MICROSTATE COUNTING for SUPERSYMMETRIC AdS BLACK HOLES via HOLOGRAPHY

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New Frontiers in Theoretical Physics

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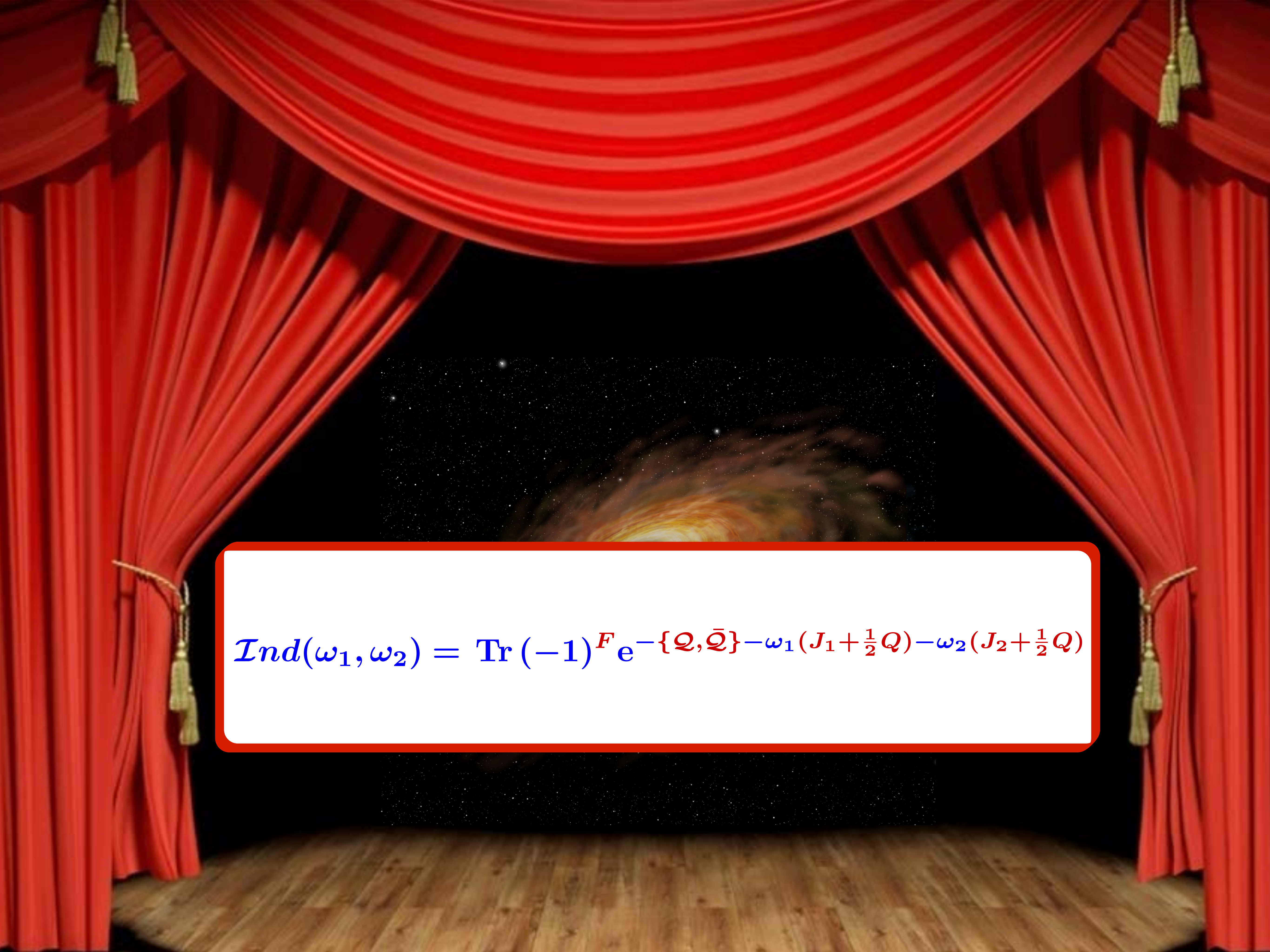


$$\mathcal{I}nd(\omega_1,\omega_2) = \mathrm{Tr}\,(-1)^F \mathrm{e}^{-\{\mathcal{Q},\bar{\mathcal{Q}}\}-\omega_1(J_1+\frac{1}{2}Q)-\omega_2(J_2+\frac{1}{2}Q)}$$

