Spatial Extremes

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https://www.epfl.ch/labs/stat/

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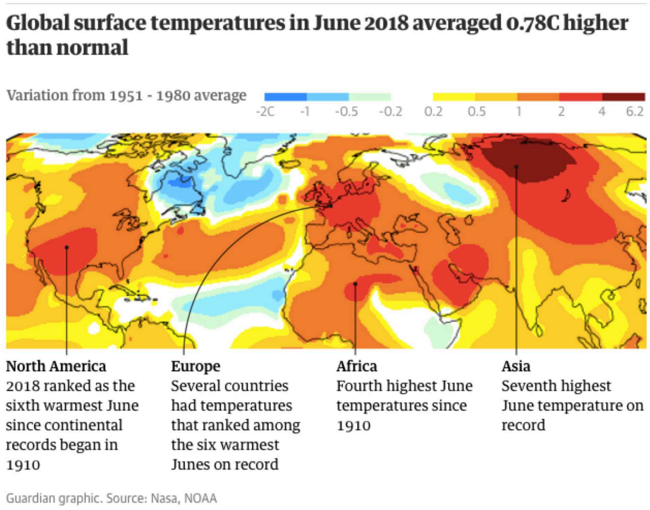
- ☐ Motivation: heatwaves, wildfires, drought, heavy rainfall . . .



Motivations for spatial modelling

- ☐ **Assessment** of significance of ‘hot spots’.
- ☐ **Attribution** of events to possible causes.
- ☐ **Estimation** of changes in extremes of time series, accounting for dependence between related series.
- ☐ **Risk assessment** at a single important site, borrowing strength from sites nearby.
- ☐ **Risk estimation** for large spatial events.

Why specialised models?

- ☐ Basic problem is generally **extrapolation** to rare(r) events.
- ☐ Spatial statistics is mostly based on multivariate normal distributions, inappropriate for modelling tails of distributions, rare events, etc.
- ☐ Extrapolation from a fit to the entire distribution can be misleading:
  - different mechanisms may apply in the extremes
  - different fits to the bulk may give very different tail estimates—in particular, the light tails of the Gaussian density can grossly underestimate probabilities of rare events
  - Gaussian models for multivariate data predict independence of very rare events (‘the formula that killed Wall Street’)
- ☐ Use of standard copulas can deal with transformations to marginal distributions, but not with joint dependence.

Setup

- ☐ Focus on extremes of $Y(x)$ for x in some space or space/time domain $\mathcal{X}$.
- ☐ Aim to estimate probabilities of the form

$$\mathrm{P}\left\{Y(x) \in \mathcal{R}\right\},$$

  where $\mathcal{R}$ represents rare event of interest.
- ☐ Data are available at only a finite subset $\mathcal{X}'$:
  - a few long series (long-term observations, space-poor/time rich)
  - many short series (satellite data, space rich/time poor)
  - many longer series (15-minute radar data, space rich/time rich)

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Extreme-Value Theory 6

Poisson process

- ☐ Random point pattern $\mathcal{P}$ in a state space $\mathcal{E}$ defined by properties of counts

$$N(\mathcal{A}) = |\{x : x \in \mathcal{P} \cap \mathcal{A}\}|, \quad \mathcal{A} \subset \mathcal{E}:$$

  - $N(\mathcal{A}_1), \ldots, N(\mathcal{A}_k)$ independent for disjoint $\mathcal{A}_1, \ldots, \mathcal{A}_k$,
  - $N(\mathcal{A}) \sim \mathrm{Poiss}\{\mu(\mathcal{A})\}$,

  where the measure μ is non-atomic (diffuse), and often has an **intensity** $\dot{\mu}$.
- ☐ **Mapping theorem**: if $g : \mathcal{E} \to \mathcal{E}^*$ does not create atoms, then $\mathcal{P}^* = g(\mathcal{P})$ is also a Poisson process.
- ☐ Restriction of $\mathcal{P}$ to $\mathcal{E}' \subset \mathcal{E}$ is also Poisson.

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Classical extremal models

- Use random sample $X_1, \ldots, X_n \overset{\text{iid}}{\sim} F$ and for $b_n \in \mathbb{R}$ and $a_n > 0$ define point process

$$\mathcal{P}_n = \{(X_j - b_n)/a_n : j = 1, \ldots, n\}, \quad \mathcal{E} = \mathbb{R}.$$

- Then the rescaled maximum $\{\max(X_1, \ldots, X_n) - b_n\}/a_n$ has a non-degenerate limiting distribution iff $\mathcal{P}_n$ converges to a Poisson process with mean measure

$$\Lambda\{(y,\infty)\} = \left(1 + \xi \frac{y - \eta}{\tau}\right)_+^{-1/\xi}, \quad y \in \mathbb{R},$$

where $u_+ = \max(u, 0)$, and η and τ are location and scale parameters.

- The shape parameter ξ determines the rate of tail decay, with
  - $\xi > 0$ giving the heavy-tailed (Fréchet) case,
  - $\xi = 0$ giving the light-tailed (Gumbel) case—corresponds to Gaussian data,
  - $\xi < 0$ giving the short-tailed (reverse Weibull) case.
- Limiting distributions:
  - for maxima, **generalized extreme-value (GEV)**, $G(y) = \exp\{-\Lambda(y)\}$;
  - for excesses over threshold u, **generalized Pareto (GPD)**, $H(y) = 1 - \Lambda(y + u)/\Lambda(u)$.

