Strong-coupling results in $\mathcal{N} = 2$ superconformal gauge theories

Paolo Vallarino

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This talk is mainly based on

- M. Billò, M. Frau, A. Lerda, A. Pini, P.V. "Localization vs holography in 4d $\mathcal{N}=2$ quiver theories" (2022) arXiv: 2207.08846, JHEP 10 (2022) 020

- M. Billò, M. Frau, A. Lerda, A. Pini, P.V. "Strong coupling expansions in $\mathcal{N}=2$ quiver gauge theories" (2022) arXiv: 2211.11795, JHEP 01 (2023) 119

Purpose

- The analysis of the strong-coupling regime in an interacting gauge theory is a very difficult problem but, when there is a high amount of symmetry, remarkable progress can be made.

- In particular this happens for $\mathcal{N} = 4$ SYM theory, where many exact results have been found over the years, especially in the planar limit

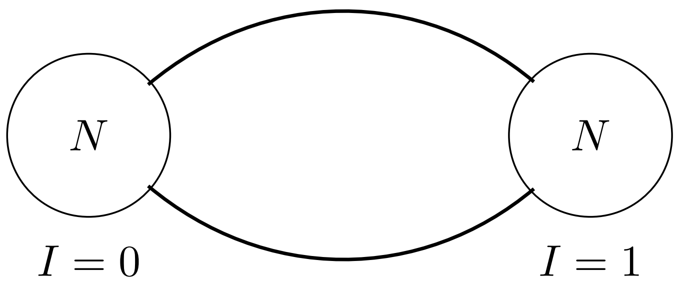
$$N \to \infty \quad \text{and} \quad \lambda \equiv g_{YM}^2 N \quad \text{is fixed}$$

  However we will see cases where exact results are found in $\mathcal{N} = 2$ quiver gauge theories.

- Main tools: holography, supersymmetric localization and integrability.

$\mathcal{N} = 2$ quiver gauge theory

We consider the case with 2 nodes



- In each of the two nodes there is a SU(N) gauge group with its vector multiplet while the links represent hypermultiplets in the bifundamental representation.
- It arises as a $\mathbb{Z}_2$ orbifold projection from $\mathcal{N} = 4$ SYM.
- Given the matter content, this is a conformal gauge theory.

$\mathcal{N} = 2$ quiver gauge theory

An important class of operators consists of **chiral operators**

$$\mathcal{O}_{\mathbf{n},I}(x) = \mathrm{tr}\phi_I(x)^{n_1} \dots \mathrm{tr}\phi_I(x)^{n_k}$$

These are local, gauge invariant, chiral primary operators, i.e.

$$\left[\bar{Q}^{A\,\dot{\alpha}}, \mathcal{O}_{\mathbf{n},I}\right] = 0 \quad \Rightarrow \quad \text{conformal dimension } \Delta_{\mathbf{n}} = \sum_{i=1}^{k} n_i$$

We focus on single-trace operators $\mathcal{O}_{n,I}(x)$. In this quiver theory we can define 2 different combinations of chiral operators

$$\boxed{U_k(x) = \frac{1}{\sqrt{2}}\left[\mathrm{tr}\phi_0(x)^k + \mathrm{tr}\phi_1(x)^k\right]} \qquad \textbf{untwisted}$$

$$\boxed{T_k(x) = \frac{1}{\sqrt{2}}\left[\mathrm{tr}\phi_0(x)^k - \mathrm{tr}\phi_1(x)^k\right]} \qquad \textbf{twisted}$$