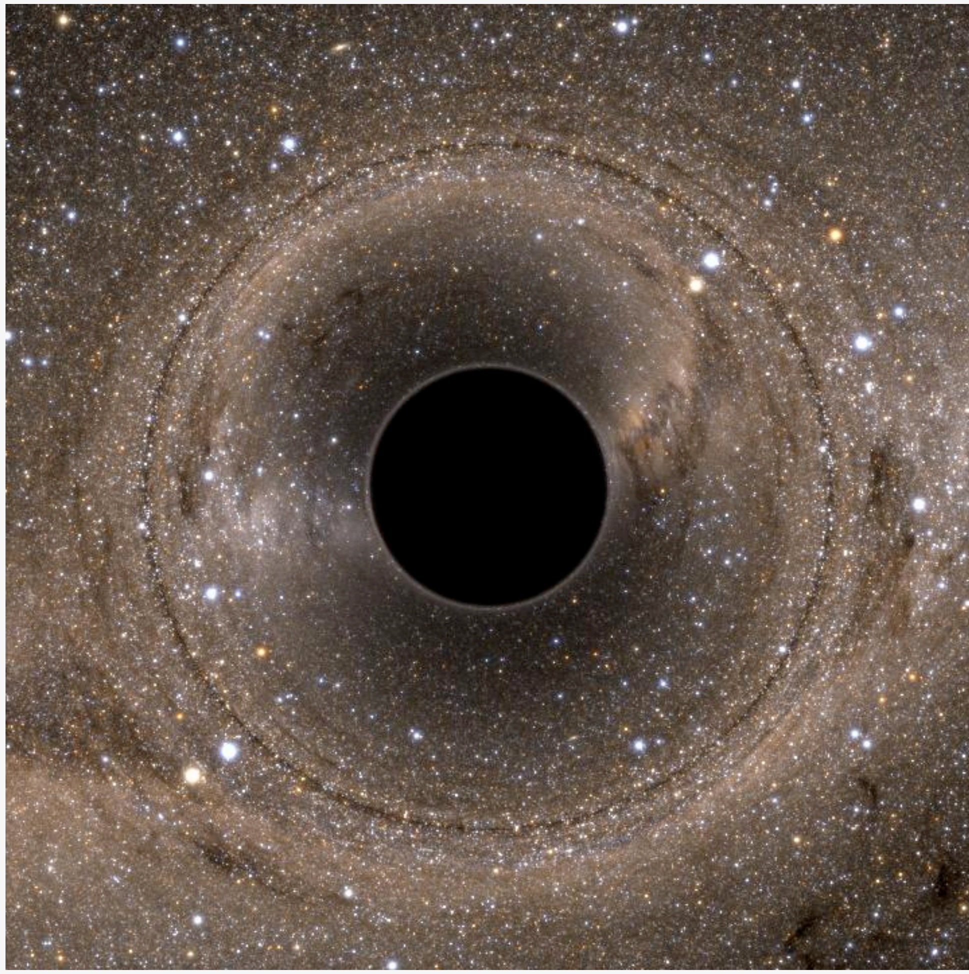
Black holes and quantum gravity

- **Black holes** are "theoretical laboratories" useful to test any theory of **quantum gravity**

Thermodynamic properties

Entropy ➜ Bekenstein-Hawking formula

$$S = \frac{\mathbf{Area}}{\mathbf{4}}$$



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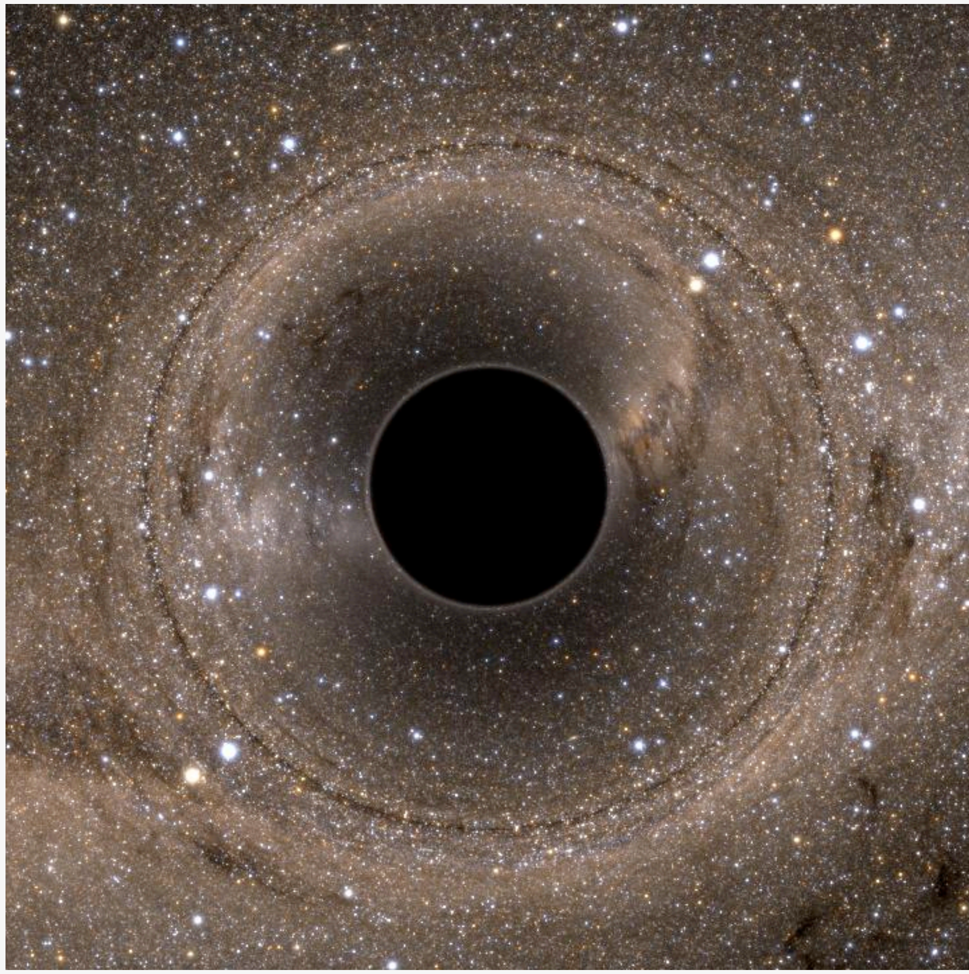


Thermodynamic properties

Entropy ➜ Bekenstein-Hawking formula

$$S = \frac{\text{Area}}{4}\,\frac{k_B c^3}{\hbar G}$$

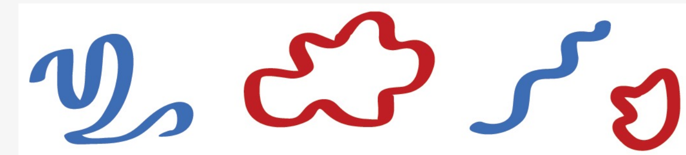
- Microscopic statistical derivation **?**

➜ **quantum gravity**

- Quantum corrections **?**

String theory and black hole entropy

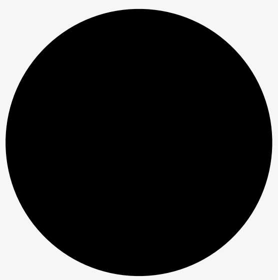
✤ A major success of **string theory**



➜ microstates for asymptotically flat **supersymmetric black holes**

Strominger, Vafa '96

macroscopic



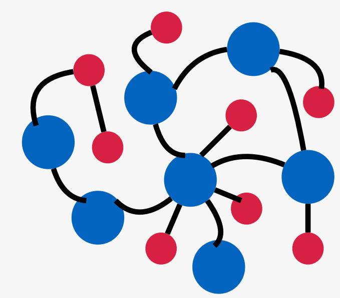
supersymmetric black hole

$$S = \frac{\textbf{Area}}{4} + \dots$$

$$\xrightarrow{G_N \to 0}$$

$$\xleftarrow[\text{thermodynamic limit}]{}$$

microscopic



bound state of many D-branes

$$S = \log\left(\sharp \text{ degenerate bound states}\right)$$

String theory and black hole entropy

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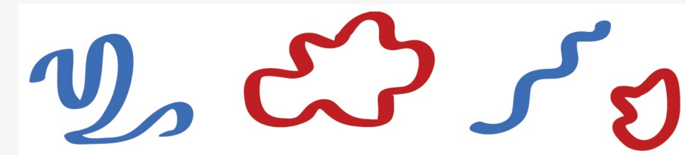

