Gravitational Waves from Scattering Amplitudes

The Eikonal Operator and Angular Momentum Losses

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Based on

P. Di Vecchia, C.H., R. Russo, G. Veneziano [2210.12118] [2306.16488]

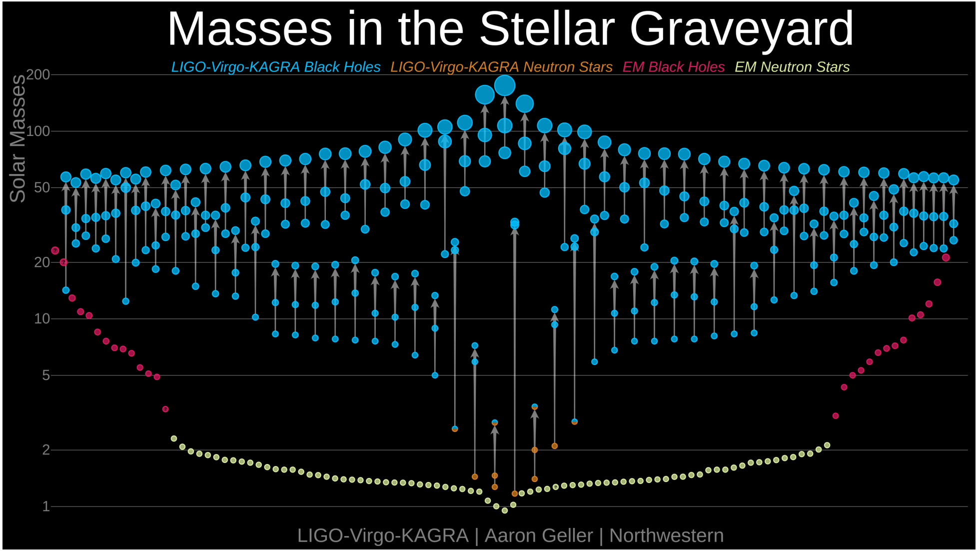
A. Georgoudis, C.H., I. Vazquez–Holm [2303.07006]

C.H. [2210.15689] [2308.11470]

Outline

Outline

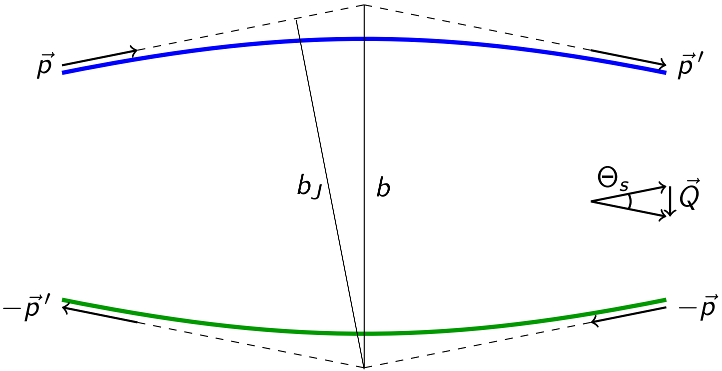
Gravitational Wave Astronomy



Post-Minkowskian (PM) Scattering

[see e.g. Kosower, Maybee, O'Connell '18; Bern et al. '19; Di Vecchia, C.H., Russo, Veneziano '21; Bellazzini, Isabella, Riva '22]

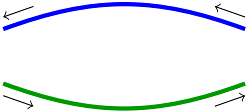
$$\boxed{\frac{\hbar}{m} \ll Gm \ll b}$$



$$\frac{Gm^2}{\hbar} \underset{CL}{\gg} 1\,, \qquad \frac{Gm}{b} \underset{\mathrm{PM}}{\ll} 1\,, \qquad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic).}$$

General Relativity from Scattering Amplitudes

- Weak-coupling expansion ↔ PM expansion
- Lorentz invariance ↔ generic velocities

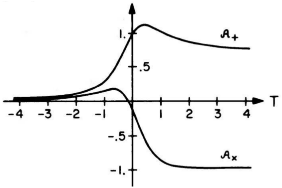


$\mathcal{A}_0 = \mathcal{O}(G)$ — Θ_{1PM} — [Geissler '59]

$\mathcal{A}_1 = \mathcal{O}(G^2)$ — Θ_{2PM} — [Westpfahl '85] [Cheung, Rothstein, Solon '18] [Collado, Di Vecchia, Russo '19]

$\mathcal{A}_2 = \mathcal{O}(G^3)$ — Θ_{3PM} — [Bern et al. '19] [Di Vecchia, C.H., Russo, Veneziano '20, '21]

$\mathcal{A}_3 = \mathcal{O}(G^4)$ — $\Theta_{4PM}^{(i=1,2)}$ — [Bern et al. '21] [Dlapa et al. '21, '22]