$\mathcal{N} = 2$ quiver gauge theory

Our aim is to study 2- and 3-point functions of these chiral primary operators in the 't Hooft limit.
These correlators are constrained by conformal invariance, conservation of the $U(1)$ R-charge and $\mathbb{Z}_2$ symmetry

$$\big\langle U_k(x)\,\overline{U}_k(y)\big\rangle = \frac{G_{U_k}}{|x-y|^{2k}} \qquad \big\langle T_k(x)\,\overline{T}_k(y)\big\rangle = \frac{G_{T_k}}{|x-y|^{2k}}$$

$$\big\langle U_k(x)\,U_\ell(y)\,\overline{U}_p(z)\big\rangle = \frac{G_{U_k,U_\ell,\overline{U}_p}}{|x-z|^{2k}\,|y-z|^{2\ell}}$$

$$\big\langle U_k(x)\,T_\ell(y)\,\overline{T}_p(z)\big\rangle = \frac{G_{U_k,T_\ell,\overline{T}_p}}{|x-z|^{2k}\,|y-z|^{2\ell}} \qquad \text{with} \quad p = k+\ell$$

$\mathcal{N} = 2$ quiver gauge theory

Then the normalized 3-point functions

$$C_{U_k,U_\ell,\overline{U}_p} = \frac{G_{U_k,U_\ell,\overline{U}_p}}{\sqrt{G_{U_k}G_{U_\ell}G_{U_p}}} \qquad C_{U_k,T_\ell,\overline{T}_p} = \frac{G_{U_k,T_\ell,\overline{T}_p}}{\sqrt{G_{U_k}G_{T_\ell}G_{T_p}}}$$

These correlation functions can be computed in perturbation theory using Feynman diagrams, but only few terms can be found in an explicit form due to the difficulty of the evaluation of the loop integrals. A much more efficient way to compute them is by using **localization**.

Localization

- Supersymmetric localization maps the computation of these correlators in the gauge theory to a **matrix model** on $\mathbb{S}^4$ [Pestun, 2007]

$$\text{Path integrals} \quad \Longrightarrow \quad \text{Finite dimensional integrals}$$

- Matrix model can be used to evaluate different kinds of observables in $\mathcal{N} = 2$ conformal gauge theories, for instance

$$\left\langle O_k(0)\,\overline{O}_k(\infty)\right\rangle_{\mathbb{R}^4} \xrightarrow[\textit{map}]{\text{conformal}} \left\langle O_k(N)\,\overline{O}_k(S)\right\rangle_{\mathbb{S}^4} \xrightarrow{\text{localization}} \textbf{matrix model}$$

[Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu, 2016]

- There exists a precise correspondence between the operators defined in the QFT and the operators defined in the matrix model.

2-point functions

- In the matrix model one can easily prove that 2- and 3- point functions of untwisted operators are planar equivalent to their $\mathcal{N} = 4$ counterpart.
- Correlators with twisted operators require much more efforts. The final result for 2-point functions reads

$$\boxed{\frac{G_{T_{2n+1}}}{\mathcal{G}_{T_{2n+1}}} = \frac{\det\left(\mathbb{1} - \mathsf{X}^{odd}_{[n+1]}\right)}{\det\left(\mathbb{1} - \mathsf{X}^{odd}_{[n]}\right)}}$$

$$\boxed{\frac{G_{T_{2n}}}{\mathcal{G}_{T_{2n}}} = \frac{\det\left(\mathbb{1} - \mathsf{X}^{even}_{[n+1]}\right)}{\det\left(\mathbb{1} - \mathsf{X}^{even}_{[n]}\right)}}$$

[Billò, Frau, Galvagno, Lerda, Pini, 2021]

2-point functions

where

$$\mathsf{X}_{n,m} = -8(-1)^{\frac{n+m+2nm}{2}} \sqrt{n\,m} \int_0^\infty \frac{dt}{t} \frac{e^t}{(e^t-1)^2}\, J_n\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big) J_m\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)$$

and

$$\mathsf{X}_{2n,2m+1} = 0$$

so that it is convenient to use the notation

$$(\mathsf{X}^{\text{even}})_{n,m} = \mathsf{X}_{2n,2m} \qquad \text{and} \qquad (\mathsf{X}^{\text{odd}})_{n,m} = \mathsf{X}_{2n+1,2m+1}$$

The perturbative expansion is entirely resummed and we have the exact dependence on the coupling λ for the correlator through the X matrix!

