Maxwell theory of fractons

Erica Bertolini

Università di Genova and I.N.F.N. Sezione di Genova

Cortona, 27^{th} September 2023

Based on :

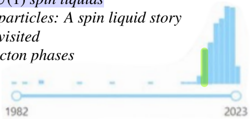
E.B. and N.Maggiore, Phys. Rev. D 106 (2022) no.12, 125008, *Maxwell theory of fractons*
A.Blasi and N.Maggiore, Phys. Lett. B 833 (2022), 137304, *The theory of symmetric tensor field*
E.B., A.Blasi, A.Damonte and N.Maggiore, Symmetry, 15 (2023), no.4, 945, *Gauging fractons and linearized gravity*
E.B., N.Maggiore, G.Palumbo, Phys. Rev. D 108 (2023), 025009, *Covariant fracton gauge theory with boundary*

What are fractons?

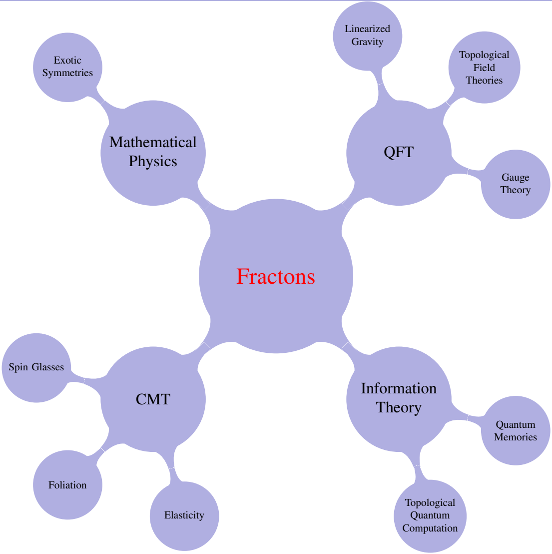
It's a wonderfully rich world

C.Chamon, Phys.Rev.Lett. 94, 040402 (2005) *Quantum glassiness in strongly correlated clean systems: An example of topological overprotection*
J.Haah, Phys.Rev.A 83, 042330 (2011) *Local stabilizer codes in three dimensions without string logical operators*
B.Yoshida, Phys.Rev.B 88, 125122 (2013) *Exotic topological order in fractal spin liquids*
S.Vijay, J.Haah, L.Fu, Phys.Rev.B 92, 235136 (2015) *A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations*
S.Vijay, J.Haah, L.Fu, Phys.Rev.B 94, 235157 (2016) *Fracton topological order, generalized lattice gauge theory and duality*
H.Ma, E.Lake, X.Chen, M.Hermele, Phys.Rev.B 95, 245126 (2017) *Fracton topological order via coupled layers*
W.Shirley, K.Slagle, Z.Wang, X.Chen, Phys.Rev.X 8, 031051 (2018) *Fracton models on general 3-dimensional manifolds*
H.Ma, M.Hermele, X.Chen, Phys.Rev.B 98, 035111 (2018) *Fracton topological order from the Higgs and partial-confinement mechanisms of rank-two gauge theory*
D.Bulmash and M.Barkeshli, Phys.Rev.B 97, 235112 (2018) *Higgs mechanism in higher-rank symmetric $U(1)$ gauge theories*
N.Seiberg, SciPost Phys. 8, 050 (2020) *Field theories with a vector global symmetry*
N.Seiberg, S.-H.Shao, SciPost Phys. 9, 046 (2020) *Exotic $U(1)$ symmetries, duality and fractons in 3+1-dimensional QFT*
P.Gorantla, H.T.Lam, N.Seiberg, S.-H.Shao, SciPost Phys. 9, 073 (2020) *More exotic field theories in 3+1 dimensions*
N.Seiberg, S.-H.Shao, SciPost Phys. 10, 003 (2021) *Exotic $\mathbb{Z}_N$ symmetries, duality, and fractons in 3+1-dimensional QFT*
P.Gorantla, H.T.Lam, N.Seiberg, S.-H.Shao, Phys.Rev.B 106, 045112 (2022) *Global dipole symmetry, compact Lifshitz theory, tensor gauge theory, and fractons*
P.Gorantla, H.T.Lam, N.Seiberg, S.-H.Shao, Phys.Rev.B 107, 125121 (2023) *Gapped lineon and fracton models on graphs*
...

