A Statistical Formulation of Conjunction Assessment

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Motivation

With the new space race

- Hundreds of satellites are launched into Earth's orbit every year for different purposes.
- In total, there are nearly 6,000 satellites circling our planet and this number keeps growing.
- About 60% of these satellites are defunct, contributing to the growing problem of space debris or "space junk".
- Managing the increasing number of space objects poses a significant challenge in collision avoidance.

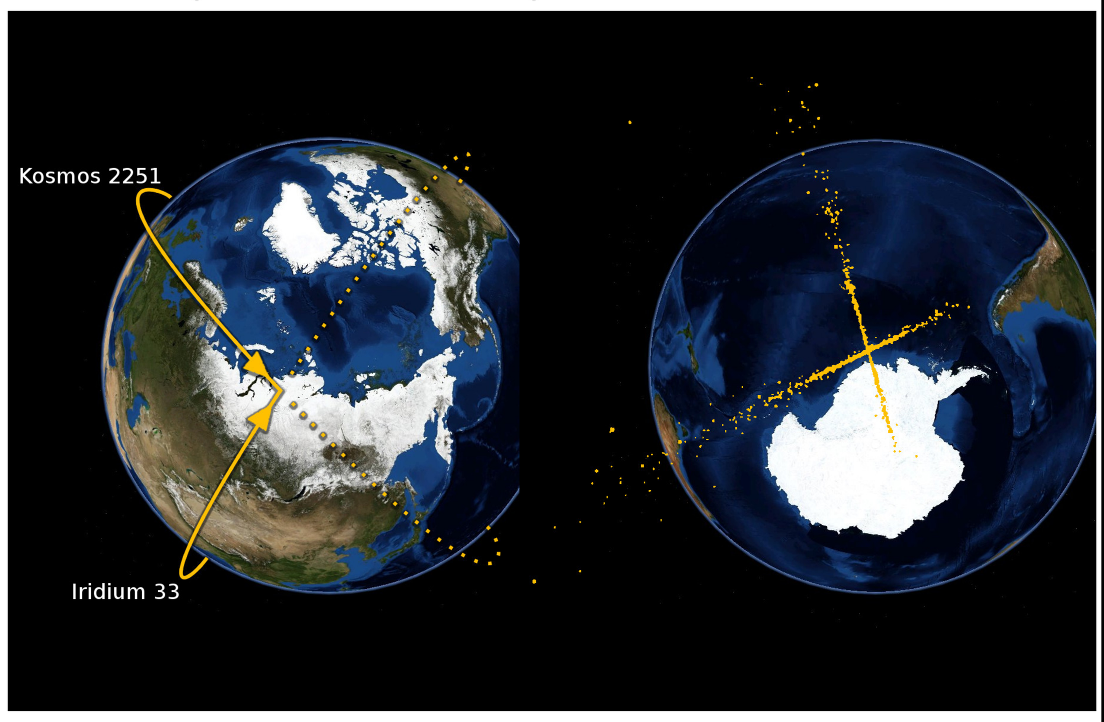


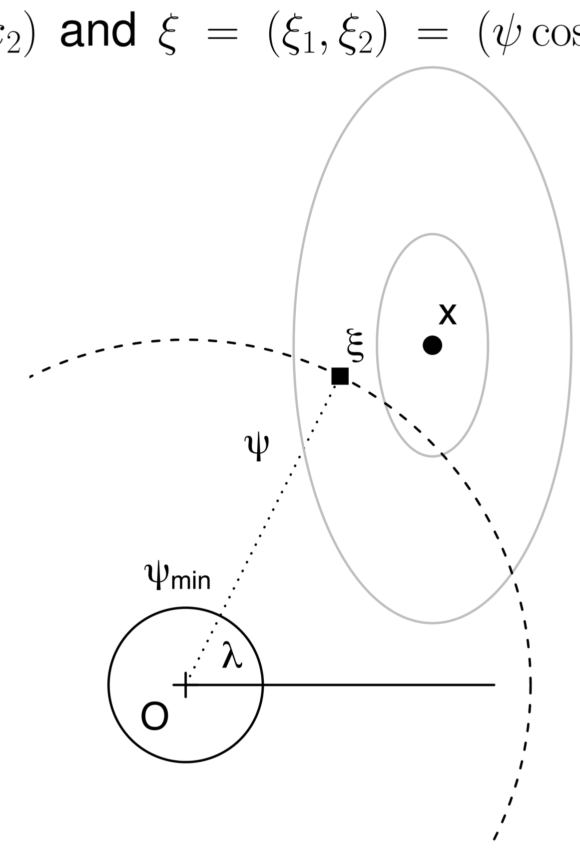
Figure 1: Collision of Kosmos 2251 and Iridium 33 on February 10, 2009.