Strong-coupling regime

- From the asymptotic expansion of the Bessel functions, one can derive the behavior of the X matrix and then of the 2-point functions, when the 't Hooft coupling becomes large.
- At the leading term at strong coupling the 2-point functions read

$$\boxed{\frac{G_{T_n}}{\mathcal{G}_{T_n}} \underset{\lambda\to\infty}{\sim} \frac{4\pi^2}{\lambda}\, n\,(n-1) + O\left(\frac{1}{\lambda^{3/2}}\right)}$$

[Billò, Frau, Galvagno, Lerda, Pini, 2021]

Strong-coupling regime

However, even the full **strong-coupling expansion** of the 2-point functions has been derived

$$\frac{G_{T_n}}{\mathcal{G}_{T_n}} \underset{\lambda\to\infty}{\sim} \frac{4\pi^2}{\lambda}\, k(k-1)\left(\frac{\lambda'}{\lambda}\right)^{k-1}\Bigg[1+(k-1)(2k-1)(2k-3)\,\frac{\zeta_3}{\lambda'^{\,3/2}}$$
$$-(k-1)(2k-3)(2k-5)(4k^2-1)\,\frac{9\,\zeta_5}{16\lambda'^{\,5/2}}$$
$$+(k-1)(2k-1)(2k-3)(2k-5)(4k^2-20k-3)\,\frac{\zeta_3^2}{4\lambda'^{\,3}}+O\left(\frac{1}{\lambda'^{\,7/2}}\right)\Bigg]$$
$$+\ldots$$

with

$$\sqrt{\lambda'} \equiv \sqrt{\lambda} - 4\log 2$$

[Beccaria, Korchemsky, Tseytlin, 2022]

3-point functions

- Let's now consider the 3-point functions.
- **Crucial observation**: the 3-point functions are related to the 2-point functions through exact identities that are valid for all values of the coupling constant

$$\boxed{G_{U_k,T_\ell,\overline{T}_p} = \frac{\mathcal{G}_{k,\ell,p}}{\sqrt{\ell\,\mathcal{G}_\ell\,p\,\mathcal{G}_p}}\,\sqrt{\big(\ell+\lambda\partial_\lambda\big)\,G_{T_\ell}}\,\sqrt{\big(p+\lambda\partial_\lambda\big)\,G_{T_p}}}$$

$$\boxed{G_{T_k,T_\ell,\overline{U}_p} = \frac{\mathcal{G}_{k,\ell,p}}{\sqrt{k\,\mathcal{G}_k\,\ell\,\mathcal{G}_\ell}}\,\sqrt{\big(k+\lambda\partial_\lambda\big)\,G_{T_k}}\,\sqrt{\big(\ell+\lambda\partial_\lambda\big)\,G_{T_\ell}}}$$

[Billò, Frau, Lerda, Pini, PV, 2022]

3-point functions

- Finally, from these results it is straightforward to obtain also the large-λ expansions of the normalized 3-point functions

$$\boxed{C_{U_k,T_\ell,\overline{T}_p} = \frac{\sqrt{k}}{\sqrt{2}\,N}\sqrt{\ell + \lambda\partial_\lambda\big(\log G_{T_\ell}\big)}\,\sqrt{p + \lambda\partial_\lambda\big(\log G_{T_p}\big)}}$$

- At $\lambda = 0$ they read $C_{U_k,T_\ell,\overline{T}_p} = \frac{\sqrt{k\,\ell\,p}}{N\sqrt{2}}$

[Lee, Minwalla, Rangamani, Seiberg, 1998]

- At the leading term at strong coupling $C_{U_k,T_\ell,\overline{T}_p} = \frac{\sqrt{k\,(\ell-1)\,(p-1)}}{N\sqrt{2}}$

  Result also checked using AdS/CFT correspondence!