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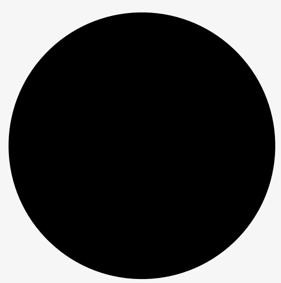


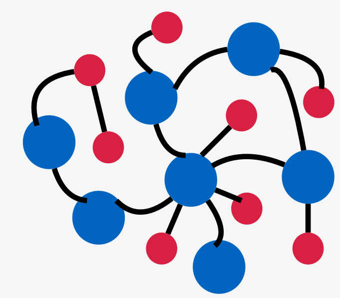
supersymmetric black hole

$$S = \frac{\mathbf{Area}}{4} + \ldots$$

$G_N \to 0$

thermodynamic limit

microscopic



bound state of many D-branes

$$S = \log\left(\sharp \text{ degenerate bound states}\right)$$

This is in the microcanonical ensemble (fix charges E, J, Q and count degeneracy)

Open questions:

- ◆ characterization of individual microstates for $G_N \neq 0$
- ◆ compute corrections when $G_N \neq 0$ Sen, . . .

Gravitational grand-canonical ensemble

- Also interesting to study quantum gravity in grand-canonical ensemble

Black hole in a bath with temperature T , angular velocity Ω , electric potential Φ

$\longrightarrow$ **phase transitions?**

Gravitational grand-canonical ensemble

- Also interesting to study quantum gravity in grand-canonical ensemble

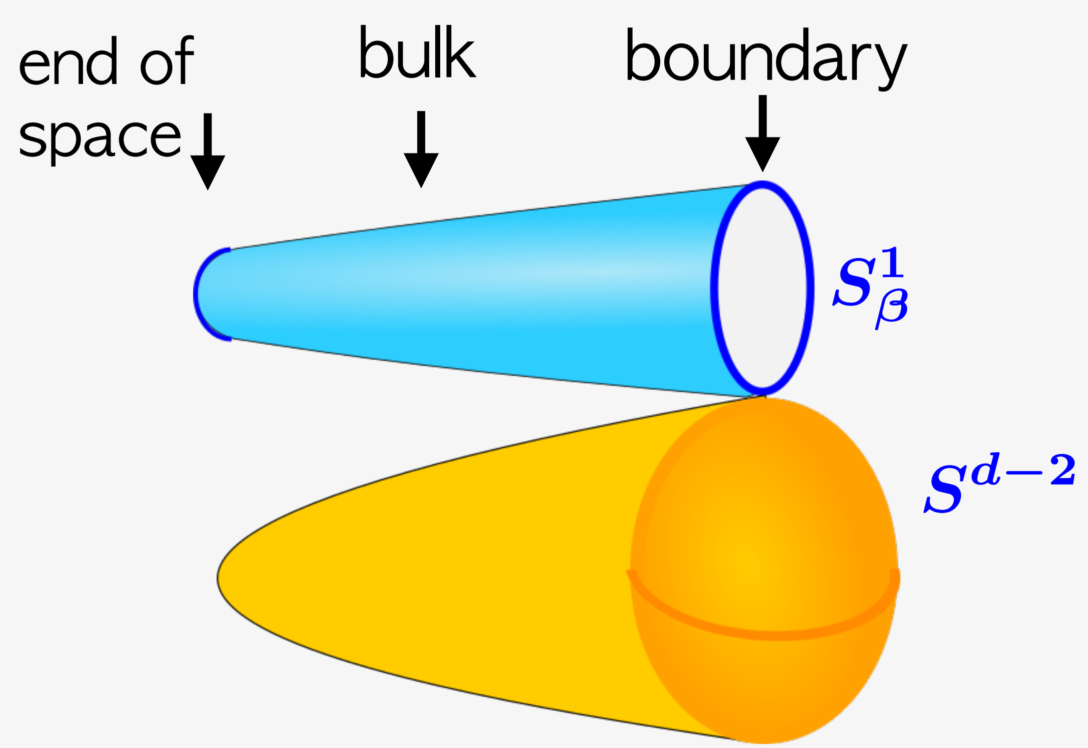
Black hole in a bath with temperature T, angular velocity Ω, electric potential Φ

→ **phase transitions?**

Grand-canonical ensemble well-defined in **Anti de Sitter space (AdS)**

T, Ω, Φ specified by metric and Maxwell field at asymptotic boundary in Euclidean signature. E.g. $\beta = T^{-1}$ = length of Euclidean time circle

