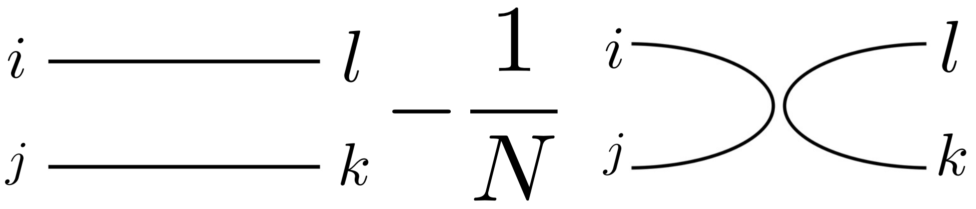
$$\mathcal{W}_{\text{1-loop}} = \sum_{\times}$$  $$+\,\text{new topologies} \quad \sim + \log \mathrm{Det}$$

To understand where the extra topologies come from, we need to recall the proof of the ‘t Hooft topological expansion.

In SU(N) Yang-Mills theory the color dependence of the propagator has a leading and subleading contribution

$$\langle A_{ij} A_{lk} \rangle \propto \frac{1}{2} \left(\delta_{il}\delta_{jk} - \frac{1}{N} \delta_{ij}\delta_{lk} \right)$$

In the double line representation [14][15]



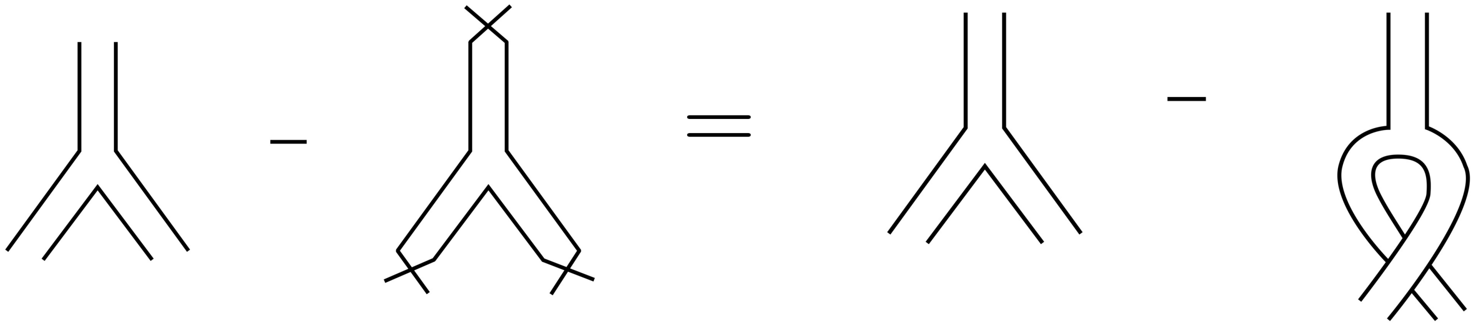
[14] M. Marino, Instantons and Large N: An Introduction to Non-Perturbative Methods in Quantum Field Theory,' Cambridge University Press, 2015.

[15] F. Maltoni, K. Paul, T. Stelzer and S. Willenbrock, Color Flow Decomposition of QCD Amplitudes, Phys. Rev. D 67 (2003), [hep-ph/0209271].

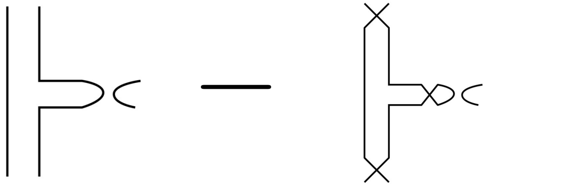
The vertices of the Lagrangian are $V_3 \propto \frac{g}{\sqrt{N}} f^{abc}$ and $V_4 \propto \frac{g^2}{N} f^{abl} f^{cdl} \propto V_3^2$

They can be drawn as [14]

$V_3 \propto$



The subleading part of the propagator does not contribute when attached to the vertices of the Lagrangian. Indeed, the subleading part of the propagator corresponds to the subtraction of a U(1) factor, while the vertices are purely nonabelian. Graphically



$= 0$

Therefore, the topology of vacuum-vacuum diagrams corresponds to the ‘t Hooft original identification in the double line representation [14]. The same holds for all the gluon composite operators (in the adjoint representation) whose local vertices are proportional to the product of some f^{abc} [15].

We conclude that in these cases we may drop the subleading contribution to the propagator and the proof of ‘t Hooft theorem that holds for U(N) also holds for SU(N) [14][15].

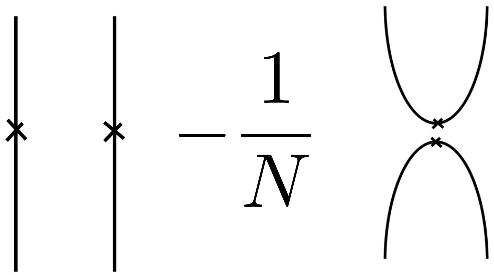
Yet, we point out that the above proof does not apply to the 2-gluon operators -for example, our twist-2 operators in the light-cone gauge. Indeed, even in the adjoint representation, their local vertex involves δ^{ab} as opposed to f^{abc}.