In this work

- We point out that the usual collision probability estimate can be badly biased.
- We formulate an approach to satellite conjunction assessment based on a statistical model.
- We discuss inference on the miss distance between two space objects.

1. Probability of collision

We consider two space objects in close conjunction described relative to an origin 'O' at the first satellite. We assume that the velocity vector is known, and we consider x and ξ as the projections of the observed position and the true position into the plane perpendicular to the velocity vector (encounter plane), i.e., $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2) = (\psi\cos\lambda, \psi\sin\lambda)$.



Most existing literature focuses on the probability of collision p_c, i.e., the integral of the density of x over the Hard body Ratio (HBR or ψ_{min}) disk, i.e.,

$$\widehat{p}_c = p_c(x), \quad \text{where} \quad p_c(\xi) = \int_{\{x':\|x'\|\leq\psi_{\min}\}} f(x';\xi)\,\mathrm{d}x'$$

which can be seen as a plug-in estimate of $p_c(\xi)$.

What is the problem with $\hat{p}_c$?

1) $p_c(x)$ is the probability that the observation x lies inside the HBR, not the probability that the second object will hit the first object.

2) $\mathrm{P}\{\widehat{p}_c < p_c(\xi)\} > 1/2$, i.e., $\widehat{p}_c$ is too small more often than not.

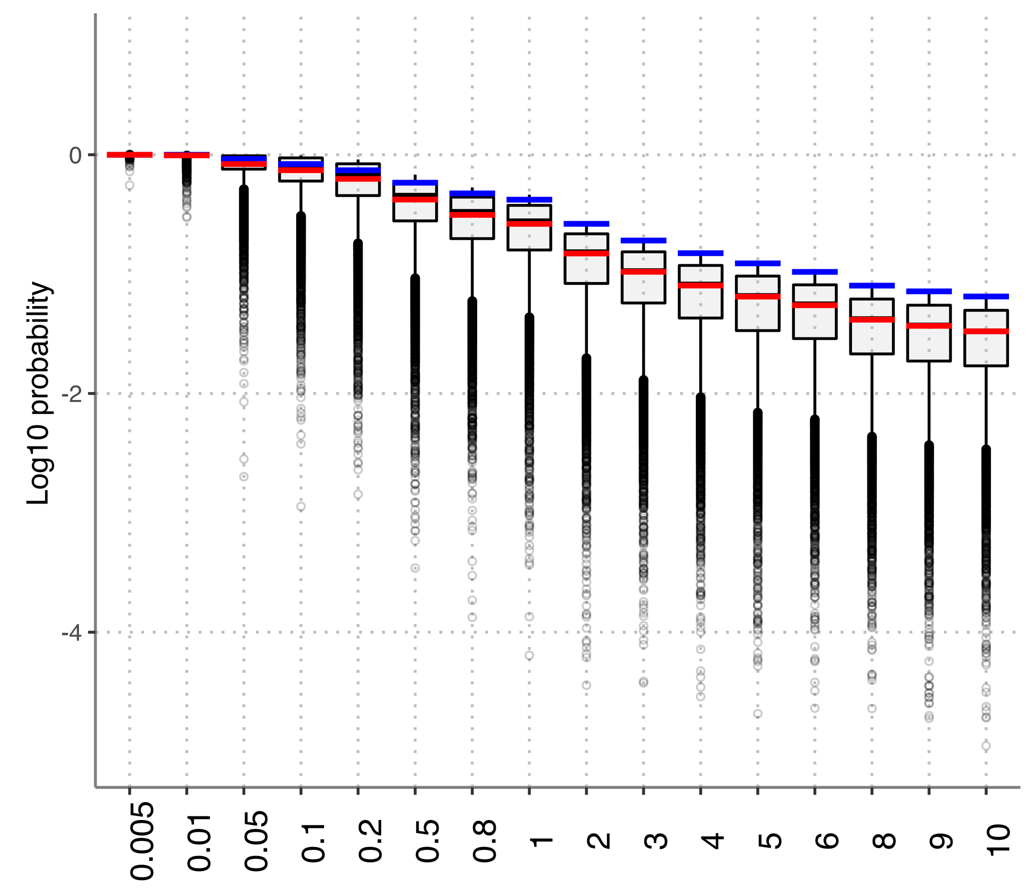


Figure 2: Simulated $\hat{p}_c$ when $\psi \approx 12$, HBR=20m. The blue lines show $p_c(\xi)$, the red the averaged $\hat{p}_c$.