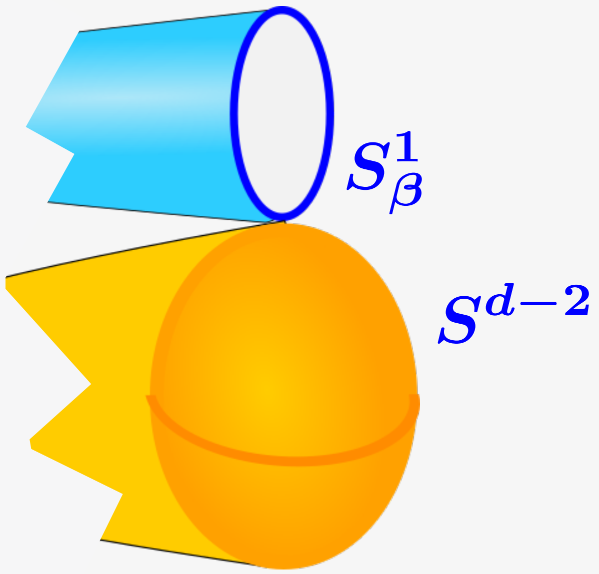
Euclidean AdS$_d$



Gravitational phase transition

fix the boundary conditions ➜ different possible fillings

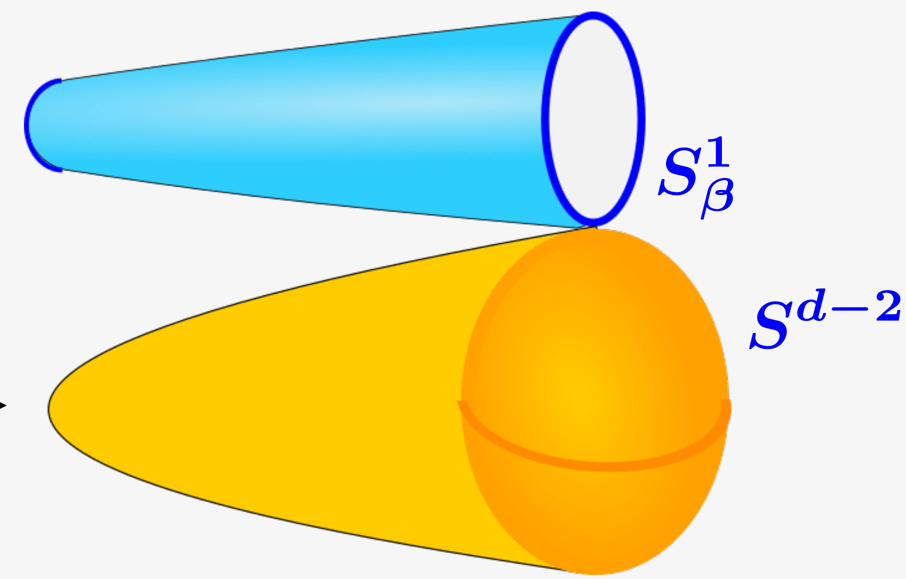
?



Gravitational phases

Euclidean AdS

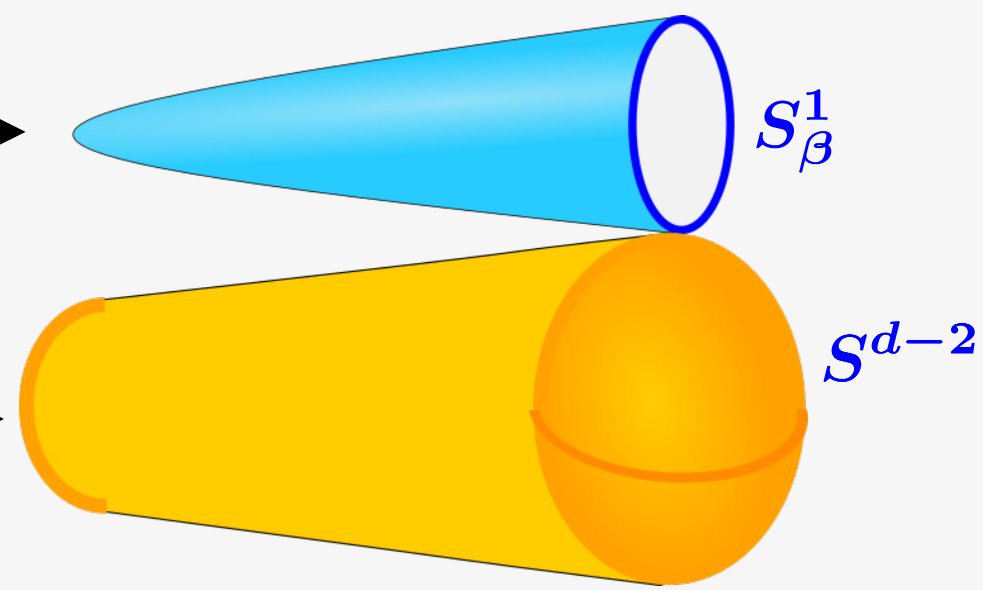
spatial sphere
shrinks to zero size
➜ entropy $= \mathbf{0}$



Black hole

time circle
shrinks

sphere with
finite area,
entropy $\neq \mathbf{0}$



Gravitational phases

Which solution dominates the ensemble?

- **Partition function**

$$Z(\beta) \;=\; \int \mathcal{D}g_{\mu\nu}\, \mathrm{e}^{-\mathbf{Action}[g_{\mu\nu}]}$$ with fixed boundary fields

semiclassical approx $\rightarrow$ $$\simeq\; e^{-I_{\text{AdS}}(\beta)} + e^{-I_{\text{black hole}}(\beta)} + \dots$$

dominates at large β dominates at small β

Hawking-Page '83

Euclidean quantum gravity

- **On-shell action** satisfies $I(\beta) = \beta E - S$ quantum statistical relation

  ➜ $\beta \times$ free energy

  Gibbons-Hawking '77

Suggests an interpretation as:

$$Z(\beta) = \text{“Tr”}\, \mathrm{e}^{-\beta E} = \sum_E d(E)\, \mathrm{e}^{-\beta E} \simeq \mathrm{e}^{S-\beta E} = \mathrm{e}^{-I(\beta)}$$