As a consequence, when the subleading part of the propagator is attached to the local vertex, a nonzero contribution is obtained.

Equivalently, in the SU(N) theory we may keep only the leading part of the propagator -in order to maintain the ‘t Hooft double line representation- provided that we project the above operators on their traceless part

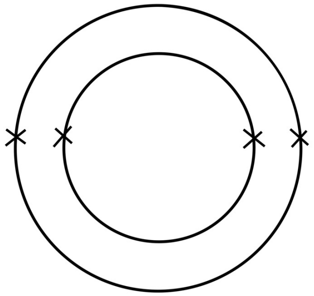
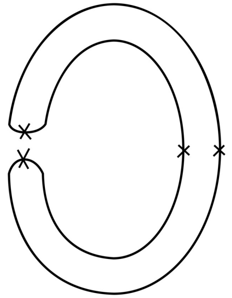
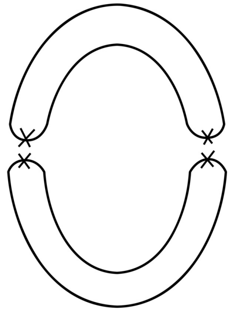
$$\mathrm{Tr}(\bar{A}A) \rightarrow \mathrm{Tr}(\bar{A}A) - \frac{1}{N}\mathrm{Tr}(\bar{A})\mathrm{Tr}(A)$$

Hence, the 2-gluon operators can be represented as a 2-point vertex with a subleading contribution

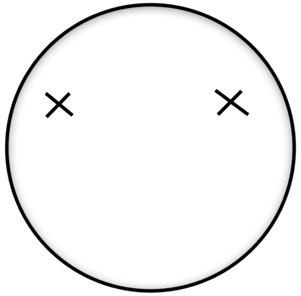
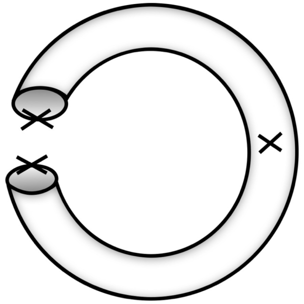
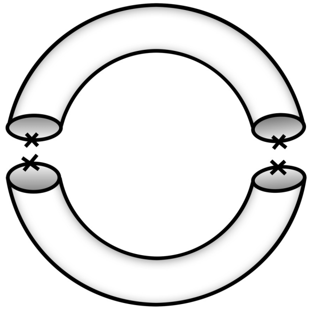


while keeping the propagator in the standard double line representation 

This subleading contribution to the operator vertex gives rise to new topologies. To see this we consider the color structure of the 2-point correlator of twist-2 operators to the leading perturbative order

$$\langle \mathbb{O}_s(x)\mathbb{O}_s(0)\rangle =$$  $$-\frac{2}{N}$$  $$+\frac{1}{N^2}$$ 

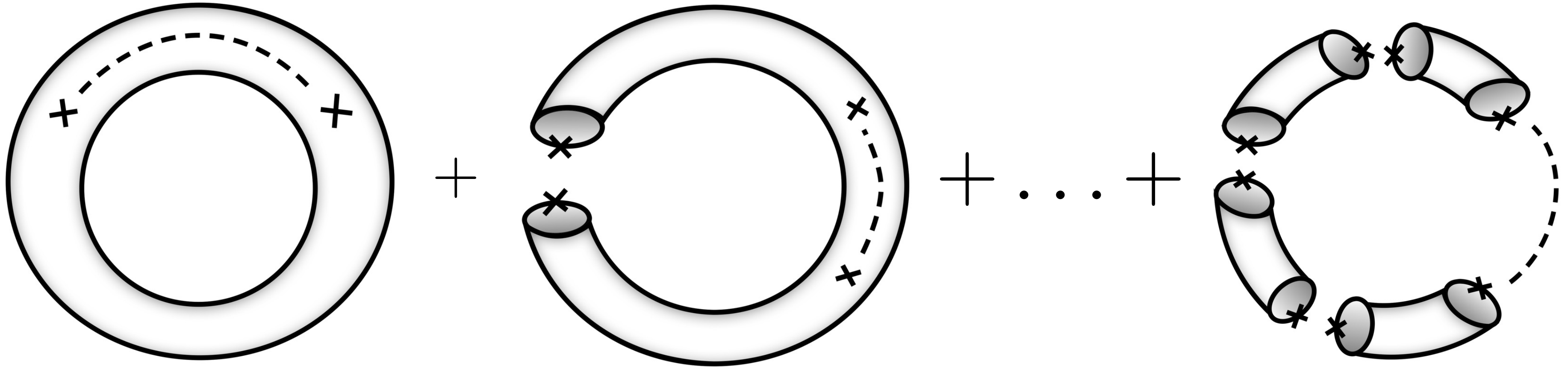
Hence, new planar subleading diagrams arise -in addition to first one above- corresponding to possibly color-disconnected (but space-time connected) punctured disks. Following 't Hooft prescription the first one corresponds to a punctured sphere

$$\langle \mathbb{O}_s(x)\mathbb{O}_s(0)\rangle =$$  $$-2$$  $$+$$ 

while we have identified the topology of the remaining ones by requiring that their Euler characteristic matches the correct 1/N dependence and that they remain planar and possibly color-disconnected.