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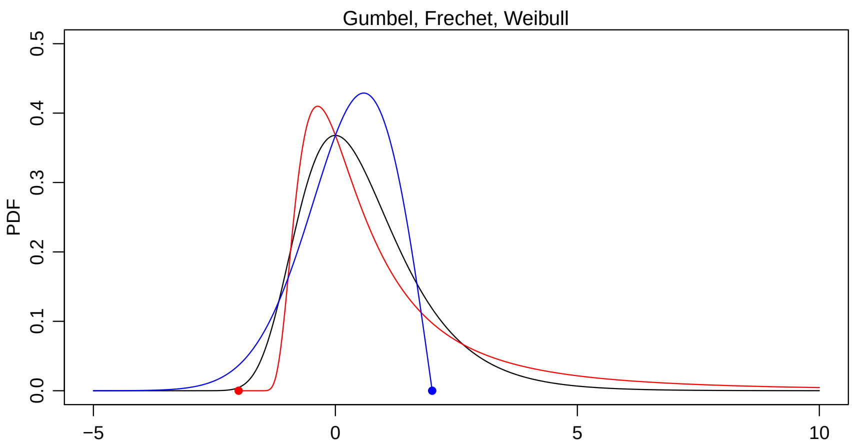
GEV and shape parameter ξ



- PDFs of **Gumbel** ($\xi = 0$), **Fréchet** ($\xi = 0$) and (reverse) **Weibull** ($\xi < 0$).
- The Fréchet is bounded below, and the reverse Weibull is bounded above.
- The standard Weibull is a distribution for minima.

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Extrapolation

- ☐ Extreme value theory gives **limiting** models:
  - – GEV applies for maxima of an infinite sample,
  - – GPD applies for exceedances of an 'infinite' threshold.
- ☐ Extrapolation to high levels is based on the fact that the GEV is **max-stable**:

$$G(y)^t = G(b_t + a_t y), \quad t > 0,$$

or equivalently

$$\max(X_1, \ldots, X_t) \stackrel{\mathrm{D}}{=} b_t + a_t X_1$$

for known functions $a_t > 0$ and b_t.

- ☐ For the standard Fréchet, GEV(1,1,1), distribution, $e^{-1/z}$, $(z > 0)$, we have $b_t \equiv 0$, $a_t = t$.
- ☐ Likewise the GPD is **threshold-stable**.
- ☐ Could fit other models, but with weaker mathematical justification.
- ☐ In practice we have finite samples, so the extremal models are approximate and extrapolation may be vulnerable.
- ☐ Now generalize the above **extremal paradigm** to complex settings ...

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Max-stable processes

- ☐ Can transform maxima to have limiting standard Fréchet distribution, so

$$\max\{Z_1, \ldots, Z_n\} \stackrel{\mathrm{D}}{=} nZ, \quad n = 1, 2, \ldots$$

- ☐ Want processes $Z(x)$ with standard Fréchet margins such that if $Z_1(x), \ldots, Z_n(x) \stackrel{\mathrm{iid}}{\sim} Z(x)$, we can base extrapolation on **max-stability**

$$\max\{Z_1(x), \ldots, Z_n(x)\} \stackrel{\mathrm{D}}{=} nZ(x), \quad x \in \mathcal{X}.$$

- ☐ Let $W(x) \geq 0$ be a random process with $\mathrm{E}\{W(x)\} = 1$ $(x \in \mathcal{X})$, and consider the Poisson process on $\mathbb{R}_+ \times \mathcal{C}_+(\mathcal{X})$:

$$\{(R_j, W_j(x)) : j = 1, 2, \ldots, \}, \quad R_j = (E_1 + \cdots + E_j)^{-1}, \quad E_i \stackrel{\mathrm{iid}}{\sim} \exp(1) \perp\!\!\!\perp W_j \stackrel{\mathrm{iid}}{\sim} W.$$

- ☐ Setting $Q_j(x) = R_j W_j(x)$ gives a Poisson process on $\mathcal{C}_+(\mathcal{X})$, and any **max-stable process** has a **spectral representation** (de Haan, 1984)

$$Z(x) = \sup_{j=1}^{\infty} Q_j(x), \quad x \in \mathcal{X}, \tag{1}$$

with $Q_j(x)$ interpreted as the jth event, with overall size R_j and profile $W_j(x)$.