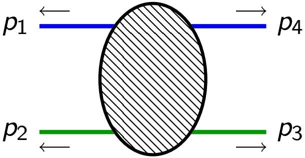


$\mathcal{A}_0^{\mu\nu} = \mathcal{O}(G^{\frac{3}{2}})$ — $\tilde{\mathcal{W}}_{LO}^{\mu\nu}$ — [Kovacs, Thorne '78] [Jakobsen, Mogull, Plefka, Steinhoff '21] [Mougiakakos, Riva, Vernizzi '21]

$\mathcal{A}_1^{\mu\nu} = \mathcal{O}(G^{\frac{5}{2}})$ — $\mathcal{W}_{NLO}^{\mu\nu}$ — [Brandhuber et al. '23] [Herderschee, Roiban, Teng '23] [Georgoudis, C.H., Vazquez-Holm '23]

- Study scattering events, then export to bound trajectories (analytic continuation)

The Elastic Eikonal



$$s = -(p_1+p_2)^2 = E^2$$
$$= m_1^2 + 2m_1 m_2\,\sigma + m_2^2\,,$$
$$t = -(p_1+p_4)^2 = -q^2\,.$$

- From q to b: Fourier transform $[q \sim \mathcal{O}(\frac{\hbar}{b})]$

$$\tilde{\mathcal{A}}(b) = \frac{1}{4Ep}\int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\mathcal{A}(s,q)\,, \qquad \boxed{1 + i\tilde{\mathcal{A}}(b) = e^{2i\delta(b)}}$$

with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim \frac{Gm^2}{\hbar}\left(\log b + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \cdots\right)$

- From b to Q: stationary-phase approximation $[Q \sim \mathcal{O}(p\cdot\frac{Gm}{b})]$

$$\int d^{D-2}b\, e^{-ib\cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial\,\mathrm{Re}\,2\delta}{\partial b^\mu}$$

The 3PM Eikonal in General Relativity

[Di Vecchia, C.H., Russo, Veneziano '20, '21]

[Related work at 3PM: Bern al.'19; Damour '20; Herrmann et al. '21, Bjerrum-Bohr et al.'21; Brandhuber et al.'21]

- Eikonal phase:

$$\mathrm{Re}\,2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\frac{s\left(12\sigma^4-10\sigma^2+1\right)}{2m_1 m_2\left(\sigma^2-1\right)^{\frac{3}{2}}}-\frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}}-\frac{4\sigma^4-12\sigma^2-3}{\sigma^2-1}\,\mathrm{arccosh}\sigma\right] + \mathrm{Re}\,2\delta_2^{\mathrm{RR}}$$

with

$$\mathrm{Re}\,2\delta_2^{\mathrm{RR}} = \frac{G}{2}Q_{1\mathrm{PM}}^2\mathcal{I}(\sigma)\,,\quad \mathcal{I}(\sigma)\equiv\frac{8-5\sigma^2}{3(\sigma^2-1)}+\frac{\sigma\left(2\sigma^2-3\right)}{(\sigma^2-1)^{3/2}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

$$\mathrm{Im}\,2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon}+\log(\sigma^2-1)\right]\mathrm{Re}\,2\delta_2^{\mathrm{RR}}+\cdots$$

Elastic Final State

- Final state (schematically):

$$\boxed{|\text{out}\rangle = e^{2i\delta(b)}|\text{in}\rangle}$$

- Impulse:

$$Q_\mu = \Big(-i\langle\text{out}|\frac{\overleftrightarrow{\partial}}{\partial b^\mu}|\text{out}\rangle\Big)/\langle\text{out}|\text{out}\rangle = \frac{\partial\,\text{Re}\,2\delta}{\partial b^\mu}\,.$$

Problems:

1. How do we restore (nonperturbative) unitarity?

$$\langle\text{out}|\text{out}\rangle = e^{-\,\text{Im}\,2\delta}\langle\text{in}|\text{in}\rangle \to 0 \qquad \text{as } D \to 4$$

2. What about gravitational wave emissions?

Outline

Operator Exponential

- **Option 1**: Look for an exponential structure in the full S-matrix [Damgaard, Planté, Vanhove '21]

$$S = 1 + iT = e^{iN} \qquad \boxed{|\mathrm{out}\rangle = e^{iN}|\mathrm{in}\rangle}$$

The "N-matrix" elements $\mathcal{B}$ are better behaved in the classical limit.

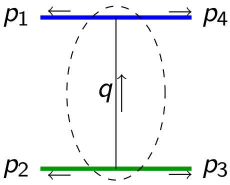
- **Option 2**: Look for an exponential structure in the classical limit,

$$\boxed{|\mathrm{out}\rangle \simeq e^{2i\hat{\delta}}|\mathrm{in}\rangle}$$

with $2\hat{\delta}$ a suitable eikonal operator [Cristofoli et al. '21] [Di Vecchia, CH, Russo, Veneziano '22].