M. Pretko (et al.) **:**
Phys.Rev.B 95, 115139 (2017) *Subdimensional particle structure of higher rank $U(1)$ spin liquids*
Phys.Rev.B 96, 035119 (2017) *Generalized electromagnetism of subdimensional particles: A spin liquid story*
Phys.Rev.D 96, 024051 (2017) *Emergent gravity of fractons: Mach's principle revisited*
Phys.Rev.B 96, 125151 (2017) *Higher-spin Witten effect and two-dimensional fracton phases*
Phys.Rev.Lett. 120, 195301 (2018) *Fracton-Elasticity Duality*
Phys.Rev.B 98, 115134 (2018) *The fracton gauge principle*
Phys.Rev.X 9, 021003 (2019) *Localization in Fractonic Random Circuits*
...



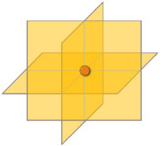
Fractons are everywhere!

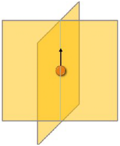


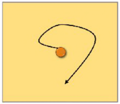
What are fractons? The defining property

Restricted mobility

- fracton (0D)
- lineon (1D)
- planon (2D)







What does it mean?

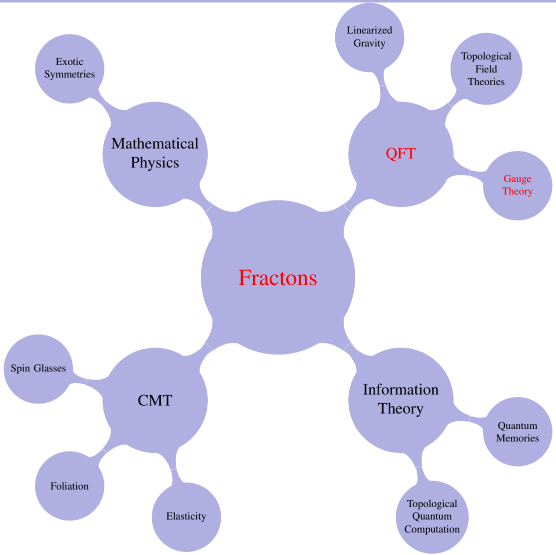
- Dispersion ($\omega = 0$) ?
- Propagators ?
- Kinetic constraints ?

Many Languages

Descriptions :

- Lattice theory :
  - exactly solvable spin models
  - quantum error correcting codes
  - “gapped”
  - Type I : fractons, lineons and planons
  - Type II : fractons only.
- Tensor gauge theory (Pretko) :
  - Higher moment conservations
  - A_{ij}
  - “gapless”
  - Maxwell-like ($\mathscr{H} = E^2 + B^2$)
  - Gauss → limited mobility .

Fractons' Gauge theories



The Gauss constraint

Maxwell Theory :

$$\text{Gauss}: \quad \partial_i E^i = \rho \quad \Rightarrow \quad \text{charge conservation}$$

E^i electric field.

Fractons - scalar charge theory :

$$\text{Gauss}: \quad \partial_i \partial_j E^{ij} = \rho \quad \Rightarrow \quad \text{dipole } (x^i \rho) \text{ conservation}$$

E^{ij} symmetric tensor "electric field".