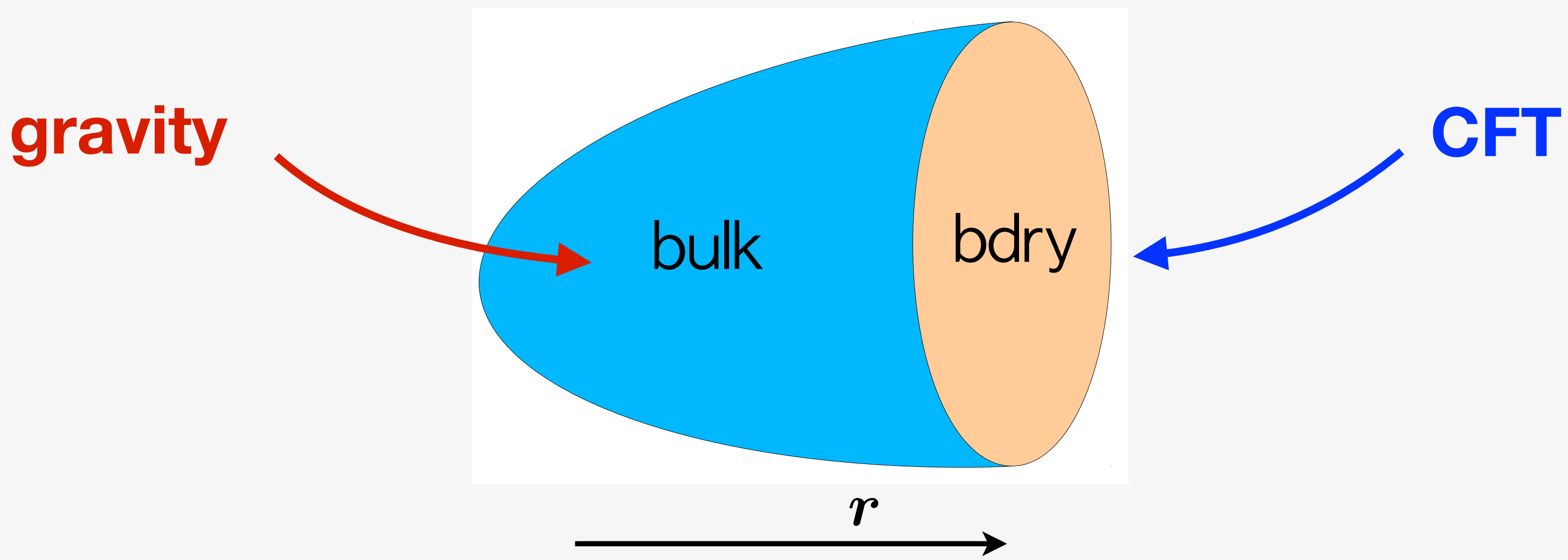
degeneracy thermodynamic limit

But we don't know the Hilbert space of quantum gravity!

This is made precise by the **AdS/CFT correspondence.**

AdS/CFT correspondence

Non-perturbative definition of Quantum Gravity in asymptotically AdS space



$$Z_{\text{gravity}} = Z_{\text{CFT}}$$

AdS gravity boundary conditions $\Longleftrightarrow$ CFT background fields

semiclassical gravity & strongly coupled CFT

- AdS/CFT: a weak/strong coupling duality

gravity in semiclassical regime (weakly coupled and close to Einstein gravity)

$\Updownarrow$

CFT at "large N" and strong coupling

$\Downarrow$

need non-perturbative QFT methods:

supersymmetry

semiclassical gravity & strongly coupled CFT

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CFT at "large N" and strong coupling

$\Downarrow$

need non-perturbative QFT methods:

supersymmetry

- supersymmetric localization: path integral "collapses" to ordinary integral

Witten '88, Nekrasov '02, Pestun '07, . . .

- in large N limit, look for saddle points of the simplified integral

➜ good approximation to path integral $Z_{\mathbf{CFT}} = Z_{\mathbf{gravity}}$

Strategy towards exact results

- analyze gravity solution ➜ identify suitable generating function ***Z*** : an index
- use AdS/CFT and compute ***Z*** in CFT, rather than in gravity
- result is **exact** ➜ **full Quantum Gravity *Z***
- take semi-classical limit, check Bekenstein-Hawking entropy
- go beyond semiclassical gravity ➜ towards **full Quantum Entropy**

AdS$_5$ Black Hole

- minimal AdS$_5$ supergravity

$$\mathcal{L} = R + \frac{12}{\ell^2} - \frac{1}{3} F_{\mu\nu} F^{\mu\nu} - \frac{2}{27} \epsilon^{\lambda\mu\nu\rho\sigma} A_\lambda F_{\mu\nu} F_{\rho\sigma} \quad + \textit{ fermions}$$

- General black hole solutions:

Chong, Cvetic, Lu, Pope '05

energy

angular momenta

R-charge

4 conserved charges $\quad E\,,\ J_1\,,\ J_2\,,\ Q$

$\updownarrow$

4 potentials $\quad \beta\,,\ \Omega_1\,,\ \Omega_2\,,\ \Phi$

inverse temperature

angular velocities

electrostatic potential

Supersymmetric AdS$_5$ Black Hole

- Preserves:

Gutowski, Reall '04,
Chong, Cvetic, Lu, Pope '05, . . .

2 supercharges, $\mathcal{Q}, \overline{\mathcal{Q}}$

E, J_1, J_2, Q with $0 = \{\mathcal{Q}, \overline{\mathcal{Q}}\} = E - J_1 - J_2 - \frac{3}{2}Q$

Bekenstein-Hawking entropy $S = \frac{\text{Area}}{4G} = \pi\sqrt{3Q^2 - 8a(J_1 + J_2)}$

$a = \frac{\pi}{8g^3 G}$

- Potentials: $\beta = \infty$ extremal $\quad \Omega_i = 1 \quad \Phi = \frac{3}{2}$ frozen!

➜ confusion, is there a thermodynamics left?

Reaching the index

Cabo-Bizet, Murthy, DC, Martelli '18

Look for a partition fct that only receives contributions from susy states

- better under control
- susy black hole should be saddle

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Look for a partition fct that only receives contributions from susy states

- better under control
- susy black hole should be saddle

$$Z(\beta,\Omega_i,\Phi) = \mathrm{Tr}\, e^{-\beta(E-\Omega_i J_i-\Phi Q)}$$

Use:
- $E = \{\mathcal{Q},\overline{\mathcal{Q}}\} + 1\, J_1 + 1\, J_2 + \frac{3}{2}\, Q$ superalgebra
- difference between the potentials and their frozen values

$$\omega_1 = \beta(\Omega_1 - 1)\ , \qquad \omega_2 = \beta(\Omega_2 - 1)\ , \qquad \varphi = \beta(\Phi - \tfrac{3}{2})$$ Silva

➜ $$Z(\beta,\omega_i,\varphi) = \mathrm{Tr}\, e^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+\omega_i J_i+\varphi Q}$$

Reaching the index

$$Z(\beta, \omega_i, \varphi) = \mathrm{Tr}\; e^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+\omega_i J_i+\varphi Q}$$

- supersymmetry : $$\boxed{\varphi = \frac{\omega_1 + \omega_2}{2} - \pi i}$$

Hosseini, Hristov, Zaffaroni

Cabo-Bizet, Murthy, DC, Martelli

$$\rightarrow \quad Z(\beta, \omega_i) = \mathrm{Tr}\; e^{-\pi i Q}\; e^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+\omega_i (J_i+\frac{1}{2}Q)}$$