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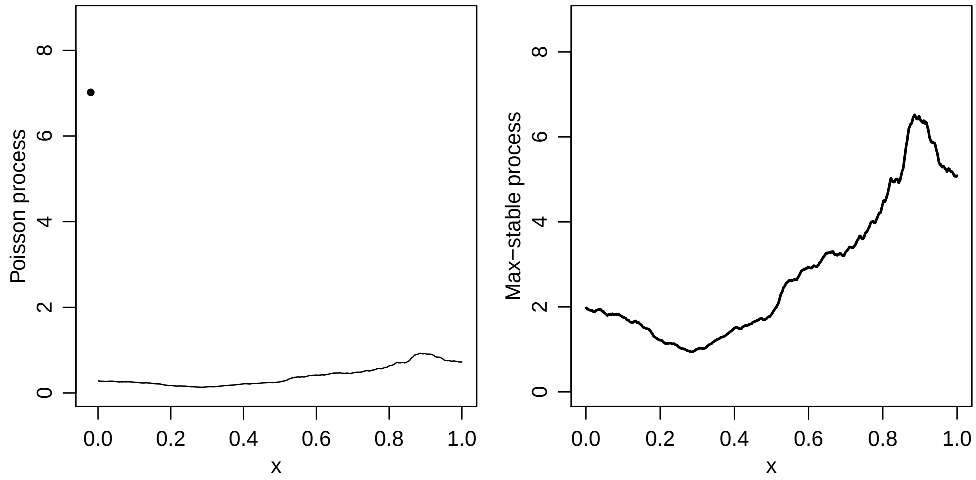
Spectral representation



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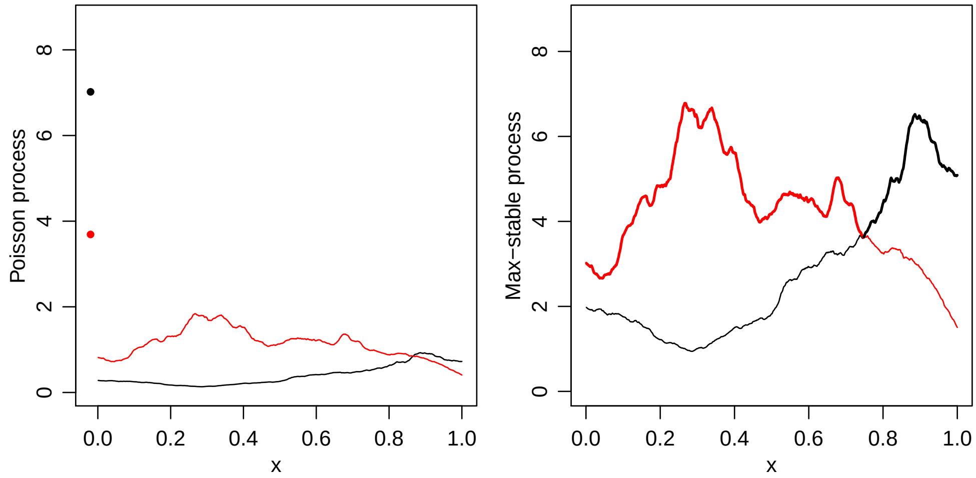
Spectral representation



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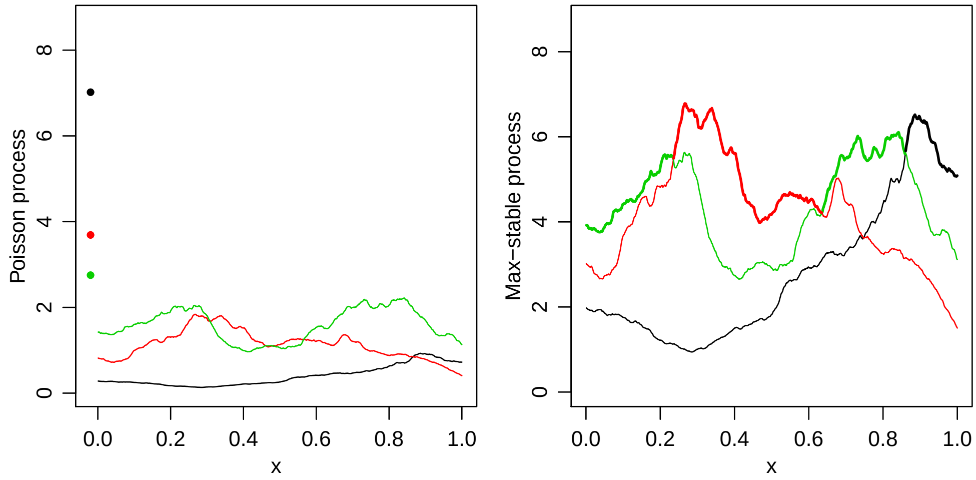
Spectral representation



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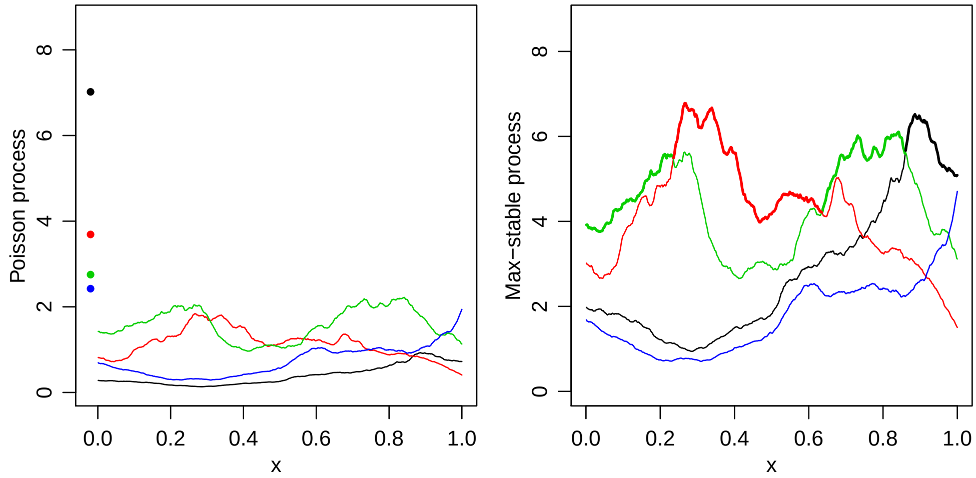
Spectral representation