M. Pretko, Phys.Rev.B 95, 115139 (2017) *Subdimensional particle structure of higher rank $U(1)$ spin liquids*
M. Pretko, Phys.Rev.B 96, 035119 (2017) *Generalized electromagnetism of subdimensional particles*

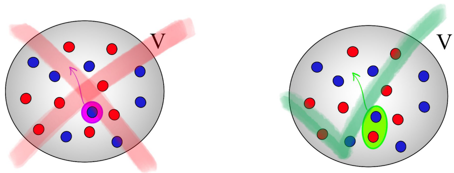
The dipole conservation

Dipole ($x^k\rho$) conservation

$$\int dV\, x^k \partial_j \partial_i E^{ji} = -\int dV\, \partial_j E^{kj} = 0 = \int dV\, x^k \rho$$

$\Rightarrow$ single charges ρ = **fractons**

while dipoles are free to move



The ingredients for the scalar charge theory

Symmetric rank-2 tensor :

$$A_{ij} = A_{ji}$$

Symmetry and Gauss constraint :

$$\delta_{fract} A_{ij} = \partial_i \partial_j \Lambda \quad ; \quad \partial_i \partial_j E^{ij} = \rho$$

Hamiltonian : Maxwell-like

$$H \sim E^2 + B^2$$

"Electric" and "magnetic" fields :

$$E_{ij} = \partial_i \partial_j A_0 - \partial_0 A_{ij} \quad ; \quad B_{ij} = \epsilon_{ikl} \partial^k A^l_j$$

M. Pretko, Phys.Rev.B 96, 035119 (2017) *Generalized electromagnetism of subdimensional particles*
M. Pretko, Phys.Rev.B 96, 125151 (2017) *Higher-spin Witten effect and two-dimensional fracton phases*

Weaknesses → Motivations

But (from a QFT point of view) :

- ad hoc definitions...
- A_0?
- non-covariant ;
- inhomogeneous # of ∂ ;
- bizarre mass dimensions.

Maxwell theory of fractons

Approach

QFT covariant (symmetry-based) technique

$$\delta_{fract}A_{ij} = \partial_i\partial_j\Lambda \quad \to \quad \delta_{fract}A_{\mu\nu} = \partial_\mu\partial_\nu\Lambda \ ,$$

Symmetry	$\to$	**Action**	$\to$	**EoM**
$\delta_{gauge}A_\mu = \partial_\mu\Lambda(x)$	$\to$	$\int d^4x \ F^{\mu\nu}F_{\mu\nu}$	$\to$	$\partial_\mu F^{\mu\nu} = 0$
$\delta_{fract}A_{\mu\nu} = \partial_\mu\partial_\nu\Lambda(x)$	$\to$	?	$\to$	?

The symmetry is the only ingredient !
(together with covariance, locality and power-counting)

The covariant symmetry

The most general invariant action ($\delta_{fract} S_{inv} = 0$) is

$$S_{inv} = g_1 S_{fract} + g_2 S_{LG} ,$$

where

$$S_{fract} = \int d^4x \left(\partial_\rho A_{\mu\nu} \partial^\rho A^{\mu\nu} - \partial_\rho A_{\mu\nu} \partial^\mu A^{\nu\rho} \right)$$

$$S_{LG} = \int d^4x \left(\partial_\mu A \partial^\mu A - \partial_\rho A_{\mu\nu} \partial^\rho A^{\mu\nu} - 2\partial_\mu A \partial_\nu A^{\mu\nu} + 2\partial_\rho A_{\mu\nu} \partial^\mu A^{\nu\rho} \right) .$$

diffs $\qquad$ **fract**

$$\delta A_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \xrightarrow{\xi_\mu = \frac{1}{2}\partial_\mu \Lambda} \delta A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda$$

M.Pretko, Phys.Rev.D 96, 024051 (2017) *Emergent gravity of fractons: Mach's principle revisited*