The $2 \to 2$ Amplitude

[Parra-Martinez, Ruf, Zeng '20]

$\mathcal{A}^{(4)}(q) =$ 

$$
\begin{aligned}
p_1^\mu &= -\bar{m}_1 u_1^\mu + q^\mu/2 \\
p_4^\mu &= +\bar{m}_1 u_1^\mu + q^\mu/2 \\
p_2^\mu &= -\bar{m}_2 u_2^\mu - q^\mu/2 \\
p_3^\mu &= +\bar{m}_2 u_2^\mu - q^\mu/2 \\
q &\sim \hbar/b
\end{aligned}
$$

- The "average velocities" $u_{1,2}^\mu$ obey

$$u_1^2 = -1 = u_2^2 \qquad \& \qquad u_1 \cdot q = 0 = u_2 \cdot q$$

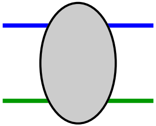
- We consider the physical variables

$$y = -u_1 \cdot u_2 \geq 1\,, \qquad q^2\,.$$

Note that $y = \sigma + \mathcal{O}(\hbar^2)$ and $\bar{m}_{1,2} = m_{1,2} + \mathcal{O}(\hbar^2)$.

Tree-Level $2 \to 2$ Amplitude

- $2 \to 2$ amplitude in momentum space

$$\mathcal{A}_0^{(4)} = \mathcal{B}_0^{(4)} =$$  $$= \frac{32\pi G \bar{m}_1^2 \bar{m}_2^2 \left(y^2 - \frac{1}{2(1-\epsilon)}\right)}{q^2}$$

$$+ \left[\frac{4\pi G\left(\bar{m}_1^2 + \bar{m}_2^2\right)}{1-\epsilon} + \frac{\pi G(3-2\epsilon)}{\epsilon - 1} q^2\right]$$

analytic in q^2

- In impact-parameter space $\boxed{1 + i\tilde{\mathcal{A}}^{(4)} \simeq e^{2i\delta_0}}$

$$\tilde{\mathcal{A}}_0^{(4)} = \tilde{\mathcal{B}}_0^{(4)} = 2\delta_0 = \frac{4Gm_1m_2\left(\sigma^2 - \frac{1}{2(1-\epsilon)}\right)}{2\sqrt{\sigma^2-1}} \frac{\Gamma(-\epsilon)}{(\pi b)^{-\epsilon}}$$

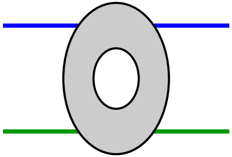
$+$ short-range

One-Loop $2 \to 2$ Amplitude

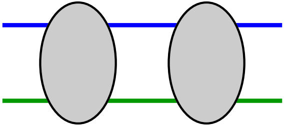
[Cheung, Rothstein, Solon '18; Collado, Di Vecchia, Russo '19]

Elastic subtraction:

$$\mathcal{A}_1^{(4)} =$$



$$= \frac{i}{2}$$

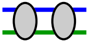


$$+ \mathcal{B}_1^{(4)}$$

with

$$\mathcal{B}_1^{(4)} = \frac{6\pi^2 G^2 \bar{m}_1^2 \bar{m}_2^2 \left(5y^2 - 1\right)\left(\bar{m}_1 + \bar{m}_2\right)}{\sqrt{q^2}}$$

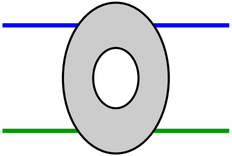
in $D = 4$

- The subtracted term $\frac{i}{2}$  is imaginary and infrared divergent
- It is also "superclassical" $\mathcal{O}(\frac{1}{\hbar^2})$
- $\mathcal{B}_1^{(4)}$ is real, finite and classical $\mathcal{O}(\frac{1}{\hbar})$ [and tr. weight 2].

Elastic Exponentiation at One Loop

[Collado, Di Vecchia, Russo '19]

Inelastic subtraction in b-space:

$$\tilde{\mathcal{A}}_1^{(4)} = \text{FT}$$  $$= \frac{i}{2}(2\delta_0)^2 + \tilde{\mathcal{B}}_1^{(4)}$$

- This matches the exponential $\boxed{1 + i\tilde{\mathcal{A}}^{(4)} \simeq e^{i(2\delta_0 + 2\delta_1)}}$
- And it identifies

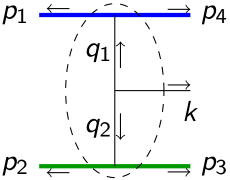
$$\tilde{\mathcal{B}}_1^{(4)} = 2\delta_1 = \frac{3\pi G^2 m_1 m_2 \left(5\sigma^2 - 1\right)(m_1 + m_2)}{4b\sqrt{\sigma^2 - 1}}$$

$D = 4$, which determines 2PM correction to the deflection angle.