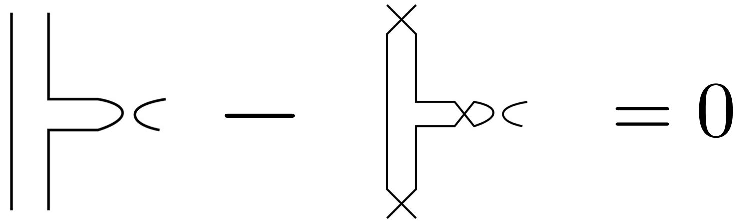
Hence, our refined topological expansion of the one-loop generating functional of twist-2 operators reads:

$$\mathcal{W}_{1-\text{loop}} = \sum_{\times}$$

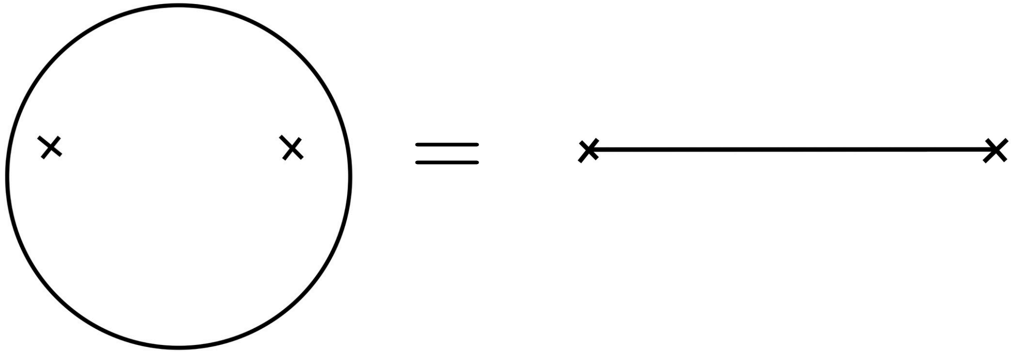


Remarkably, the torus diagram is corrected by a sum of new possibly color-disconnected objects (but not in space-time).

Interestingly, the torus contribution is suppressed in perturbation theory with respect to the remaining diagrams, since the torus inevitably involves the Lagrangian vertices (that carry powers of the coupling), while the remaining diagrams have no perturbative corrections because they cannot be attached to the Lagrangian vertices as a consequence of the identity

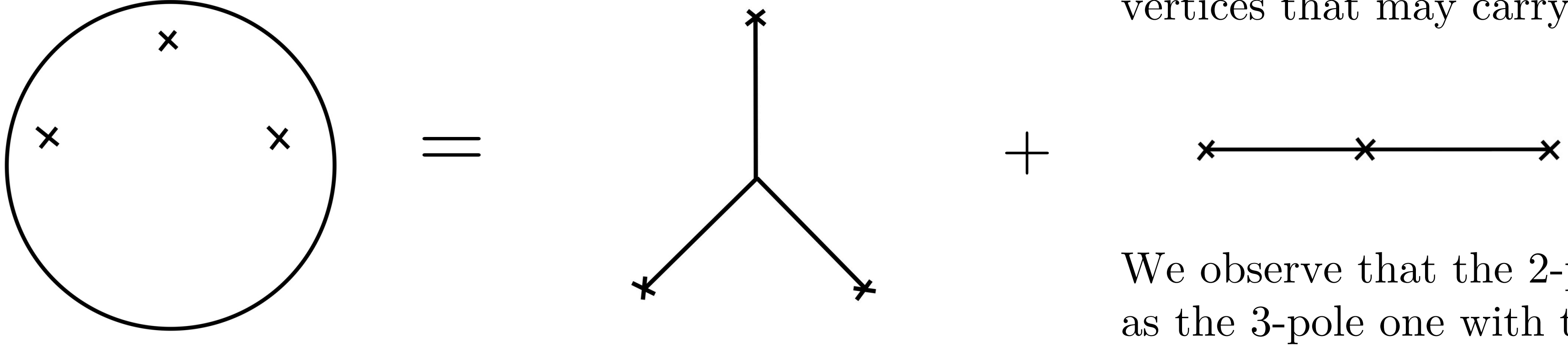


We now provide a nonperturbative interpretation of our refined topological expansion in terms of the effective theory of glueballs. It has been known for more than 40 years that punctured spheres correspond to glueball tree diagrams [16][17]



The sphere with two punctures corresponds to an infinite sum of glueball propagators [16].