[Billò, Frau, Lerda, Pini, PV, 2022]

Summary and outlook

- We summed up the perturbative expansion for 2- and 3- point functions of CPOs in $\mathcal{N} = 2$ quiver theories, finding exact expressions in the planar limit and we derived their strong-coupling expansions.

- We extended these results to the most general quiver with M nodes and also to another $\mathcal{N} = 2$ SCFT obtained through orientifold projection from the $\mathbb{Z}_2$ quiver.

- Recently also other kinds of correlation functions in these theories have been considered at strong coupling, i.e. $\langle W\, O_k \rangle$, $\langle W_{\alpha_1} \dots W_{\alpha_k} \rangle$.
  [Pini, PV, 2023]

- It would be nice to derive subleading corrections at strong coupling from AdS/CFT correspondence (string corrections, beyond supergravity limit).

Thanks for your attention!

Localization and matrix model

Matrix model

In the large-N limit, the instanton contributions are exponentially suppressed. Then in the quiver gauge theory we have

$$\mathcal{Z} = \int da_0\, da_1\; \mathrm{e}^{-\mathrm{tr}\, a_0^2 - \mathrm{tr}\, a_1^2 - S_{\mathrm{int}}}$$

where a_0 and a_1 are $N \times N$ traceless Hermitean matrices and

$$S_{int} = 2\sum_{m=2}^{\infty}\sum_{k=2}^{2m}(-1)^k \left(\frac{-\lambda}{8\pi^2 N}\right)^m \binom{2m}{k} \frac{\zeta_{2m-1}}{2m} (\mathrm{tr}\, a_0^{2m-k} - \mathrm{tr}\, a_1^{2m-k})(\mathrm{tr}\, a_0^k - \mathrm{tr}\, a_1^k)$$

Hence for a generic function $f(a)$

$$\langle f(a)\rangle = \frac{\int da_0\, da_1\; \mathrm{e}^{-\mathrm{tr}\, a_0^2 - \mathrm{tr}\, a_1^2 - S_{int}}\, f(a)}{\int da_0\, da_1\; \mathrm{e}^{-\mathrm{tr}\, a_0^2 - \mathrm{tr}\, a_1^2 - S_{int}}} = \frac{\langle e^{-S_{int}}\, f(a)\rangle_0}{\langle e^{-S_{int}}\rangle_0}\,,$$

where $\langle\,\rangle_0$ stands for the expectation value in the free matrix model.

Matrix model

Normal-ordering!

The operators that we naturally introduce in the matrix model

$$A_k^{\pm} = \frac{1}{\sqrt{2}}\left(\mathrm{tr}\, a_0^k \pm \mathrm{tr}\, a_1^k\right)$$

do not correctly correspond to the gauge theory operators $U_k(x)$ and $T_k(x)$, since they are normal-ordered, i.e.

$$\contraction{}{\phi}{_I\,}{\phi}\phi_I\,\phi_I = 0$$

whereas

$$\contraction{}{a}{_I\,}{a}a_I\,a_I \neq 0$$

<u>Solution</u>: define normal-ordered operators also in the matrix model

$$O_k^{\pm} \equiv :A_k^{\pm}: = \sum_{\ell \leq k} \mathrm{M}_{k,\ell}^{\pm} A_\ell^{\pm}$$

Free matrix model

Consider the free theory: $S_{int} = 0$.

Solving the Gaussian integrals one finds

$$\boxed{\langle O_n^{\pm} O_m^{\pm} \rangle_0 = n \left(\frac{N}{2}\right)^n \delta_{n,m} \equiv \mathcal{G}_n \, \delta_{n,m}}$$

$$\boxed{\langle O_n^{\pm} O_m^{\pm} O_p^{\pm} \rangle_0 = \frac{n\,m\,p}{2\sqrt{2}} \left(\frac{N}{2}\right)^{\frac{n+m+p}{2}-1} \delta_{n+m,p} \equiv \mathcal{G}_{n,m,p} \, \delta_{n+m,p}}$$