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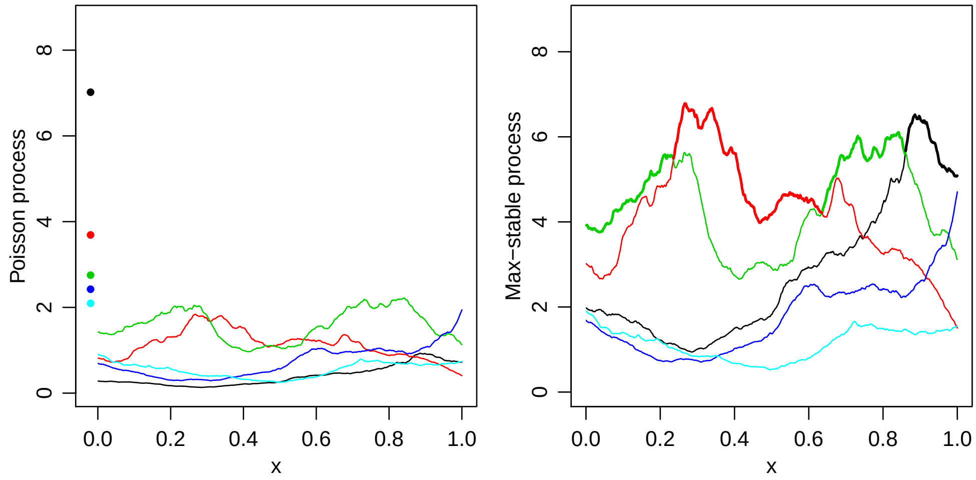
Spectral representation



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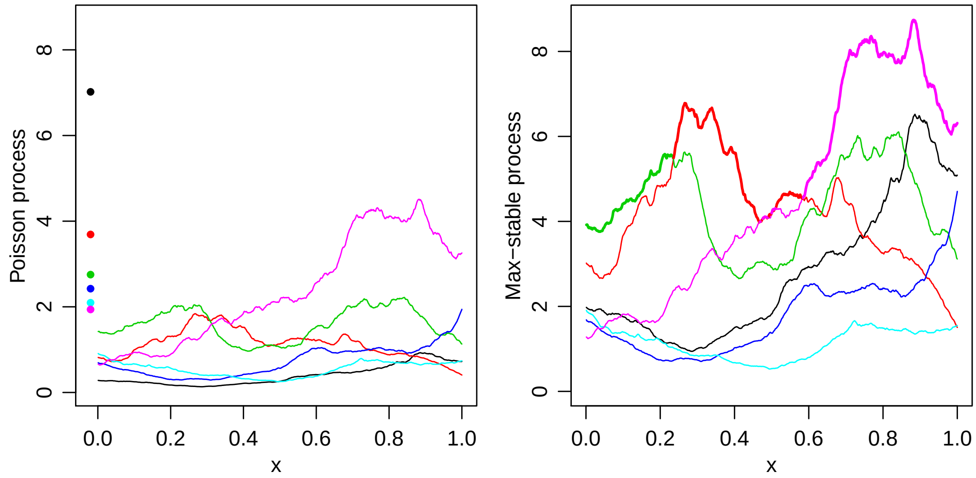
Spectral representation



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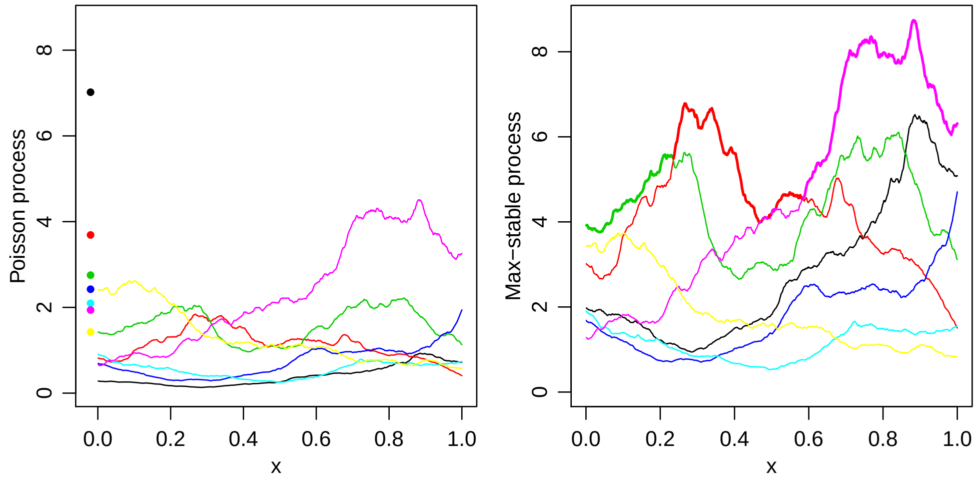
Spectral representation