The sphere with three punctures corresponds to vertices that may carry sums of 3 or 2 poles [17].

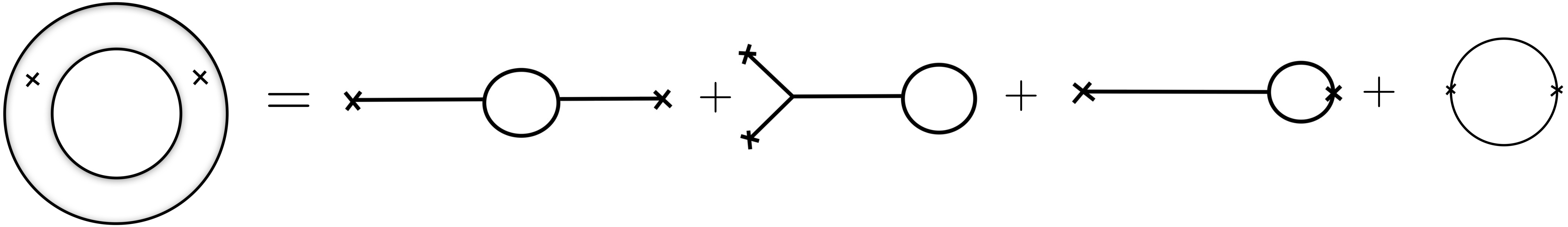


We observe that the 2-pole diagram may be interpreted as the 3-pole one with the insertion of a contact term, due to composite operators, in place of a propagator [4]. As a consequence, the second diagram contributes zero to the S matrix [4].

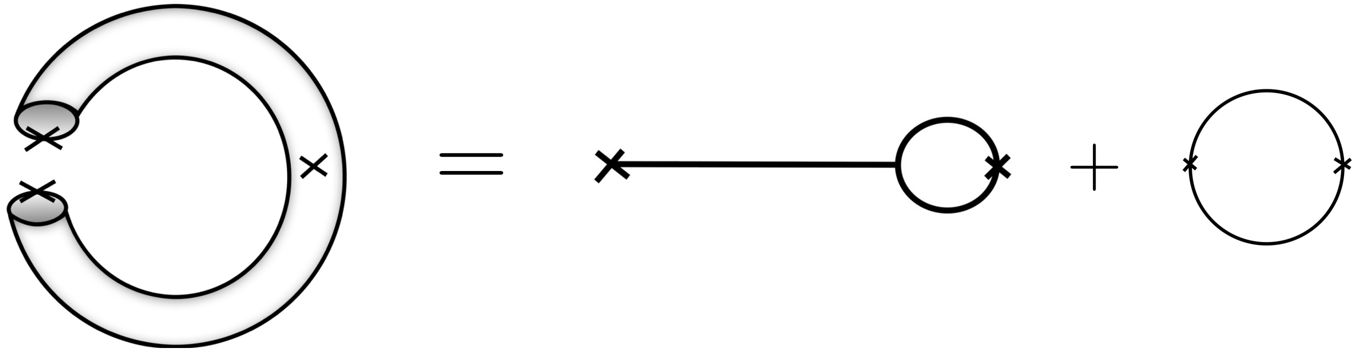
[16] A. Migdal, Multicolor QCD as Dual Resonance Theory, Annals Phys. 109 (1977) 365.

[17] E. Witten Baryons in the 1/N expansion, Nucl. Phys. B160 (1979) 57-115.

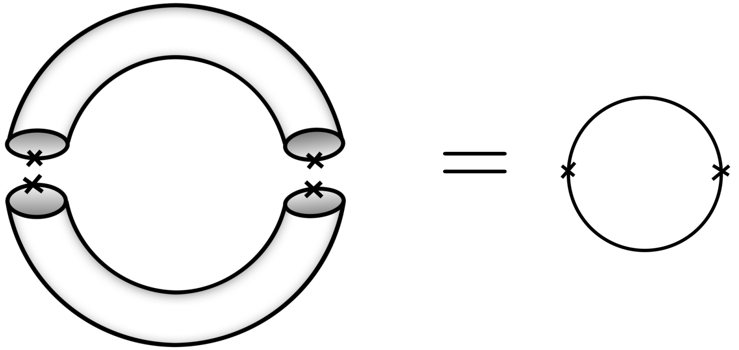
Glueball loops in the effective theory are represented by higher-genus surfaces. Specifically, 2-point glueball one-loop diagrams correspond to the punctured torus:



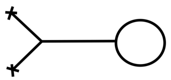
Interestingly, we observe [4] that in the effective theory on the right-hand side of the above picture also diagrams without external glueball legs occur. Moreover, for our new topologies we obtain the following interpretation



The punctures on the boundary of the cylinder are identified in spacetime and cannot represent an external glueball leg. Consequently, both diagrams contribute zero to the S matrix as well.