Free matrix model

We can conveniently define

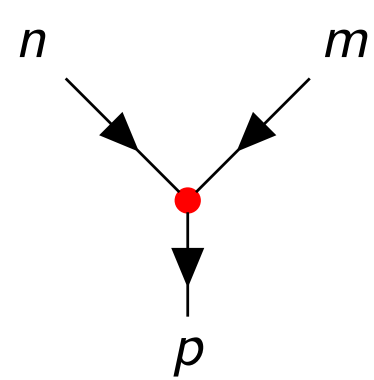
$$P_n^{\pm} = \frac{1}{\sqrt{\mathcal{G}_n}} O_n^{\pm}\Big|_{\lambda=0}$$

so that 2- and 3- point functions become

$$\langle P_n^{\pm} P_m^{\pm} \rangle_0 = \delta_{n,m} \equiv \; n \longrightarrow m$$

and

$$\langle P_n^{\pm} P_m^{\pm} P_p^{+} \rangle_0 = \frac{\sqrt{n\,m\,p}}{N\sqrt{2}} \delta_{n+m,p} \equiv$$



Interacting matrix model

- We have to treat separately **untwisted** and **twisted** operators.
- Untwisted operators are simple: they behave exactly as the operators in the free matrix model, since S_{int} just depends on **twisted** operators.
- Indeed we get

$$\left\langle P_n^+ P_m^+ \right\rangle = \frac{\left\langle P_n^+ P_m^+ \, e^{-S_{int}} \right\rangle_0}{\left\langle e^{-S_{int}} \right\rangle_0} \underset{N\to\infty}{\sim} \frac{\left\langle P_n^+ P_m^+ \right\rangle_0 \cancel{\left\langle e^{-S_{int}} \right\rangle_0}}{\cancel{\left\langle e^{-S_{int}} \right\rangle_0}}$$

$$\left\langle P_n^+ P_m^+ P_p^+ \right\rangle = \frac{\left\langle P_n^+ P_m^+ P_p^+ \, e^{-S_{int}} \right\rangle_0}{\left\langle e^{-S_{int}} \right\rangle_0} \underset{N\to\infty}{\sim} \frac{\left\langle P_n^+ P_m^+ P_p^+ \right\rangle_0 \cancel{\left\langle e^{-S_{int}} \right\rangle_0}}{\cancel{\left\langle e^{-S_{int}} \right\rangle_0}}$$

Interacting matrix model

Twisted operators $\implies$ we have to take into account S_{int}.

Rewriting S_{int} in terms of the normalized operators P_n^- one can realize that

$$\boxed{S_{int} = -\frac{1}{2}\sum_{n,m} P_n^- \mathsf{X}_{n,m} P_m^-}$$

[Beccaria, Billò, Galvagno, Hasan, Lerda, 2020]

where

$$\mathsf{X}_{n,m} = -8(-1)^{\frac{n+m+2nm}{2}}\sqrt{n\,m}\int_0^\infty \frac{dt}{t}\frac{e^t}{(e^t-1)^2}\,J_n\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)J_m\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)$$

and

$$\mathsf{X}_{2n,2m+1} = 0$$

so that it is convenient to use the notation

$$(\mathsf{X}^{\text{even}})_{n,m} = \mathsf{X}_{2n,2m} \qquad \text{and} \qquad (\mathsf{X}^{\text{odd}})_{n,m} = \mathsf{X}_{2n+1,2m+1}$$

Interacting matrix model

- Thus S_{int} in this form has all the dependence on λ inside the matrix X through the Bessel functions, differently from the original expression, where it was given through a weak-coupling expansion.
- Hence the perturbative expansion in S_{int} is resummed and we have the exact dependence on λ through the matrix X.
- Now we are able to compute 2- and 3- point functions of twisted operators in the interacting matrix model at any value of the coupling constant.