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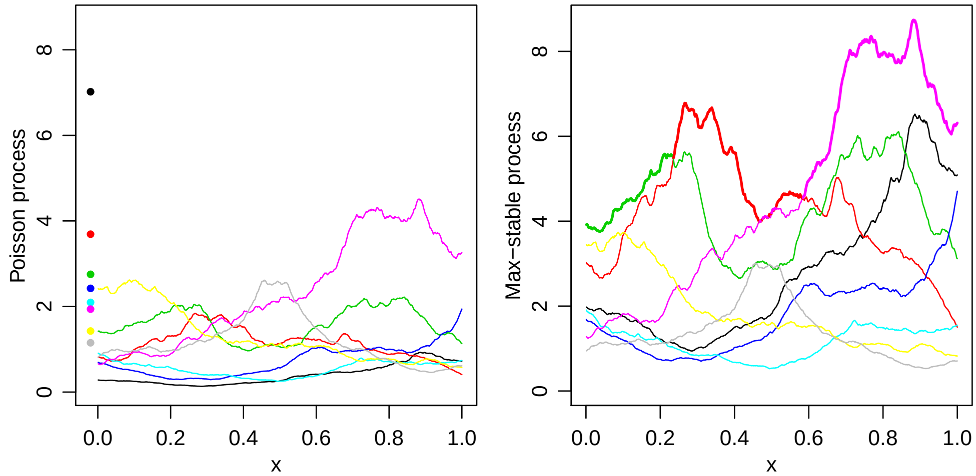
Spectral representation



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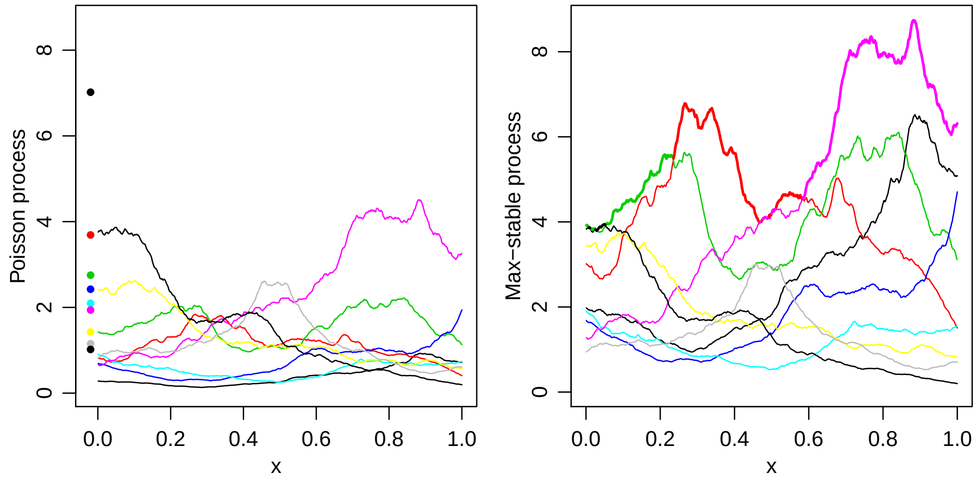
Spectral representation



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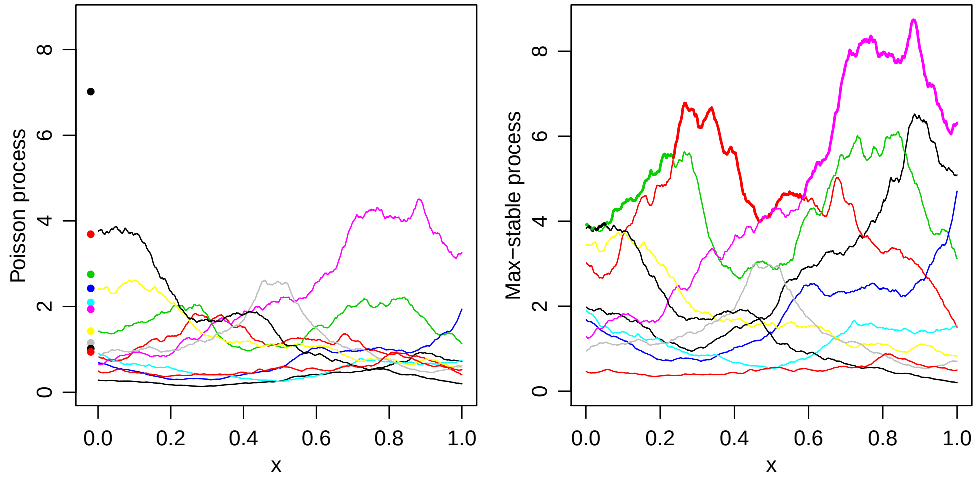
Spectral representation