The $2 \to 3$ Amplitude

[Brandhuber et al. '23; Herderschee, Teng, Roiban '23] [Georgoudis, CH, Vazquez-Holm '23]

$$\mathcal{A}^{(5)\mu\nu}(q_1, q_2, k) =$$



$$p_1^\mu = -\bar{m}_1 u_1^\mu + q_1^\mu/2$$
$$p_4^\mu = +\bar{m}_1 u_1^\mu + q_1^\mu/2$$
$$p_2^\mu = -\bar{m}_2 u_2^\mu + q_2^\mu/2$$
$$p_3^\mu = +\bar{m}_2 u_2^\mu + q_2^\mu/2$$
$$q_1 \sim q_2 \sim k \sim \hbar/b$$

- The "average velocities" $u_{1,2}^\mu$ obey

$$u_1^2 = -1 = u_2^2 \qquad \& \qquad u_1 \cdot q_1 = 0 = u_2 \cdot q_2$$

- We consider the physical variables

$$y = -u_1 \cdot u_2 \geq 1\,, \qquad \omega_{1,2} = -u_{1,2} \cdot k \geq 0\,, \qquad q_{1,2}^2\,.$$

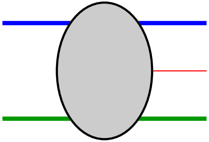
Note that $y = \sigma + \mathcal{O}(\hbar)$ and $\bar{m}_{1,2} = m_{1,2} + \mathcal{O}(\hbar^2)$.

Tree-Level $2 \to 3$ Amplitude

[2 → 3 amplitude: Goldberger, Ridgway '17; Luna, Nicholson, O'Connell, White '17] [Di Vecchia, CH, Russo, Veneziano, '21]

Gauge-invariant amplitude $k_\mu \mathcal{A}_0^{(5)\mu\nu} = 0$,

$$\mathcal{A}_0^{(5)\mu\nu} = \mathcal{B}_0^{(5)\mu\nu} =$$



$$= (8\pi G)^{\frac{3}{2}} \Bigg\{ 4 m_1^2 m_2^2 \left(\sigma^2 - \tfrac{1}{D-2}\right) \Bigg[- \frac{p_1^\mu p_1^\nu (k\cdot q_1)}{(p_1\cdot k)^2 q_2^2} - \frac{p_2^\mu p_2^\nu (k\cdot q_2)}{(p_2\cdot k)^2 q_1^2} + \frac{p_1^{(\mu}(q_1-q_2)^{\nu)}}{2(p_1\cdot k) q_2^2} - \frac{p_2^{(\mu}(q_1-q_2)^{\nu)}}{2(p_2\cdot k) q_1^2}$$

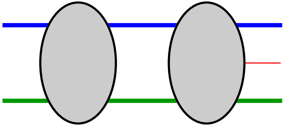
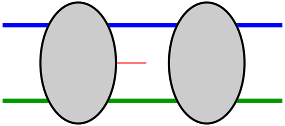
$$+ \frac{(q_1-q_2)^\mu (q_1-q_2)^\nu}{2 q_1^2 q_2^2} \Bigg] + 8 \frac{\left((p_1\cdot k) p_2^\mu - (p_2\cdot k) p_1^\mu\right)\left((p_1\cdot k) p_2^\nu - (p_2\cdot k) p_1^\nu\right)}{q_1^2 q_2^2}$$

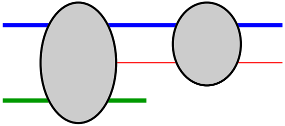
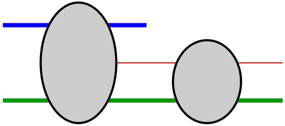
$$+ (2 p_1\cdot p_2) \left(\frac{4 p_1^\mu p_1^\nu \frac{k\cdot p_2}{k\cdot p_1} - 2 p_1^{(\mu} p_2^{\nu)}}{q_2^2} + \frac{4 p_2^\mu p_2^\nu \frac{k\cdot p_1}{k\cdot p_2} - 2 p_1^{(\mu} p_2^{\nu)}}{q_1^2} + \frac{(q_1-q_2)^{(\mu}\left(-2(p_1\cdot k) p_2 + 2(p_2\cdot k) p_1\right)^{\nu)}}{q_1^2 q_2^2} \right) \Bigg\}.$$

One-Loop 2 → 3 Amplitude

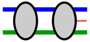
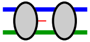
[Brandhuber et al. '23; Herderschee, Roiban, Teng '23] [Georgoudis, C.H., Vazquez-Holm '23]

Inelastic subtraction:

$\mathcal{A}_1^{(5)} = \frac{i}{2}$  $+ \frac{i}{2}$ 

$+ \frac{i}{2}$  $+ \frac{i}{2}$  $+ \mathcal{B}_1^{(5)}$

with $\mathcal{B}_1^{(5)}$ a rational function of ($\sqrt{\ }$ of) y, ω_1, ω_2, q_1^2, q_2^2.

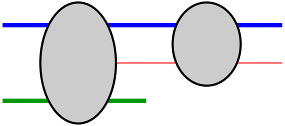
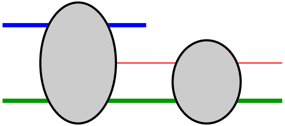
- The subtracted terms are all imaginary and infrared divergent
- $\frac{i}{2}$  $+ \frac{i}{2}$  are also "superclassical" $\mathcal{O}(\frac{1}{\hbar^2})$
- The leftover $\mathcal{B}_1^{(5)}$ is real, finite and classical $\mathcal{O}(\frac{1}{\hbar})$ [and tr. weight 2].