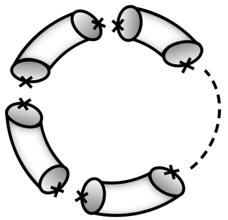


The maximally color-disconnected diagrams are the only ones that have a 1-to-1 correspondence with the effective theory, since they do not carry any external leg.

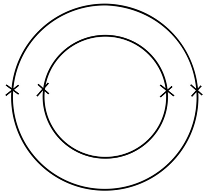
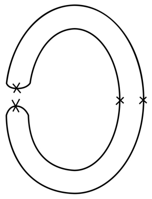
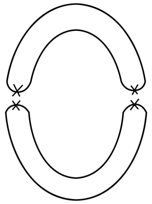
From the point of view of the effective theory, its not obvious how to sum on the number of insertions of external tree diagrams on the glueball loop, over which we have no explicit control -for example,  -

but for the maximally color-disconnected diagrams, which only carry the insertion of contact terms in the effective theory.

Indeed, the only contribution that it is easy to resum is the maximally color-disconnected one, which is one loop in space-time

$\sum_{\times}$ 

Remarkably, by direct evaluation we find that it has the opposite sign with respect to the total result.

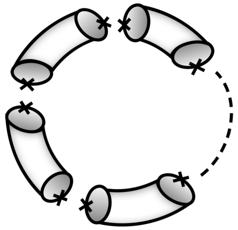
Indeed, for example $\langle \mathbb{O}_s(x)\mathbb{O}_s(0)\rangle =$  $-\frac{2}{N}$  $+\frac{1}{N^2}$  (notice the opposite sign in the last term)

Therefore, perturbatively

$$\mathcal{W}^{E}_{\text{maximally c-disconnected conf}}[J_{\mathbb{O}}] = -\log \operatorname{Det}\left(\mathbb{I} - \frac{1}{2N}\Delta^{-1} J_{\mathbb{O}^E_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

Hence,

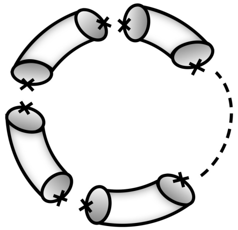
$$\mathcal{W}^{E}_{\text{maximally c-disconnected}}[J_{\mathbb{O}}] = \sum_{\times}$$



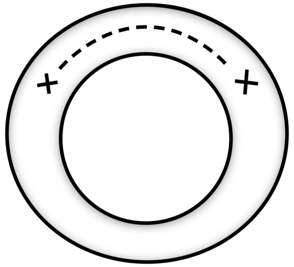
$$\sim \quad -\log \operatorname{Det}\left(\mathbb{I} - \frac{1}{2N}\frac{1}{\lambda^{s+2}}\Delta^{-1} J_{\mathbb{O}^E_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

that, though asymptotically conformal, in the SU(N) theory must depend nonperturbatively on the RG—invariant scale. After the RG-improvement it reads

$$\mathcal{W}^{E}_{\text{RG-improved maximally c-disconnected}}[J_{\mathbb{O}}] = \sum_{\times}$$



$$+$$



$$\Big|_{\text{certain legless contributions}}$$

$$\sim -\log \operatorname{Det}\left(\mathbb{I} - \frac{1}{2N}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} J_{\mathbb{O}^E_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

that now includes certain contributions from the torus without external legs in the effective theory.

Therefore, the introduction of the extra topologies solves the sign puzzle, in the sense that the only object that we really know how to resum nonperturbatively carries a sign consistent with the spin-statistics theorem.

Conclusions

In the SU(N) YM theory it is necessary to introduce a new topological sector for twist-2 operators that refines the ‘t Hooft topological expansion, both perturbatively and nonperturbatively.

Instead, in the U(N) YM theory the new sector is absent perturbatively.

Yet, this would not solve the aforementioned sign problem nonperturbatively, since in the U(N) theory the asymptotics of the generating functional is the sum of the RG-improved SU(N) result and the free U(1) part

$$\mathcal{W}_{U(N)}[J_{\mathbb{O}}] \sim -(N^2-1)\log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2N}\sum_s \frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} J_{\mathbb{O}_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$
$$-\log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2N}\sum_s \frac{1}{\lambda^{s+2}}\Delta^{-1} J_{\mathbb{O}_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

with the U(1) contribution in the second line being actually exactly conformal and not only asymptotic, since the U(1) theory is free.
Hence, nonperturbatively -even in the U(N) theory- the solution of the sign puzzle is the one outlined above in the SU(N) theory.

The new topological sector dominates the UV of the correlators of twist-2 operators in the SU(N) theory, and its maximally color-disconnected diagrams can be re-summed asymptotically in the UV into a functional determinant with the sign agreeing with the spin-statistics theorem.

However, nonperturbatively the entire new sector does not contribute to the S matrix, since in the effective theory it consists of diagrams with at least one external glueball leg missing.

Hence, by limiting ourselves to the nonperturbative S matrix only, the original ‘t Hooft topological expansion is complete even for the twist-2 operators.