The idea of the building block

New field strength $F \sim \partial A$

	Maxwell	**Fractons**
invariance	$\delta_{gauge} F_{\mu\nu} = 0$	$\delta_{fract} F_{\mu\nu\rho} = 0$
cyclicity	$F_{\mu\nu} + F_{\nu\mu} = 0$	$F_{\mu\nu\rho} + F_{\nu\rho\mu} + F_{\rho\mu\nu} = 0$
Bianchi	$\epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0$	$\epsilon_{\alpha\mu\nu\rho} \partial^\mu F^{\beta\nu\rho} = 0$

where

$$F_{\mu\nu\rho} = F_{\nu\mu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\mu\rho} - 2\partial_\rho A_{\mu\nu} \ .$$

Now

$$S_{fract} \sim \int d^4x \ F^{\mu\nu\rho} F_{\mu\nu\rho} \sim \int F^2$$

$$S_{LG} = \int d^4x \left(\frac{1}{4} F^{\mu}{}_{\mu\nu} F_{\rho}{}^{\rho\nu} - \frac{1}{6} F^{\mu\nu\rho} F_{\mu\nu\rho} \right) \ .$$

Maxwell theory for fractons ($g_2 = 0$)

The action :

$$S_{fract} = \frac{g_1}{6} \int d^4x \, F^{\mu\nu\rho} F_{\mu\nu\rho}$$

Equations of Motion (EoM) **:**

$$0 = \frac{\delta S_{fract}}{\delta A_{\alpha\beta}} = \partial_\mu F^{\alpha\beta\mu} \ .$$

EoM^{00} :

$$\partial_i F^{00i} = 2\partial_i \left(\partial^0 A^{0i} - \partial^i A^{00} \right) = 0$$

with solution

$$\boxed{A^{0\mu} = A^{\mu 0} \equiv \partial^\mu A^0 \ .}$$

Maxwell theory for fractons

"Electric" tensor field (canonical momentum) **:**

$$E^{ij} \equiv \frac{\partial \mathscr{L}_{fract}}{\partial(\partial_t A_{ij})} = -g_1 F^{ij0} = g_1 \left(2\partial^0 A^{ij} - \partial^j A^{0i} - \partial^i A^{0j}\right)$$

$$= 2g_1 \left(\partial^0 A^{ij} - \partial^i \partial^j A^0\right)$$

"Magnetic" tensor field :

$$B_i^{\ j} \equiv \frac{1}{3}\epsilon_{0ikl} F^{jkl} = \epsilon_{0ilk} \partial^l A^{jk} \ .$$

Claim : fractons are embedded in the covariant theory

Summary of "Maxwell" equations

EoM + Bianchi + A_0 :

Maxwell	**Fractons**
$\vec{\nabla}\cdot\vec{E}=0$	$\partial_j E^{ij}=0 \quad \left(\partial_i\partial_j E^{ij}=0\ ⚠\right)$
$\vec{\nabla}\cdot\vec{B}=0$	$\partial^a B_a^{\ p}=0$
$\vec{\nabla}\times\vec{E}-\partial_t\vec{B}=0$	$\epsilon_{0lmj}\partial^m E^{ij}+\partial^0 B_l^{\ i}=0$
$\vec{\nabla}\times\vec{B}-\partial_t\vec{E}=0$	$-\partial_0 E^{ij}+\frac{1}{2}\left(\epsilon^{0ikl}\partial_k B_l^{\ j}+\epsilon^{0jkl}\partial_k B_l^{\ i}\right)=0$,

same as the Literature ! [M.Pretko, Phys.Rev.B 96, 035119 (2017)]

Energy-momentum tensor

$$T_{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{fract}}{\delta g^{\alpha\beta}}\bigg|_{g^{\alpha\beta}=\eta^{\alpha\beta}} = \frac{g_1}{6}\eta_{\alpha\beta}F^2 - \frac{g_1}{3}\left(2F_{\alpha\nu\rho}F_\beta{}^{\nu\rho} + F^{\mu\nu}{}_\beta F_{\mu\nu\alpha}\right),$$

whose components are :