Inelastic Exponentiation at One Loop

[Brandhuber et al. '23; Herderschee, Roiban, Teng '23] [Georgoudis, C.H., Vazquez-Holm '23]

Inelastic subtraction in b-space:

$$\tilde{\mathcal{A}}_1^{(5)} = 2i\delta_0\, \tilde{\mathcal{B}}_0^{(5)}$$

$+ \frac{i}{2}$ FT  $+ \frac{i}{2}$ FT  $+ \tilde{\mathcal{B}}_1^{(5)}$

- This matches the exponential (here $\int_k \simeq \int \frac{d^Dk}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)$)

$$\boxed{i\tilde{\mathcal{A}}_1^{(5)} = \langle \text{in}|\, a(k)\, e^{i(2\delta_0+2\delta_1)} e^{i\int_k[\tilde{\mathcal{W}}(k)a^\dagger(k)+\tilde{\mathcal{W}}^*(k)a(k)]}\, |\text{in}\rangle}$$

with $\mathcal{B}_0^{(5)} = \mathcal{W}_0$.

Inelastic Final State

[Di Vecchia, C.H., Russo, Veneziano 2210.12118]

[cf. Kosower, Maybee, O'Connell '18; Damgaard, Planté, Vanhove '21; Cristofoli et al. '21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{i\,\mathrm{Re}\,2\delta(b)}\,e^{i\int_k\left[\tilde{\mathcal{W}}(k)a^\dagger(k)+\tilde{\mathcal{W}}^*(k)a(k)\right]}\,.$$

- Final state (again, schematically):

$$\boxed{|\mathrm{out}\rangle = e^{2i\hat{\delta}(b_1,b_2)}|\mathrm{in}\rangle}$$

- Unitarity:

$$\langle\mathrm{out}|\mathrm{out}\rangle = \langle\mathrm{in}|\mathrm{in}\rangle = 1$$

- Consistency with the elastic exponentiation: by the BCH formula,

$$\langle\mathrm{in}|\mathrm{out}\rangle = e^{i\,\mathrm{Re}\,2\delta(b)}\,e^{-\,\mathrm{Im}\,2\delta(b)} = e^{2i\delta(b)}$$

because, by unitarity, $\mathrm{Im}\,2\delta_2 = \frac{1}{2}\int_k \tilde{\mathcal{A}}_0^{(5)}\tilde{\mathcal{A}}_0^{(5)*}$.

Outline

Warm-Up: Radiated Energy-Momentum

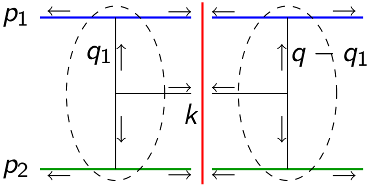
[Kosower, Maybee, O'Connell '18; Herrmann, Parra-Martinez, Ruf, Zeng '21] [Di Vecchia, C.H., Russo, Veneziano 2210.12118]

- $\langle \text{out} | \hat{P}^\alpha | \text{out} \rangle = \boldsymbol{P}^\alpha$
- In terms of the waveform

$$\boldsymbol{P}^\alpha = \int_k k^\alpha \tilde{A}^{\mu\nu}(k) \left(\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} \right) \tilde{A}^{*\rho\sigma}(k) \equiv \int_k k^\alpha \tilde{A}\tilde{A}^* \,.$$

- Recast as the FT of a cut in momentum-space (reverse unitarity)

$$\boldsymbol{P}^\alpha = \text{FT} \int d(\text{LIPS}) k^\alpha$$



Same integrals appearing in the $2 \to 2$ amplitude!

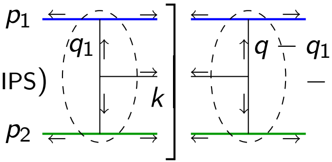
Radiated Angular Momentum

- $\langle \text{out}|\hat{J}^{\alpha\beta}|\text{out}\rangle = \boldsymbol{J}^{\alpha\beta} + \mathcal{J}^{\alpha\beta}$ where [Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo '22]

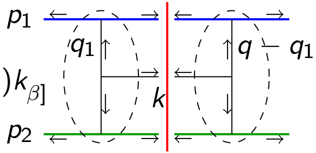
$$\boldsymbol{J}_{\alpha\beta} = \boldsymbol{J}^{(o)}_{\alpha\beta} + \boldsymbol{J}^{(s)}_{\alpha\beta}\,, \qquad i\boldsymbol{J}^{(o)}_{\alpha\beta} = \int_k k_{[\alpha} \frac{\partial \tilde{\mathcal{A}}^{(5)}}{\partial k^{\beta]}} \tilde{\mathcal{A}}^{(5)*}\,, \qquad \boldsymbol{J}^{(s)}_{\alpha\beta} = 2i \int_k \tilde{\mathcal{A}}^{(5)\mu}_{[\alpha} \tilde{\mathcal{A}}^{(5)*}_{\beta]\mu}\;.$$