3) Probability of collision has a dilution region, seen in the literature as a 'paradox' in need of clarification, see [1] on false confidence.

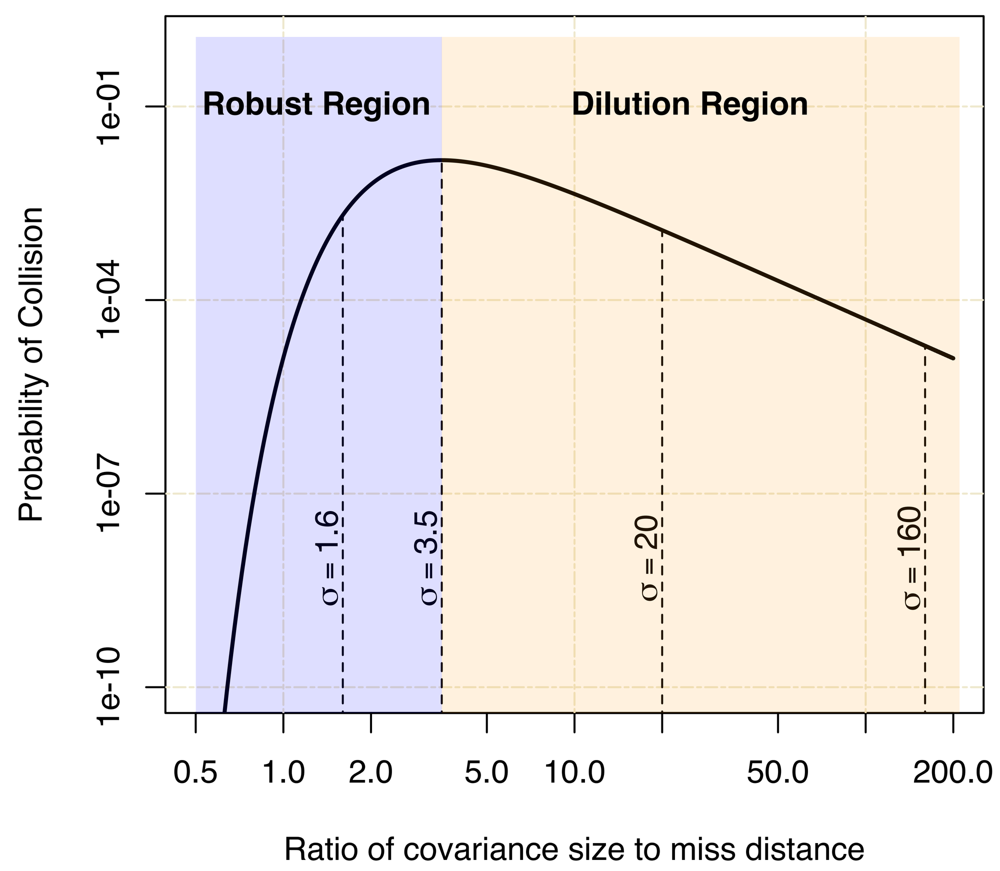


Figure 3: Dilution region of p_c.

2. Inference on the miss distance

2.1 First-order likelihood theory

The measured data in the encounter plane is assumed to be normally distributed $x \sim \mathcal{N}_2(\xi, D)$, where D is a known diagonal matrix. The minimum distance between the two objects is

$$\psi = \|\xi\|\cos\beta,$$

where $\beta \in (\pi/2, 3\pi/2)$. For $\vartheta = (\psi, \lambda)$, the log likelihood is

$$\ell(\vartheta) = -\frac{1}{2}(x-\xi)^{\mathrm{T}}D(x-\xi).$$

Computations are more complicated in the general $6\mathrm{D}$ case.