Fractons	**Maxwell**
$T_{00} = -\frac{1}{4g_1}\left(E^{ij}E_{ij} + B^{ij}B_{ij}\right) \quad \Rightarrow g_1 < 0$	$u = \frac{1}{2}(E^iE_i + B^iB_i)$
$T_{0i} = \frac{1}{2}\epsilon_{0ijk}E^{ja}B^k{}_a$	$S_i = \epsilon_{0ijk}E^jB^k$
$T_{ij} = \eta_{ij}T_{00} - E_{ia}E_j{}^a - \frac{1}{2}\left(B_{ia}B_j{}^a - B_{aj}B^a{}_i\right)$	$\sigma_{ij} = \eta_{ij}u - E_iE_j - B_iB_j$

Continuity equation

EoM with matter coupling :

$$\partial_\mu F^{\alpha\beta\mu} = -J^{\alpha\beta} \ .$$

Continuity equation

$$\partial_\alpha\partial_\beta J^{\alpha\beta} = 2\partial_0\partial_i J^{0i} + \partial_i\partial_j J^{ij} = \boxed{\partial_0\rho + \partial_i\partial_j J^{ij} = 0} \ ,$$

with charge $\rho \sim \partial_i J^{0i}$ and Gauss now is

$$\partial_i\partial_j E^{ij} = \rho \ .$$

“Lorentz” force not *by intuition*

From the on-shell “conservation” of $T^{\mu\nu}$

$$\partial_\nu T^{\mu\nu} + f^\mu = 0 \ ,$$

we identify the 4D “Lorentz” force to be

$$\textbf{Fractons:} \quad f^0 = E_{ab}J^{ab} \quad ; \quad f^i = 2J_{a0}E^{ia} - \tfrac{1}{2}\epsilon^{0imn}J_{am}B_n^{\ a}$$

$$\textbf{Maxwell:} \quad f^0 = \vec{E}\cdot\vec{J} \quad ; \quad \vec{f} = q\vec{E} + q\vec{v}\times\vec{B} \ ,$$

it is the force on a dipole, where charge, dipole and (dipole) current are identified as

$$\partial_i J^{0i} \sim \rho \quad ; \quad J^{0i} \sim p^i \equiv x^i\rho \quad ; \quad J^{ik} \sim v^i p^k + v^k p^i \ .$$

Conclusions and not

Discussion of results

- Covariant (Maxwell) theory for fractons :

Symmetry	$\rightarrow$	**Action**	$\rightarrow$	**EoM**
$\delta A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda(x)$	$\rightarrow$	$\int d^4x \; F^{\mu\nu\rho} F_{\mu\nu\rho}$	$\rightarrow$	$\partial_\mu F^{\alpha\beta\mu} = 0$;

Discussion of results

- Results of Literature from first principles of QFT :
  - ✓ electric and magnetic tensor fields ;
  - ✓ A_0 ;
  - ✓ "Maxwell" equations ;
  - ✓ energy density ;
  - ✓ continuity equation ;
  - ✓ "Lorentz" force on a dipole ;
- Energy-momentum tensor ;
- Strong relation with LG.

Future

- 3D and elasticity (in progress - stay tuned!) ;
- Non-abelian (“YM”?) ;
- Traceless theory ;
- Matter coupling (fracton QED) ;
- Fermionic fractons (twist?) ;
- LG,...$\rightarrow \infty$

Thanks for your attention!

Remark : looks familiar

$F_{\mu\nu\rho}$ appears in a Wu and Zee paper of '88, out of context!

Y.S.Wu and A.Zee, Phys. Lett. B **207** (1988), 39-43

Membranes, higher Hopf maps, and phase interactions

Volume 207, number 1 PHYSICS LETTERS B 9 June 1988

dimensional spatial "laplacian". The work done by this magnetic field on a moving magnetic monopole is represented precisely by the Hopf integral.