- Reverse unitarity: $q_{\parallel 2} = -u_2 \cdot q$ [Di Vecchia, CH, Russo, Veneziano '22]

$$i\boldsymbol{J}^{(o)}_{\alpha\beta} = \text{FT}\int k_{[\alpha} \frac{\partial}{\partial k^{\beta]}} \Big[d(\text{LIPS}) \Big]$$



$$- u_{2[\alpha}\, \text{FT} \frac{\partial}{\partial q_{\parallel 2}} \int d(\text{LIPS})\, k_{\beta]}$$



- Inclusion of **tidal** effects [CH '22] by using the $t^{\mu\nu}_{E,B}$ given in [Mougiakakos, Riva, Vernizzi '22].
- Inclusion of **spin** effects [CH '23] by using the $t^{\mu\nu}_{S_1,S_2}$ given e.g. in [Riva, Vernizzi, Wong '22].

Angular Momentum of the Static Gravitational Field $\mathcal{J}_{\alpha\beta}$

[Di Vecchia, CH, Russo '22] [see also: Veneziano, Vilkovisky '22; Javadinezhad, Porrati '22; Riva, Vernizzi, Whong '23].

Angular momentum/mass dipole loss due to static modes

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2}\sum_{n,m}\left[\left(\sigma_{nm}^2 - \frac{1}{2}\right)\frac{\frac{\sigma_{nm}\,\text{arccosh}\,\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}} - 1}{\sigma_{nm}^2 - 1} - \frac{2\sigma_{nm}\,\text{arccosh}\,\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}}\right](\eta_n - \eta_m)p_n^{[\alpha}p_m^{\beta]}\,.$$

- Here $-\eta_n\eta_m p_n\cdot p_m = m_n m_m \sigma_{nm}$ with $\eta_n = +1$ ($\eta_n = -1$) if n is outgoing (incoming).
- Matches [Damour '20; Manohar, Ridgway, Shen '22; Bini, Damour '22] up to $\mathcal{O}(G^3)$ upon expanding

$$\mathcal{J}^{\alpha\beta} = -\frac{G}{2}(p_1 - p_2)^{[\alpha}Q^{\beta]}\mathcal{I}(\sigma) + \mathcal{O}(G^4)\,, \qquad Q^\mu = Q^\mu_{1\text{PM}} + Q^\mu_{2\text{PM}} + \mathcal{O}(G^3)$$

- Valid also in the case of tidally deformable [CH '22] or spinning objects [Alessio, Di Vecchia '22] [CH '23].

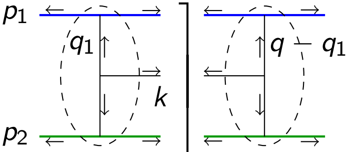
Mechanical Angular Momentum [Di Vecchia, CH, Russo, Veneziano '22]

- $\langle \text{out}|\hat{L}_2^{\alpha\beta}|\text{out}\rangle - \langle \text{in}|\hat{L}_2^{\alpha\beta}|\text{in}\rangle = \Delta \boldsymbol{L}_2^{\alpha\beta} + \Delta L_{2\text{cons}}^{\alpha\beta} + \Delta \mathcal{L}_2^{\alpha\beta}$ where

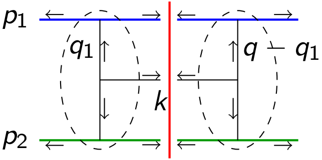
$$\Delta \boldsymbol{L}_2^{\alpha\beta} = \text{Im}\, \boldsymbol{J}_2^{\alpha\beta} + b_2^{[\alpha} \Delta \boldsymbol{P}_2^{\beta]} , \qquad \boldsymbol{J}_{2\alpha\beta} = \int_k p_{2[\alpha} \frac{\partial \tilde{\mathcal{A}}^{(5)}}{\partial p_2^{\beta]}} \tilde{\mathcal{A}}^{(5)*} .$$

- Reverse unitarity:

$$\boldsymbol{J}_{2\alpha\beta} = \text{FT} \int u_{2[\alpha} \frac{\partial}{\partial u_2^{\beta]}} \Big[d(\text{LIPS}) \quad \Big]$$



$$+ u_{2[\alpha} \text{FT} \frac{\partial}{\partial q_{\|2}} \int d(\text{LIPS}) (q_1 + k)_{\beta]}$$



Angular Momentum Balance

- Convenient functions: $\mathcal{C}\sqrt{\sigma^2-1} = -\mathcal{E}_+ + \sigma\mathcal{E}_-$, $\mathcal{F} = \pm\mathcal{E}_\pm \mp \frac{1}{2}\,\mathcal{E}$.
- Radiated angular momentum [Manohar, Ridgway, Shen '21]

$$\boldsymbol{J}^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\,\mathcal{F}\left(b^{[\alpha}\check{u}_1^{\beta]} - b^{[\alpha}\check{u}_2^{\beta]}\right).$$