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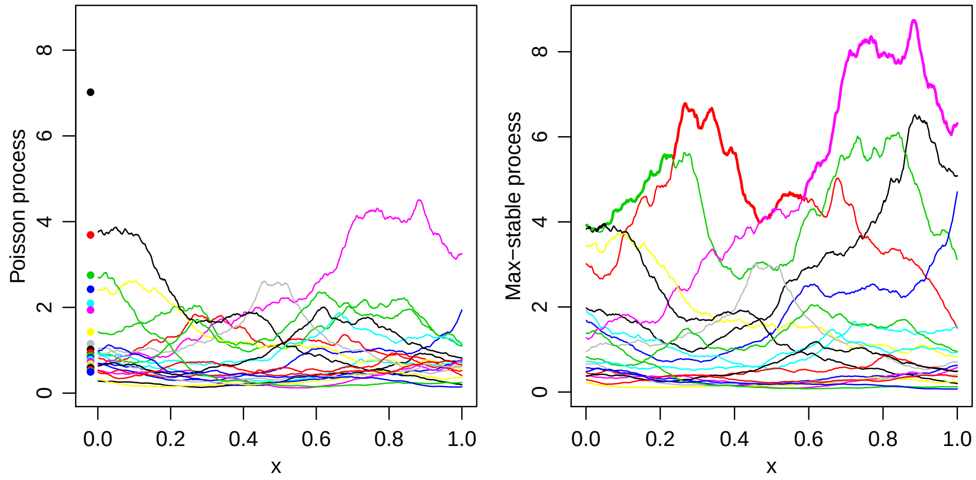
Spectral representation



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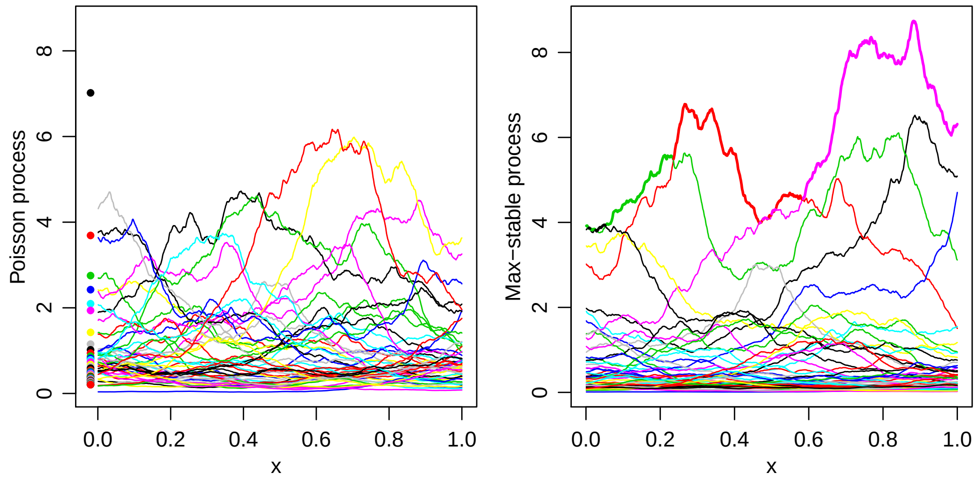
Spectral representation



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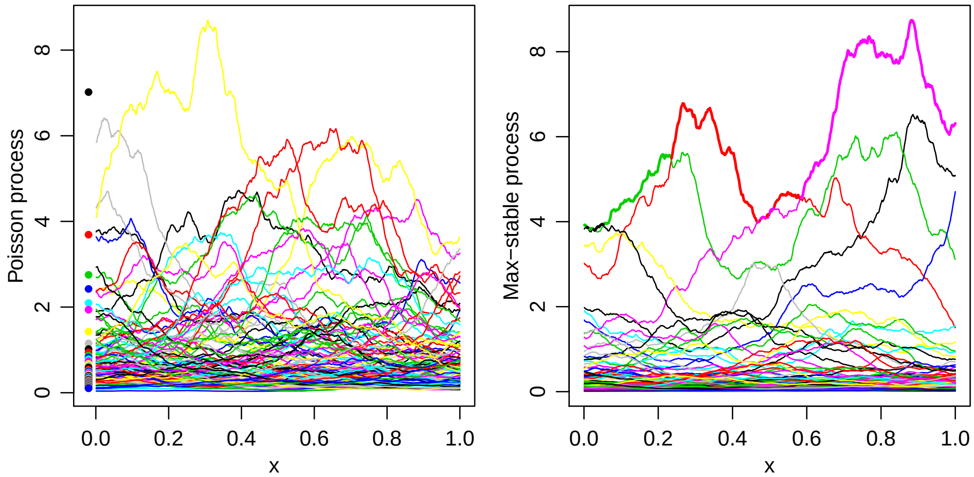
Spectral representation



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Exponent function

- For a function $z(x)$, one can show that

$$\mathrm{P}\left\{Z(x) \le z(x), x \in \mathcal{D}\right\} = \exp\left[-V\{z(x) : x \in \mathcal{D}\}\right], \quad \mathcal{D} \subset \mathcal{X},$$

  where the **exponent function**

$$V\{z(x) : x \in \mathcal{D}\} = \mathrm{E}\left[\sup_{x\in\mathcal{D}}\left\{\frac{W(x)}{z(x)}\right\}\right] = \mu[\{q : q(x) \le z(x), x \in \mathcal{D}\}^c]$$

  is derived from the mean measure μ of the Poisson process $\{Q_j\}$, and expectation is over the '**angular measure**' of W.