Interacting matrix model

Let's focus on the 2-point functions. We have

$$
\begin{aligned}
\left\langle P_n^- P_m^- \right\rangle &= \frac{\left\langle P_n^- P_m^- \, e^{-S_{int}} \right\rangle_0}{\left\langle e^{-S_{int}} \right\rangle_0} \\
&= \left\langle P_n^- P_m^- \right\rangle_0 + \left(\left\langle P_n^- P_m^- \right\rangle_0 \left\langle S_{int} \right\rangle_0 - \left\langle P_n^- P_m^- \, S_{int} \right\rangle_0 \right) + \dots \\
&\underset{N\to\infty}{\sim} \delta_{n,m} + \frac{1}{2} \sum_{k,\ell} \left\langle P_n^- P_m^- \left(P_k^- \mathrm{X}_{k,\ell} P_\ell^- \right) \right\rangle_0 \Big|_{conn} + \dots \\
&= \delta_{n,m} + \mathrm{X}_{n,m} + \mathrm{X}^2_{n,m} + \dots = \left(\frac{1}{1-\mathrm{X}} \right)_{n,m} \equiv \mathrm{D}_{n,m}
\end{aligned}
$$

$$
\Longrightarrow \qquad \boxed{\left\langle P_n^- P_m^- \right\rangle = \mathrm{D}_{n,m}}
$$

Strong-coupling regime

- From the asymptotic expansion of the Bessel functions, one can derive the behavior of the X matrix and then of the 2-point functions, when the 't Hooft coupling becomes large.

- One can prove that at strong coupling X is a three-diagonal infinite matrix

$$\mathsf{X}^{odd}_{n,m} \underset{\lambda\to\infty}{\sim} -\frac{\lambda}{8\pi^2}(-1)^{n+m}\sqrt{\frac{2m+1}{2n+1}}\left(\frac{\delta_{n-1,m}}{n(2n-1)}+\frac{\delta_{n,m}}{n(n+1)}+\frac{\delta_{n+1,m}}{(n+1)(2n+3)}\right)$$

[Beccaria, Dunne, Tseytlin, 2021]
[Beccaria, Billò, Frau, Lerda, Pini, 2021]