A crucial consequence is that a canonical string solution matching the topology of closed punctured Riemann surfaces cannot exist for the Yang-Mills correlators, but it may exist for the S matrix.

Yet, a noncanonical string solution may exist also for the correlators provided that it contains extra couplings to D-branes [8] ([M.B.] to appear).

Indeed, the existence of the new topological sector that contributes zero to the S matrix, but nontrivially to the correlators, opens the way to an exact solution limited to the new sector ([M.B.] to appear), since the new sector may be completely integrable [8] thanks to the vanishing of the S matrix.

[8] M. Bochicchio, An asymptotic solution of large-N QCD, for the glueball and meson spectrum and the collinear S-matrix, HADRON 2015, AIP Conf. Proc. (2016).

BACKUP

Defining

$$\langle \mathcal{O}_{k_1}(x_1)\ldots\mathcal{O}_{k_n}(x_n)\rangle = G^{(n)}_{k_1\ldots k_n}(x_1,\ldots,x_n;\mu,g(\mu))$$

in general, the above operators mix with derivatives of lower spin operators of twist-2.
As a consequence we get from the Callan-Symanzik equation as $\lambda \to 0$

$$G^{(n)}_{k_1\ldots k_n}(\lambda x_1,\ldots,\lambda x_n;\mu,g(\mu)) = \sum_{j_1\ldots j_n} Z_{k_1 j_1}(\lambda)\ldots Z_{k_n j_n}(\lambda)\,\lambda^{-\sum_{i=1}^{n} D_{\mathcal{O}_i}} G^{(n)}_{j_1\ldots j_n}(x_1,\ldots,x_n;\mu,g\Big(\frac{\mu}{\lambda}\Big))$$

where $Z(\lambda) = P\exp\Big(\int_{g(\mu)}^{g(\frac{\mu}{\lambda})}\frac{\gamma(g')}{\beta(g')}dg'\Big)$ is the renormalized mixing matrix that involves the computation of a path-ordered exponential.

The corresponding UV asymptotics for $\lambda \to 0$ is

$$G^{(n)}_{k_1\ldots k_n}(\lambda x_1,\ldots,\lambda x_n;\mu,g(\mu)) \sim \sum_{j_1\ldots j_n} Z_{k_1 j_1}(\lambda)\ldots Z_{k_n j_n}(\lambda)\,\lambda^{-\sum_{i=1}^{n} D_{\mathcal{O}_i}} G^{(n)}_{conf\; j_1\ldots j_n}(x_1,\ldots,x_n)$$

provided that $G^{(n)}_{conf\; j_1\ldots j_n}(x_1,\ldots,x_n)$, which can be computed at the lowest order of perturbation theory, is not zero.

The idea in [2] is to find a (formal) holomorphic gauge transformation depending on the coupling that defines a finite change of renormalization scheme

$$\mathcal{O}'(x) = S(g)\mathcal{O}(x)$$

Under the above gauge transformation $-\frac{\gamma(g)}{\beta(g)} = \frac{1}{g}\left(\frac{\gamma_0}{\beta_0} + \sum_{n=1}^{\infty} C_n g^{2n}\right)$ transforms as a gauge connection with a pole at g=0 [2].

In [2] it demonstrated, by means of the Poincaré-Dulac theorem, that under the "non-resonant" condition

$$\lambda_i - \lambda_j - 2k \neq 0$$

with i>j, k a positive integer and λ_i the eigenvalues of $\frac{\gamma_0}{\beta_0}$ in non-increasing order, an S(g) exists such that

to all orders of perturbation theory $-\frac{\gamma(g)}{\beta(g)} = \frac{1}{g}\frac{\gamma_0}{\beta_0}$ is one-loop exact, so that only the pole part survives.

Besides, if γ_0 is diagonalizable, $Z(\lambda)$ is diagonalizable as well with eigenvalues $Z_{\mathcal{O}_i}(\lambda) = \left(\frac{g(\mu)}{g(\frac{\mu}{\lambda})}\right)^{\frac{\gamma_0 \mathcal{O}_i}{\beta_0}}$

The minus sign in front of the logDet in $\mathcal{W}_{\text{glueball}}[j]$ arises from the spin-statistics theorem, since all the gauge-invariant glueball interpolating fields have integer spin, and thus the glueballs should be bosons.

The corresponding glueball one-loop effective action, $\Gamma_{\text{glueball}}[\phi] = \int j * \phi - \mathcal{W}_{\text{glueball}}[j]$, reads [8]

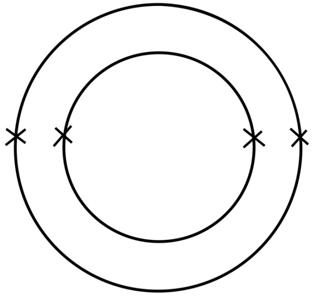
$$\Gamma_{\text{glueball}}[\phi] = \frac{1}{2}\int \phi *(-\Delta + M^2)\,\phi + \frac{1}{N}\frac{1}{3!}\int \phi * \phi * \phi + \frac{1}{2}\log\text{Det}\left(*(-\Delta + M^2) + \frac{1}{N} * \phi * + \ldots\right) + \ldots$$

From the perspective of the ‘t Hooft expansion, the one-loop part of the glueball effective action can be thought of as being the nonperturbative resummation -involving Λ_{YM} that all the glueball masses must be proportional to- of the sum of the amputated punctured toruses.