Evidently, by using differential forms, we can write many of our equations much more compactly. Thus, for example, the current four-form in eq. (4) may be written as $J+\mathrm{tr}\,n(\mathrm{d}n)^4$ where $n=n^a\gamma^a$ and the identity in eq. (12) may be written as $A=\mathrm{tr}(BdB+\frac{2}{3}B^3)$.

Finally, we comment briefly on how the construction of a phase interaction may be extended to symmetric tensor fields. Let $\phi_{\mu\nu}$ be a two-indexed tensor field with the transformation law $\phi_{\mu\nu}\to\phi_{\mu\nu}+\partial_\mu\partial_\nu\Lambda$. Then $F_{\lambda\mu\nu}=2\partial_\lambda\phi_{\mu\nu}-\partial_\mu\phi_{\lambda\nu}-\partial_\nu\phi_{\lambda\mu}$ is invariant and satisfies the Bianchi identity $\epsilon_{...\rho\lambda\mu}\partial_\rho F_{\lambda\mu\nu}=0$. In (2+1) dimensions we may consider in analogy with eq. (1) the action

$$S=\int \mathrm{d}^3x\, g\epsilon^{\mu\nu\lambda}\phi_{\mu\rho}F_{\nu\lambda}{}^{\rho}+\phi_{\mu\nu}T^{\mu\nu},$$

where we couple to the stress–energy tensor $T^{\mu\nu}$ of a point particle, a field, or a soliton. The potential around a point particle is easily found to be $\phi_{0i}\sim\epsilon_{ij}x_j/r^2$. Thus, a second particle going around the first will acquire a phase given by $\exp(\mathrm{i}\int\phi_{0i}T_{0i}\,\mathrm{d}^3x)$. This phase depends, however, on the velocity of the particle. This construction holds for flat space.

Incidentally, in analogy with eq. (4) we can also construct a solitonic string in an SO(4) nonlinear model in (4+1)-dimensional spacetime with the current

ment of Physics at the University of Utah for hospitality. This research was supported in part by the National Science Foundation under Grant No. PHY82-17853, supplemented by funds from the National Aeronautics and Space Administration, at the University of California at Santa Barbara, and by NSF Grant No. PHY87-06501 at the University of Utah.

References

[1] S. Deser, R... (1982) 975 ... see also Y.S...
[2] B. Zumino, ... 477.
[3] F. Wilczek, ...
[4] F. Wilczek ... Santa Barbara ITP preprint 84-25 (1984).
[5] Y.S. Wu, Phys. Rev. Lett. 52 (1984) 2103; 53 (1984) 111.
[6] D. Arovas, R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. B 251 (FS13) (1985) 117.
[7] A. Zee, in: Particles and the universe, eds. G. Lazarides and Q. Shafi, p. 257.
[8] Y.S. Wu and A. Zee, Phys. Lett. B 147 (1984) 325.
[9] A.M. Din and W.J. Zakrewski, Phys. Lett. B 146 (1984) 341.
[10] T. Jarosewicz, Phys. Lett. B 159 (1985) 299.
[11] P. Wiegmann, Phys. Rev. Lett. 60 (1988) 821; see however X.G. Wen and A. Zee, to be published.
[12] A.M. Polyakov, Mod. Phys. Lett. A 3 (1988) 325.
[13] H.C. Tze, preprint VPI-IHEP-88/1.
[14] R.I. Nepomechie and A. Zee, in: Quantum field theory and quantum statistics, eds. I.A. Batalin et al. (Hilger, London)

Finally, we comment briefly on how the construction of a phase interaction may be extended to symmetric tensor fields. Let $\phi_{\mu\nu}$ be a two-indexed tensor field with the transformation law $\phi_{\mu\nu}\to\phi_{\mu\nu}+\partial_\mu\partial_\nu\Lambda$. Then $F_{\lambda\mu\nu}=2\partial_\lambda\phi_{\mu\nu}-\partial_\mu\phi_{\lambda\nu}-\partial_\nu\phi_{\lambda\mu}$ is invariant and