Reaching the index

$$Z(\beta,\omega_i,\varphi) \;=\; \mathrm{Tr}\; e^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+\omega_i J_i+\varphi Q}$$

- supersymmetry: $\boxed{\varphi \,=\, \dfrac{\omega_1+\omega_2}{2} - \pi i}$

Hosseini, Hristov, Zaffaroni

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➜ $$Z(\beta,\omega_i) \;=\; \mathrm{Tr}\; e^{-\pi i Q}\; e^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+\omega_i(J_i+\frac{1}{2}Q)}$$

$$= \underbrace{\mathrm{Tr}\,(-1)^F\; e^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+(\omega_1-2\pi i)(J_1+\frac{1}{2}Q)+\omega_2(J_2+\frac{1}{2}Q)}}_{\text{Witten index}}$$

$e^{\pi i F} = e^{-2\pi i J_1}$

spin-statistics

➜ **refined Witten index**

Kinney, Maldacena, Minwalla, Raju '05

Romelsberger '05

only states with $\{\mathcal{Q},\overline{\mathcal{Q}}\}=0$ contribute

$Z \,=\, Z(\omega_i)$, no dependence on β

Supersymmetric thermodynamics

- Supersymmetric solution:

- $\varphi = \dfrac{\omega_1 + \omega_2}{2} - \pi i$ is satisfied

- on-shell action $\boxed{I(\omega_i, \varphi) = \dfrac{16}{27} \dfrac{\varphi^3}{\omega_1 \omega_2} a}$ very simple, no β

- quantum statistical relation becomes :

$$I = -S - \omega_1 J_1 - \omega_2 J_2 - \varphi Q$$

- **extremal limit** $\beta \to \infty$ but $\omega_1 , \omega_2 , \varphi$ remain finite ➜ limit is **smooth**

- Legendre transform of I gives the entropy:

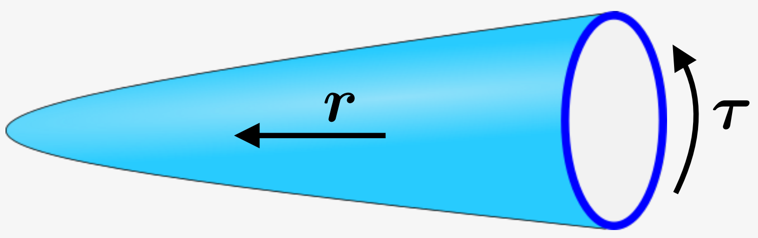
$$S = \pi \sqrt{3Q^2 - 8a(J_1 + J_2)} = \frac{\text{Area}}{4}$$

Complexification

$$\omega_1 + \omega_2 - 2\varphi = 2\pi i$$

- physical meaning?

  regularity condition ensuring the Killing spinor is antiperiodic along the shrinking thermal circle



this condition is thus necessary to include the BH saddle in ***Z***

- chemical potentials are complex!

- In fact, the susy, non-extremal BH solution has complex $g_{\mu\nu},\ A_\mu$

- What complex metrics are allowable? Witten '21

The CFT side

d=4 CFT with supersymmetry,

e.g. $\mathcal{N}=4$ super Yang-Mills, or one of the many $\mathcal{N}=1$ examples known

$$Z(\omega_i) = \mathbf{Tr}\,(-1)^F\, e^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+(\omega_1-2\pi i)(J_1+\frac{1}{2}Q)+\omega_2(J_2+\frac{1}{2}Q)}$$

well defined!

- protected by susy ➜ can be computed at weak coupling

$$\mathcal{I}nd(\sigma,\tau) = \int \prod_{i=1}^{\mathrm{rk}(G)} \mathrm{d}u_i \prod_{\alpha\in\Delta_+} \Gamma(\pm\alpha\cdot u+\sigma+\tau;\sigma,\tau) \prod_{I\in\mathrm{chirals}} \prod_{\rho\in\mathrm{R}_I} \Gamma(\rho\cdot u+\frac{r_I}{2}(\sigma+\tau);\sigma,\tau)$$

elliptic Gamma fct.

The CFT side

- For real $\omega_1 - 2\pi i\,,\ \omega_2$ ➜ index does not carry enough entropy