If ψ scalar, first order inference based on limiting $N(0,1)$ laws of

- ✻ likelihood root, $\quad r(\psi) = \mathrm{sign}(\widehat{\psi}-\psi)[2\{\ell(\widehat{\psi})-\ell(\psi)\}]^{1/2}$,
- ✻ Wald statistic, $\quad t(\psi) = j(\widehat{\psi})^{1/2}(\widehat{\psi}-\psi)$.

as $n \to \infty$ have, for instance,

$$\mathrm{P}\{r(\psi) \leq r_{\text{obs}}\} = \Phi(r_{\text{obs}}) + O\left(n^{-1/2}\right).$$

which yields tests and confidence sets for ψ_0 based on an observed $r_{\text{obs}} = r(\psi_0)$.

2.2 Higher-order likelihood theory

For continuous responses, third-order inference can be based on limiting $N(0,1)$ distribution of the modified likelihood root

$$r^*(\psi) = r(\psi) + \frac{1}{r(\psi)}\log\left\{\frac{q(\psi)}{r(\psi)}\right\},$$

where $q(\psi)$ depends on the model, and for which

$$\mathrm{P}\{r^*(\psi_0) \leq r^*_{\text{obs}}\} = \Phi(r^*_{\text{obs}}) + O\left(n^{-3/2}\right).$$

3. Decision making

Aim for interpretation of significance probabilities to be like $\widehat{p}_c$, we take

$$H_0: \psi = \psi_0 \quad H_A: \psi > \psi_0.$$

We hope that the 'null' will be rejected so we don't have to take evasive action. Under H_0, we expect

$$\text{small } 1-\Phi\{r(\psi_0)\} \equiv \text{ small } \widehat{p}_c.$$

Proposed procedure:

- ➔ choose ψ_0 to represent a 'safe' minimum distance,
- ➔ choose ε (typically 10^{-4} but it could depend on potential losses),
- ➔ if $1-\Phi\{r^*(\psi_0)\} > \varepsilon$, then plot $\Phi\{r^*(\psi)\}$ as measure of evidence about ψ relative to ψ_0,
- ➔ decide whether to take evasive action or not.

4. Example: The 2009 event

- $\widehat{p}_c = 1.14 \times 10^{-5}$ (**no action**)
- Significance probabilities for $\psi_0 = 20\ \mathrm{m}$ are 1.2×10^{-3} (Wald and r) and $7.2 \times 10^{-3}\,(r^*)$ (**take action**).

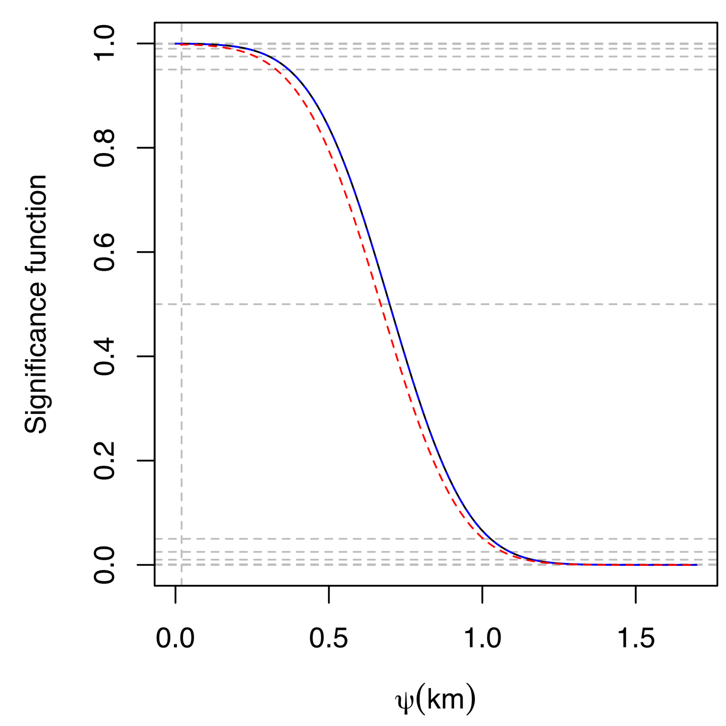


Figure 4: Significance function for the US and Russian satellite collision event.

- ✠ Promising results + can be used in parallel with conventional tools, see [3].
- ✠ Operationally, needs further testing. Conjunction Assessment and Risk Analysis (CARA) group at NASA offered to collaborate.

Acknowledgements

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References

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