- Radiative changes of mechanical angular momentum [Di Vecchia, CH, Russo, Veneziano '22]

$$\Delta\boldsymbol{L}_1^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\left[+\frac{\mathcal{E}_+ b^{[\alpha}u_1^{\beta]}}{\sigma-1} - \frac{1}{2}\,\mathcal{E}\,b^{[\alpha}\check{u}_2^{\beta]}\right],$$

$$\Delta\boldsymbol{L}_2^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\left[-\frac{\mathcal{E}_+ b^{[\alpha}u_2^{\beta]}}{\sigma-1} + \frac{1}{2}\,\mathcal{E}\,b^{[\alpha}\check{u}_1^{\beta]}\right].$$

$$\boxed{\boldsymbol{J}^{\alpha\beta} + \Delta\boldsymbol{L}_1^{\alpha\beta} + \Delta\boldsymbol{L}_2^{\alpha\beta} = 0}$$

Summary and Outlook

- An operator version of the eikonal exponentiation emerges from $2 \to 2$ and $2 \to 3$ amplitude up to one loop
- It can be used to calculate radiative observables, like the angular momentum loss(es)
- It provides a flexible framework to also include tidal and spin effects.

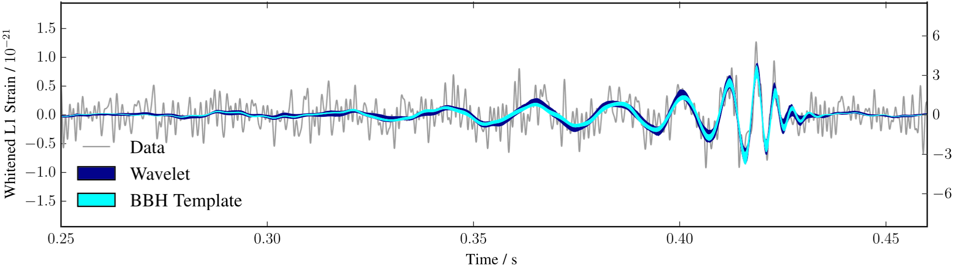
For the future:

- Analytic results for the subleading waveform? [cf. Caron-Huot, Giroux, Hannesdottir, Mizera '23]
- Beyond coherent-state approximation? E.g. $a^\dagger a$
- Differential spectrum? Integrated $P_{\rm rad}$ and $J_{\rm rad}$ at $\mathcal{O}(G^4)$?
- Subleading $\log \omega$ soft theorems? [Sen et al.'19]
- Comparison with the PN literature

ADDITIONAL MATERIAL

Waveform Templates

[LIGO Scientific Collaboration '16]



Inspiral
Weak gravity

Merger
Strong gravity

Ringdown
Small oscillations

Analytical Approximation Methods

- Post-Newtonian (PN): expansion "for small G and small v"

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,.$$

- Post-Minkowskian (PM): expansion "for small G"

$$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}\,.$$

- Self-Force: expansion in the near-probe limit $m_2 \ll m_1$ or

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2} \ll 1\,.$$

Idea

Extract the PM gravitational dynamics from scattering amplitudes.

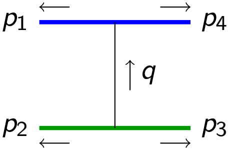
- Weak-coupling expansion $\leftrightarrow$ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM	4PM State of the art $\dots$

- Lorentz invariance $\leftrightarrow$ generic velocities
- Study scattering events, then export to bound trajectories (V_{eff}, analytic continuation...)

Example: the 1PM Eikonal

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



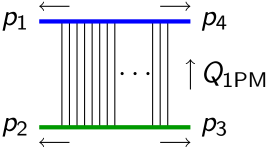
$$\mathcal{A}_0(s,q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0(s,b) = \frac{4 G m_1 m_2 (\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2-1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} \,.$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al.'18]

$$e^{2i\delta_0} \underset{\text{“small } G\text{”}}{\to} 1 + i\tilde{\mathcal{A}}_0 \implies 2\delta_0 = \tilde{\mathcal{A}}_0 \,.$$

- From $Q = \partial_b 2\delta$, we obtain the leading-order deflection



$$Q_{1PM} = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2-1}}$$

$$\Theta_{1PM} = \frac{4 G E \left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2-1)} \,.$$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy

[Di Vecchia, C.H., Russo, Veneziano 2008.12743, 2101.05772, 2104.03256]

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit:

- the *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$,
- its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

$$\mathrm{Re}\, 2\delta_2 \sim Gs \frac{\Theta_s^2}{4}\,, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

The Initial State

[Kosower, Maybee, O'Connell '18]

- We model the initial state by $|\text{in}\rangle = |1\rangle \otimes |2\rangle$, with

$$|1\rangle = \int_{-p_1} \varphi_1(-p_1)\, e^{ib_1\cdot p_1} | - p_1\rangle$$
$$|2\rangle = \int_{-p_2} \varphi_2(-p_2)\, e^{ib_2\cdot p_2} | - p_2\rangle$$

and $\int_{-p_i} = \int 2\pi\delta(p_i^2 + m_i^2)\theta(-p_i^0)\frac{d^D p_i}{(2\pi)^D}$ the LIPS measure.