- The case $\mathcal{D} = \{x_1, \ldots, x_D\}$ is key to inference, because data are observed on finite sets, and then we write $z_d = z(x_d)$,

$$V(z_1, \ldots, z_D) = \mu(\mathcal{A}_z), \quad \mathcal{A}_z = ([0, z_1] \times \cdots \times [0, z_D])^c \subset \mathcal{E}' = [0, \infty)^D \setminus \{0\}.$$

- μ and V are **homogeneous of order -1**, i.e.,

$$\mu(\mathcal{R}) = t \times \mu(t\mathcal{R}), \quad \mathcal{R} \subset \mathcal{E}, t > 0,$$

  which enables extrapolation by 'pulling down' extreme risk sets $\mathcal{R}$ to observable levels.

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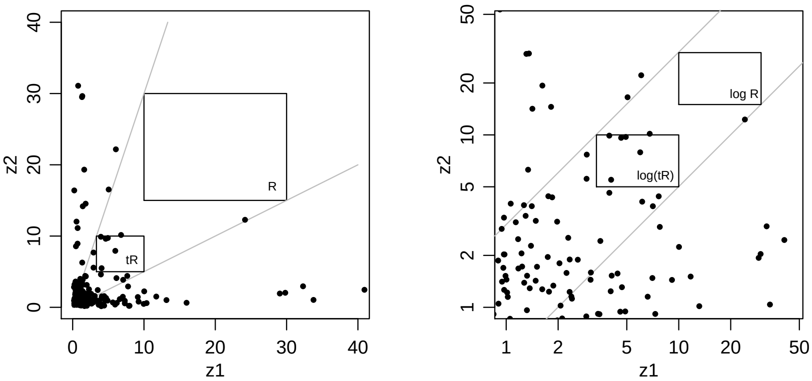
Pulling R down to the origin

Left: point process on unit Fréchet scale, with set $\mathcal{R}$ and its scaled version $\mathcal{R}$.
Right: same, but on Gumbel scale, with logarithmic axes, corresponding to translation of $\log \mathcal{R}$ by $\log t$ towards the origin.



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Extremal coefficient

- ☐ Homogeneity of V yields

$$\mathrm{P}\left\{Z(x) \le z, x \in \mathcal{D}\right\} = \exp\left\{-V_{\mathcal{D}}(z)\right\} = \exp\left\{-V_{\mathcal{D}}(1)/z\right\} = \left(e^{-1/z}\right)^{V_{\mathcal{D}}(1)}, \quad z > 0,$$

  and the **extremal coefficient**

$$\theta_{\mathcal{D}} = V_{\mathcal{D}}(1)$$

  summarises the degree of dependence of extremes within $\mathcal{D}$.

- ☐ The pairwise version,

$$\theta(x, x') = \mathrm{E}\left[\max\left\{W(x), W(x')\right\}\right], \quad x, x' \in \mathcal{X},$$

  can be regarded as an analogue of the correlation coefficient, with

$$\text{(total dependence)} \quad 1 \le \theta(x, x') \le 2 \quad \text{(independence)},$$

  and the conditional probability interpretation

$$\mathrm{P}\left\{Z(x') > z \mid Z(x) > z\right\} \sim 2 - \theta(x, x'), \quad z \to \infty.$$

- ☐ $\theta(x, x')$ is estimated nonparametrically by the **F-madogram** (Cooley *et al.*, 2006).

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Maxima and exceedances

- ☐ Inference may be based on
  - replicates of $\{Z(x) : x \in \mathcal{D}\}$, e.g., annual maximum temperatures at sites in $\mathcal{D}$,
  - individual events $\{Q_j(x) : x \in \mathcal{D}\}$, e.g., hurricanes or droughts.
- ☐ Extremal approximations may be better for maxima, but more detailed modelling is possible based on individual events.
- ☐ Choose 'extreme' events using **risk functional** ρ and retaining only events that fall into

$$\mathcal{E}' = \{q : \rho(q) > 1\}.$$

- ☐ Examples involving threshold function $u(x)$:

$$\rho_1(Q) = \sup_{x\in\mathcal{D}} Q(x)/u(x), \quad \rho_2(Q) = \inf_{x\in\mathcal{D}} Q(x)/u(x), \quad \rho_3(Q) = \int_{\mathcal{D}} Q(x)/u(x)\,\mathrm{d}x.$$

- ☐ Inference based on Poisson process likelihood for $\{q_j : q_j \in \mathcal{E}'\}$ involves $\mu(\mathcal{E}')$, which must be finite and computable.
- ☐ If $\rho(aQ) = a\rho(Q)$ for $a > 0$, then $\rho(Q) > 1$ gives $R\rho(W) > 1$; then $\mu(\mathcal{E}') = \mathrm{E}\{\rho(W)\}$ depends only on the distribution of W.

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Models

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Models

- ☐ Choice of W determines event size, orientation, smoothness, etc., with weak constraints $W \geq 0$ and $\mathrm{E}\{W(x)\} = 1$ for all $x \in \mathcal{X}$.
- ☐ Several choices are based on zero-mean Gaussian process $\varepsilon(x)$ for $x \in \mathcal{X}$ with variogram

$$\gamma(x, x') = \mathrm{var}\{\varepsilon(x) - \varepsilon(x')\}, \quad x, x' \in \mathcal{X},$$

with $\varepsilon(x)$ either intrinsically stationary or stationary; if stationary

$$0 \leq \gamma(x, x') \leq 2\mathrm{var}\{\varepsilon(x)\} = 2\sigma^2,$$

and if intrinsically stationary, then γ is unbounded but

$$\mathrm{cov}\{\varepsilon'(x_1), \varepsilon'(x_2)\} = \tfrac{1}{2}\left\{\gamma(x_1, x') + \gamma(x_2, x') - \gamma(x_1, x_2)\right\},$$

where $\varepsilon'(x) = \varepsilon(x) - \varepsilon(x')$ for some $x' \in \mathcal{X}$.