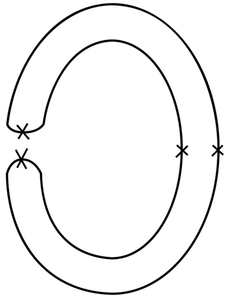
[8] M. Bochicchio, An asymptotic solution of large-N QCD, for the glueball and meson spectrum and the collinear S-matrix, HADRON 2015, AIP Conf. Proc. (2016).

This subleading contribution to the operator vertex gives rise to new topologies. To see this we consider the color structure of the 2-point correlator of twist-2 operators to the leading perturbative order

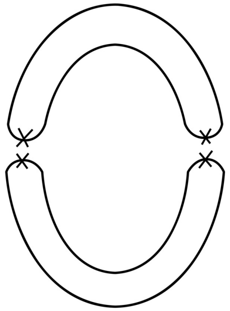
$$\langle \mathbb{O}_s(x)\mathbb{O}_s(0)\rangle =$$



$$-\frac{2}{N}$$

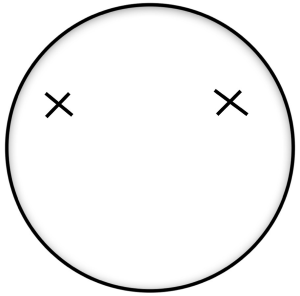


$$+\frac{1}{N^2}$$

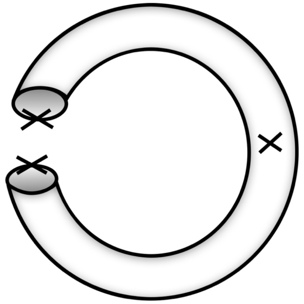


Hence, new planar subleading diagrams arise -in addition to first one above- corresponding to possibly color-disconnected (but space-time connected) punctured disks. Following ‘t Hooft prescription the first one corresponds to a punctured sphere

$$\langle \mathbb{O}_s(x)\mathbb{O}_s(0)\rangle =$$



$$-2$$



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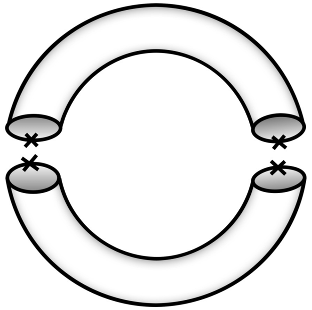


while we have identified the topology of the remaining ones by requiring that their Euler characteristic matches the correct 1/N dependence and that they remain planar and possibly color-disconnected.
Indeed, we have identified the doubly punctured disk with an infinite strip. Then, we have glued the two sides of the infinite strip, but in the neighborhood of infinity, to get a cylinder with punctures on the boundaries.

Had we glued the sides also in the neighborhood of infinity, we would have gotten a 2-punctured sphere whose Euler characteristic does not match the correct 1/N counting. Besides, the propagator associated to the sphere, instead of the punctured cylinder above, would not reproduce the correct UV asymptotics.

Nonperturbatively, we may take into account the occurrence of contact terms in the external legs by introducting in the generating functional a new source j' coupled to the equations of motion

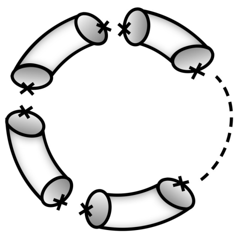
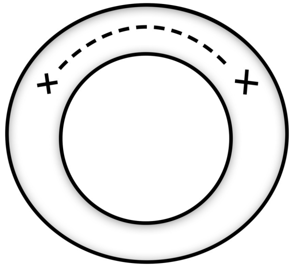
$$\mathcal{Z}_{\text{glueball}}[j,j'] = \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int \phi*(-\Delta+M^2)\phi - \frac{1}{N}\frac{1}{3!}\int \phi*\phi*\phi+\dots}\, e^{\int j\phi + j'*(-\Delta+M^2)\phi+\dots}$$

$$= e^{-\frac{1}{2}\int \phi_{j,j'}*(-\Delta+M^2)\phi_{j,j'} - \frac{1}{N}\frac{1}{3!}\int \phi_{j,j'}*\phi_{j,j'}*\phi_{j,j'}+\dots}\, e^{\int j\phi_{j,j'} + j'*(-\Delta+M^2)\phi_{j,j'}+\dots}$$