Strong-coupling regime

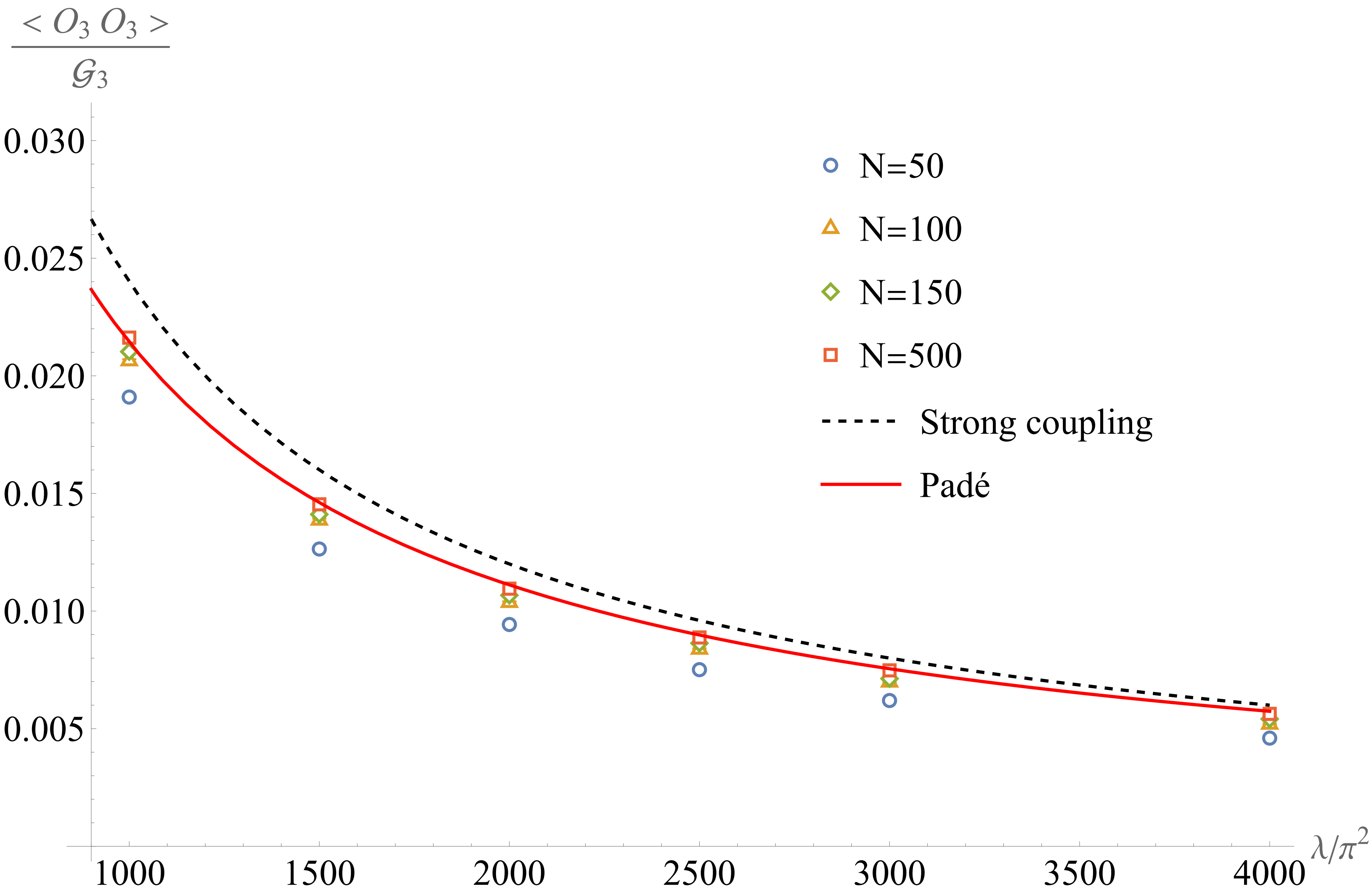
- From this result, it is possible to get the leading term for $\lambda \to \infty$ of the 2-point functions

$$\boxed{\left\langle O_n^- O_n^- \right\rangle \underset{\lambda\to\infty}{\sim} \mathcal{G}_n \frac{4\pi^2}{\lambda} n\,(n-1) + O\left(\frac{1}{\lambda^{3/2}}\right)}$$

- Exploiting the small-λ expansion of the Bessel functions it is possible to efficiently generate very long perturbative series.

- These series have a finite radius of convergence at $\lambda \simeq \pi^2$, but can be extended beyond this bound with a Padé resummation.

Strong-coupling regime



Wilson loop operator in $\mathcal{N} = 2$ superconformal gauge theories

$$W_C \equiv \frac{1}{N} \operatorname{tr} \mathcal{P} \exp \left\{ \oint_C d\tau \left[iA_\mu(x)\dot{x}^\mu(\tau) + \frac{R}{\sqrt{2}}\big(\phi(x) + \overline{\phi}(x)\big) \right] \right\}$$

A different correlator

- Exploiting localization techniques, one can also consider the correlator of a circular Wilson loop with one chiral operator $\langle W_C \, \mathcal{O}_n(x) \rangle$.
- **E**-theory: orientifold projection of the quiver by identifying the fields of the two nodes.
- Exact expression in the planar limit

$$\langle W_C \, O_{2p+1} \rangle \simeq$$

$$\frac{1}{N} \sum_{\ell=1}^{p} \sqrt{\mathcal{G}_{2\ell+1}} M_{2p+1,2\ell+1}(\lambda) \sum_{n=1}^{\infty} \sqrt{2n+1} \left(I_{2n+1}(\sqrt{\lambda}) \sum_{m=1}^{\ell} \mathrm{h}_m^{(\ell)} \, \mathrm{D}_{n,m} \right)$$

[Pini, PV, 2023]

- Leading term at strong coupling with analytical and numerical techniques.