  reason: boson - fermion cancellations due to $(-1)^F$

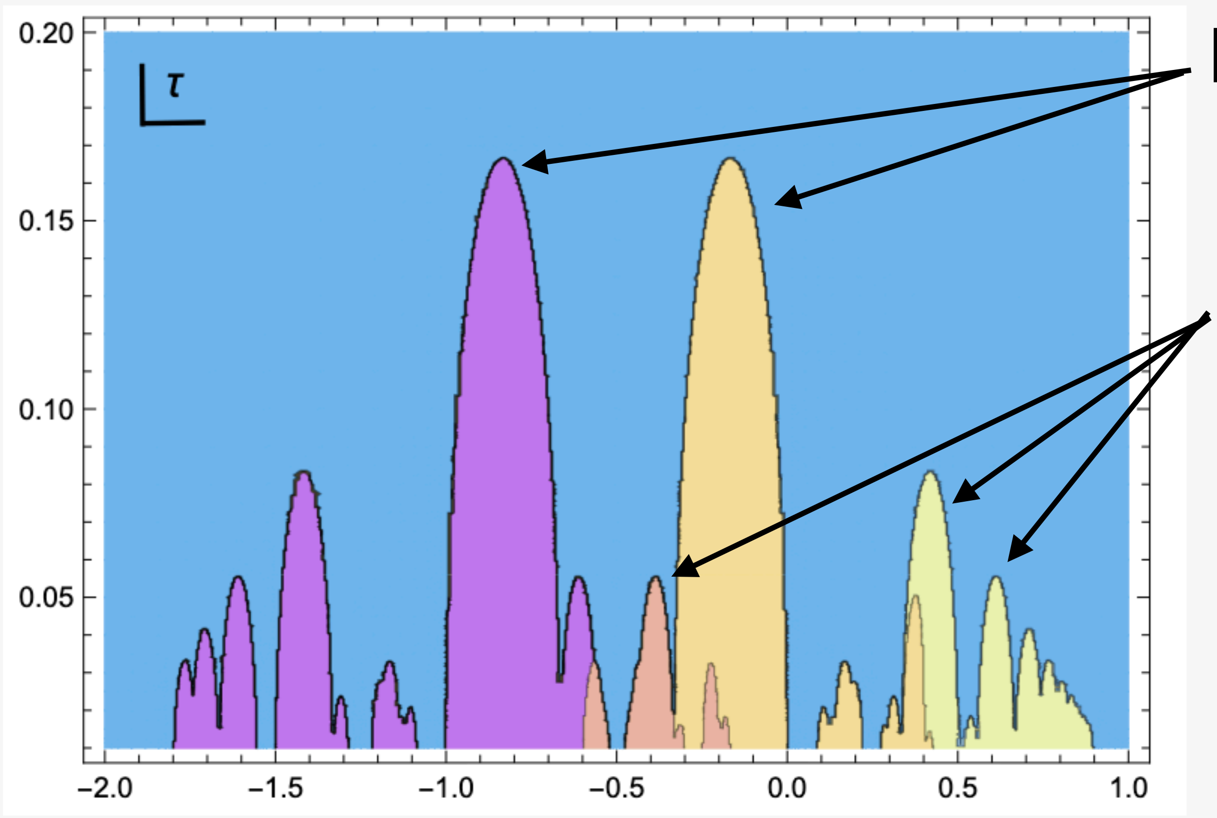
  Kinney, Maldacena, Minwalla, Raju '05

- However, ω_1, ω_2 really are complex variables

  ➜ search for complex saddles

Many competing phases

- **Large-N limit**
  - "Bethe ansatz" — Benini, Hristov, Zaffaroni '15 in AdS4, Benini, Milan '18, . . . Benini, Colombo, Soltani, Zaffaroni, Zhang '20
  - "elliptic" extension — Cabo-Bizet, Murthy '19 Cabo-Bizet, Murthy, DC, Martelli '20
  - . . . and more



Aharony, Benini, Mamroud, Milan '21

pic from Cabo-Bizet, Murthy '19

Cardy-like limit

A device to isolate black hole saddle

lots of people, starting from
Choi, J. Kim, S. Kim, Nahmgoong '18
...
DC, Komargodski '21

Small ω_1, ω_2 , with $\varphi = \dfrac{\omega_1 + \omega_2}{2} - \pi i$

$$I_{\rm CFT} = \frac{16}{27}(3c - 2a)\,\frac{\varphi^3}{\omega_1\omega_2} - \frac{4}{3}(a-c)\,\varphi\,\frac{2\pi i(\omega_1+\omega_2) - \omega_1\omega_2}{\omega_1\omega_2}$$

a , c *super*conformal anomaly coeff.

✤ Holographic SCFT at large-*N* ➜ $a = c$ ➜ $I_{\rm CFT} = \dfrac{16}{27}\dfrac{\varphi^3}{\omega_1\omega_2}\,a$

reproduces 2-derivative black-hole on-shell action!

✤ For general a , c , this is a **prediction for Quantum Gravity**

Corrected entropy

$$S = \mathrm{ext}_{\{\omega_1,\omega_2,\varphi,\Lambda\}}\left[-I - \omega_1 J_1 - \omega_2 J_2 - \varphi\, Q - \Lambda(\omega_1 + \omega_2 - 2\varphi - 2\pi i)\right]$$

linearizing in $a - c$

DC, Ruipérez, Turetta

$$\boxed{S = \pi\sqrt{3Q^2 - 8a(J_1+J_2) - 16a(a-c)\frac{(J_1-J_2)^2}{Q^2 - 2a(J_1+J_2)}} + \mathcal{O}((a-c)^2)}$$

also Bobev, Dimitrov, Reys, Vekemans

Higher-derivative corrections

DC, Ruipérez, Turetta '22

- 4-derivative corrected supergravity

$$\boxed{\begin{aligned}\mathcal{L} =\ & c_0 R + 12 c_1 g^2 - \tfrac{1}{4} c_2 F^2 - \tfrac{1}{12\sqrt{3}} c_3\, \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} A_\lambda \\ & + \lambda_1 \tfrac{1}{g^2} \left[\mathcal{X}_{\mathrm{GB}} - \tfrac{1}{2} C_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + \tfrac{1}{8} F^4 - \tfrac{1}{2\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} A_\lambda \right]\end{aligned}}$$

$$\mathcal{X}_{\mathrm{GB}} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \quad \text{Gauss-Bonnet}$$

$$c_0 = 1 + 4\lambda_2\,, \quad c_1 = 1 - 10\lambda_1 + 4\lambda_2\,, \quad c_2 = 1 + 4\lambda_1 + 4\lambda_2\,, \quad c_3 = 1 - 12\lambda_1 + 4\lambda_2$$

- Holographic anomaly matching ➜ dictionary :

$$a = \frac{\pi}{8Gg^3}(1 + 4\lambda_2)\,, \qquad c = \frac{\pi}{8Gg^3}(1 + 8\lambda_1 + 4\lambda_2)$$

- using "*susy first, extremal later*" prescription

  on-shell action I ➜ **does match CFT result!**

Outlook

- Recast a black hole microstate counting problem into a CFT computation
    - reproduced Bekenstein-Hawking entropy
    - computed corrections and reproduced them in gravity

CFT index counts states ➜ microscopic derivation

➜ a step towards **full Quantum Entropy**

example of **PRECISION HOLOGRAPHY**

- Extends to multiple charges and different *d*

asymptotics	charges	BPS entropy	"log grand-canonical partition
M, $AdS_4 \times S^7$	$J,\ Q_1, Q_2, Q_3, Q_4$		$I = -\frac{i}{2G} \frac{\sqrt{\varphi^1\varphi^2\varphi^3\varphi^4}}{\omega}$
IIB, $AdS_5 \times S^5$	$J_1, J_2,\ Q_1, Q_2, Q_3$	$S = S(J_i, Q_I)$	$I = \frac{\pi}{4G} \frac{\varphi^1\varphi^2\varphi^3}{\omega_1\omega_2}$
IIA, $AdS_6 \times_w S^4$	$J_1, J_2,\ \ Q$		$I = \frac{\pi\, i}{3G} \frac{\varphi^3}{\omega_1\,\omega_2}$
M, $AdS_7 \times S^4$	$J_1, J_2, J_3,\ Q_1, Q_2$		$I = -\frac{\pi^3}{128G} \frac{\varphi_1^2\varphi_2^2}{\omega_1\omega_2\omega_3}$

thank you !