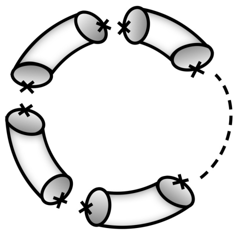
$$\text{Det}^{-\frac{1}{2}}\left(*(-\Delta+M^2)+\frac{1}{N}*\phi_{j,j'}*+\dots\right)$$

where now $\phi_{j,j'} = j' + (*(-\Delta+M^2))^{-1}j - \frac{1}{2N}(*(-\Delta+M^2))^{-1}*\phi_{j,j'}*\phi_{j,j'}+\dots$

Diagrammatically,

$\sum_{\times}$  $+$  $\Big|_{\text{certain legless contributions}}$ $= -\frac{1}{2}\log\text{Det}\left(*(-\Delta+M^2)+\frac{1}{N}*j'*\right)$

and

$\sum_{\times}$  $= -\frac{1}{2}\log\text{Det}\left(*'(-\Delta+M^2)+\frac{1}{N}*'j'*'\right)$

Color structure of the propagator in double line representation

$$\langle \bar{A}_{ij}(x) A_{kl}(y)\rangle = -\frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{lk}\right) i\Box^{-1}(x-y)$$
$$= -\frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{lk}\right) \frac{1}{4\pi^2}\frac{1}{|x-y|^2 - i\epsilon}$$

Where $D_{ij\,kl} = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{lk} = \mathbb{I}_{ijlk} - P_{ijlk}$

The generating functional of twist-2 operators in SU(N) Yang-Mills can be written as the generating functional of the constrained U(N) theory

$$\mathcal{Z}[J_{\mathbb{O}}, J_{\tilde{\mathbb{O}}}, J_{\mathbb{S}}, J_{\bar{\mathbb{S}}}] = \int \mathcal{D}A\mathcal{D}\bar{A}\,\delta(\mathrm{Tr}\,A)\delta(\mathrm{Tr}\,\bar{A})\, e^{i\int d^4x\, S_{YM}^{U(N)}(A,\bar{A})}\; e^{\int d^4x \sum_s\; J_{\mathbb{O}_s}\mathbb{O}_s + J_{\tilde{\mathbb{O}}_s}\tilde{\mathbb{O}}_s + J_{\mathbb{S}_s}\mathbb{S}_s + J_{\bar{\mathbb{S}}_s}\bar{\mathbb{S}}_s}$$

We project the operators onto the contraint with the following transformation $A_{ij} \to A_{ij} - \frac{1}{N}\delta_{ij}\,\mathrm{Tr}\,A$

We write the projected conformal generating functional for even-spin balanced operators

$$\mathcal{Z}_{\mathrm{conf}}[J_{\mathbb{O}}, 0, 0, 0] = \int \mathcal{D}A\mathcal{D}\bar{A}\;\delta(\mathrm{Tr}\,A)\delta(\mathrm{Tr}\,\bar{A})\; e^{-\int d^4x\, \bar{A}_{ij}(x)\left(\mathbb{I}_{ij\,kl}\, i\Box - \frac{1}{2} D_{ij\,kl}\sum_s J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}\right) A_{kl}(x)}$$

where we have dropped the subleading term P in the propagator, since once we have the operators projected on the constraint we can use the fact that $P \cdot D = 0$.

The connected generating functional of the rescaled operators reads

$$\mathcal{W}_{\text{conf}}[J_{\mathbb{O}},0,0,0] = -\log \operatorname{Det}\left(\mathbb{I}_{ij\,kl} - \frac{1}{2} D_{ij\,kl} \sum_s (i\Box)^{-1} \frac{J_{\mathbb{O}_s}}{N} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

which can be written as the sum of a planar term and the sum of the possibly disconnected new topologies

$$\mathcal{W}_{\text{conf}}[J_{\mathbb{O}},0,0,0] = -\log \operatorname{Det}\left(\mathbb{I}_{ij\,kl} i\Box - \frac{1}{2} \mathbb{I}_{ij\,kl} \sum_s \frac{J_{\mathbb{O}_s}}{N} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$
$$-\log \operatorname{Det}\left(\mathbb{I}_{ij\,kl} + \frac{1}{2}\left(i\Box - \frac{1}{2}\sum_s \frac{J_{\mathbb{O}_s}}{N} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)^{-1} P_{ij\,kl} \sum_s \frac{J_{\mathbb{O}_s}}{N} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

By dropping the inverse term, the maxiammly disconnected piece reads

$$\mathcal{W}_{\text{conf maximally c-disconnected}}[J_{\mathbb{O}},0,0,0] = -\log \operatorname{Det}\left(\mathbb{I}_{ij\,kl} + \frac{1}{2} P_{ij\,kl} \sum_s (i\Box)^{-1} \frac{J_{\mathbb{O}_s}}{N} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

We can recover the explicit color dependence by using the properties of P and of the identity, namely

$$P^n = P \qquad \operatorname{Tr}\mathbb{I} = N^2$$
$$\operatorname{Tr} P = 1$$