- Wavepackets $\varphi_i(-p_i)$ peaked around the classical incoming momenta.
- Impact parameter $b^\mu = b_1^\mu - b_2^\mu$ lies in the transverse plane $b \cdot p_1 = 0 = p_2 \cdot b$.

Elastic and Inelastic Fourier Transforms

- Elastic Fourier transform:

$$\text{FT}\,\mathcal{A}^{(4)} = \int \frac{d^D q}{(2\pi)^D}\, 2\pi\delta(2\bar{m}_1 u_1\cdot q)\, 2\pi\delta(2\bar{m}_2 u_2\cdot q) e^{ib\cdot q}\mathcal{A}^{(4)}(q)$$

$$\simeq \frac{1}{4Ep}\int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\mathcal{A}(s,q) = \tilde{\mathcal{A}}^{(4)}\,.$$

- Inelastic Fourier transform:

$$\text{FT}\,\mathcal{A}^{(5)} = \int \frac{d^D q_1}{(2\pi)^D}\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k)$$

$$\times\, 2\pi\delta(2\bar{m}_1 u_1\cdot q_1) 2\pi\delta(2\bar{m}_2 u_2\cdot q_2) e^{ib_1\cdot q_1 + ib_2\cdot q_2}\mathcal{A}^{(5)}(q_1,q_2,k)$$

$$\simeq \tilde{\mathcal{A}}^{(5)}(k)\,.$$

Eikonal Final State

[Di Vecchia, C.H., Russo, Veneziano 2210.12118] [cf. Cristofoli et al.'21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation

$$|\text{out}\rangle \simeq \int_{p_3}\int_{p_4} e^{-ib_1\cdot p_4 - ib_2\cdot p_3} \int \frac{d^D Q_1}{(2\pi)^D}\int \frac{d^D Q_2}{(2\pi)^D}\,\Phi_1(p_4-Q_1)\,\Phi_2(p_3-Q_2)$$
$$\times \int d^D x_1 \int d^D x_2\, e^{i(b_1-x_1)\cdot Q_1 + i(b_2-x_2)\cdot Q_2}\, e^{2i\hat{\delta}(x_1,x_2)}|p_4, p_3, 0\rangle$$

with

$$e^{2i\hat{\delta}(x_1,x_2)} = e^{i\,\mathrm{Re}\,2\delta(b)}\,e^{i\int_k \left[\tilde{\mathcal{A}}_j(x_1,x_2,k)a_j^\dagger(k)+\tilde{\mathcal{A}}_j^*(x_1,x_2,k)a_j(k)\right]}\,.$$

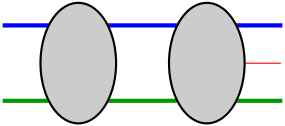
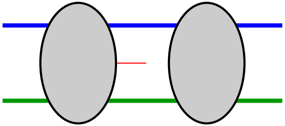
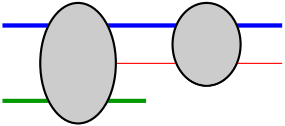
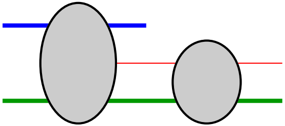
Here

- b is the projection of $x_1 - x_2$ orthogonal to $p_4 - Q_1/2$, $p_3 - Q_2/2$,
- $\tilde{\mathcal{A}}_j = \varepsilon_{j\mu\nu}\,\tilde{\mathcal{A}}^{\mu\nu}$ is the impact-parameter space $2\to 3$ amplitude.

N-Operator, T-Operator and Unitarity

- N-operator: $S = e^{iN}$
$$N = -i\log(1+iT) = T - \frac{i}{2}T^2 + \cdots$$
up to one loop.
- Unitarity: $S^\dagger S = 1$,
$$\frac{1}{2}(T - T^\dagger) = +\frac{i}{2}T^\dagger T$$
- We shall denote by $\mathcal{B}$ the N-matrix elements, just like $\mathcal{A}$ denotes the the usual amplitudes (T-matrix elements). Then, by unitarity,
$$\mathcal{B}_0 = \mathcal{A}_0\,, \qquad \mathcal{B}_1 = \mathrm{Re}\,\mathcal{A}_1\,.$$

[Caron-Huot, Giroux, Hannesdottir, Mizera '23]

$$\mathcal{W}_1^{\text{KMOC}(5)} = \frac{i}{2}$$

$$- \frac{i}{2}$$

$$+ \frac{i}{2}$$

$$+ \frac{i}{2}$$

$$+ \mathcal{B}_1^{(5)}$$

with $\mathcal{B}_1^{(5)}$ the (real) N-matrix element.