Euclidean Quantum Gravity

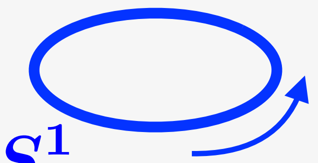
- $$Z(\beta, \Omega_i, \Phi) \; = \; \int D g_{\mu\nu} D A_\mu D \psi_\mu \, \mathrm{e}^{-\mathrm{Action}[g_{\mu\nu},\, A_\mu,\, \psi_\mu]}$$

  $$\simeq \; \mathrm{e}^{-I(\beta, \Omega_i, \Phi)}$$ on-shell action

S^3

Boundary conditions: Euclidean time circle S^1_β,

S^3 with twist parameters Ω_1, Ω_2 + gauge field $A_\tau \sim \Phi$



S^1_β

- Z should also be thermal partition fct. in grand-canonical ensemble

$$Z(\beta, \Omega_i, \Phi) \; = \; \text{“Tr”} \; \mathrm{e}^{-\beta(E - \Omega_i J_i - \Phi Q)}$$

$$\boxed{I(\beta, \Omega_i, \Phi) = \beta E - S - \beta \Omega_i J_i - \beta \Phi Q}$$ quantum statistical relation

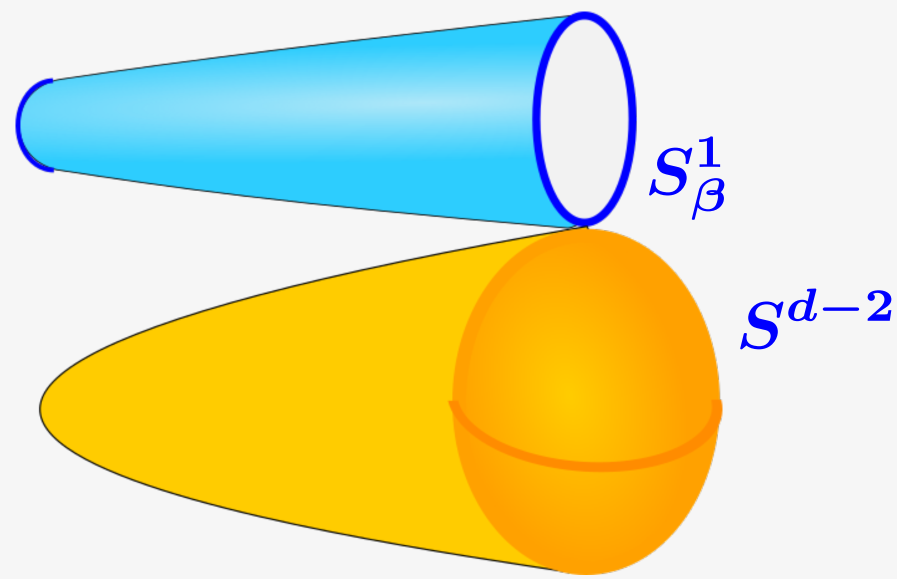
Hawking-Page is “large-N deconfinement”

Can also use the correspondence to shed light on QFT at strong coupling

Hawking-Page ⟷ confinement/deconfinement phase transition

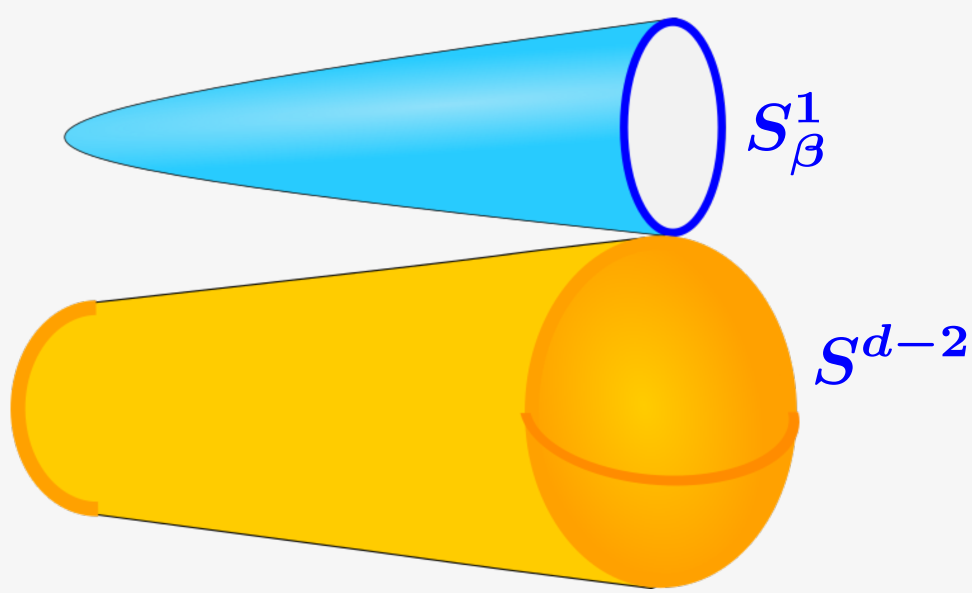
Witten ‘98

Euclidean AdS



⟷ CFT confined

Black hole



⟷ CFT deconfined

Holographic anomaly matching

R-current anomalies are matched by Chern-Simons terms in the bulk

$$\mathcal{L}_{\mathrm{grav}} \supset (5a - 3c)\, \epsilon^{\mu\nu\rho\sigma\lambda} \underbrace{F_{\mu\nu} F_{\rho\sigma} A_\lambda}_{\text{2 der.}} + \frac{9}{8}(c - a)\, \epsilon^{\mu\nu\rho\sigma\lambda} \underbrace{R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} A_\lambda}_{\text{4 der.}}$$

- 2∂ supergravity $\longleftrightarrow$ a , c large & equal

$$a = c = \frac{\pi}{8 g^3 G} + \dots$$

~ $N^2 + \ldots$ in many $SU(N)$ examples

- corrections to a , c (in particular, to $a - c = 0 + \cdots$) $\longleftrightarrow$ 4∂ susy invariants !