- ☐ Popular examples are the **Brown–Resnick** and **extremal** t processes (Brown and Resnick, 1977; Kabluchko *et al.*, 2009; Thibaud and Opitz, 2015),

$$W(x) = \exp\left[\varepsilon'(x) - \mathrm{var}\{\varepsilon'(x)\}/2\right], \quad W(x) \propto \varepsilon(x)_+^{\alpha}, \quad \alpha > 0,$$

but skew-Gaussian, skew-t, and hierarchical processes can be constructed (Tawn, 1990; Reich and Shaby, 2012; Reich *et al.*, 2014).

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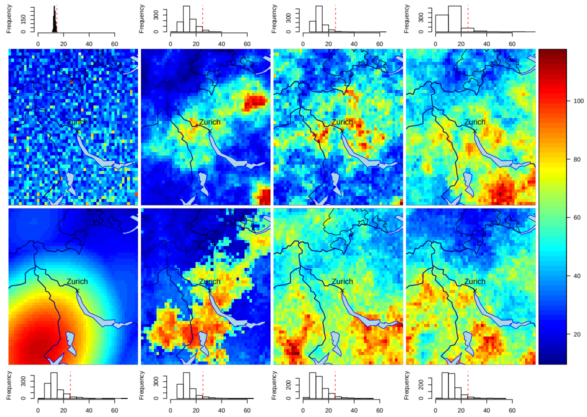
Realisations from spatial models



Top: latent variable, Student t copula, Hüsler–Reiss copula and extremal-t copula models. Bottom: Smith, Schlather, geometric Gaussian and Brown–Resnick models. The histograms are of 1000 realisations of a summary of rainfall centred on Zürich, and the vertical lines correspond to the realizations shown.
(Davison *et al.*, 2012)

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Asymptotic dependence and independence

- ☐ The models above are **asymptotically dependent (AD)**:

$$\mathrm{P}\left\{Z(x') > z \mid Z(x) > z\right\} \sim p(x, x') + O(1/z), \quad z \to \infty,$$

so dependence between extremes persists at all levels: extreme events retain the same properties at all levels.

- ☐ Many applications show **asymptotic independence (AI)**, whereby extreme events become more concentrated as they become rarer.
- ☐ Can model this through an **inverted max-stable process** (Wadsworth and Tawn, 2012)

$$Z'(x) = -1/\log\left[1 - \exp\left\{-1/Z(x)\right\}\right],$$

for which

$$\mathrm{P}\left\{Z'(x') > z \mid Z'(x) > z\right\} = \mathcal{L}(z) z^{1-\theta(x,x')}, \quad z \to \infty,$$

for some slowly-varying function $\mathcal{L}$: the rate of approach to the limiting zero probability can vary.

- ☐ Recent work (Huser and Wadsworth, 2019) combines AD and AI, looks very useful.

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Generalities

- ☐ Extremal models are always mis-specified—inferences likely biased.
- ☐ Must check stability of inferences and possible presence of AI, so vary rarity of chosen events (threshold, ...).
- ☐ Extremal index useful for
  - exploratory and confirmatory analyses based on sub-groups (especially pairs) of observation sites $\mathcal{D}$,
  - simple estimates for model parameters, e.g., by least squares (Buhl and Klüppelberg, 2017).
- ☐ Mainly focus on likelihood-based inferences for parametric models, but also use gradient score.
- ☐ Semiparametric inference preferable, but
  - models are already quite flexible;
  - low power for falsifying models, because data necessarily limited;
  - simulation often needed for risk assessment.

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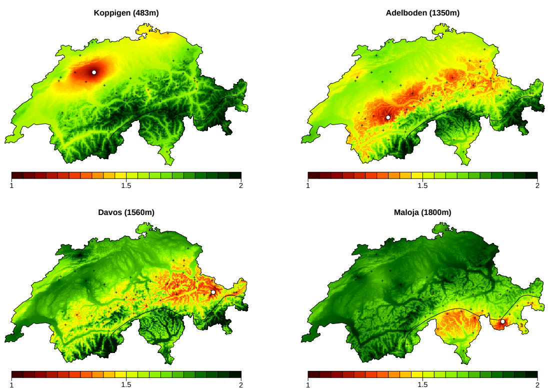
Exploratory procedures

- ☐ Exploratory procedures are mostly based on estimates of $\theta(x_1, x_2)$, with the
  - **extremogram** $2 - \widehat{\theta}(x_1, x_2)$ in time series (Davis and Mikosch, 2009),
  - **F-madogram** in spatial cases.



Blanchet and Davison (2011)

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Margins

- ☐ Extremes $Y(x)$ of original data at $\mathcal{D} = \{x_1, \ldots, x_D\}$ will have GEV/GPD distributions.
- ☐ For maxima, use **marginal transformation**

$$Z(x) = \left\{1 + \xi(x;\vartheta)\frac{Y(x) - \eta(x;\vartheta)}{\tau(x;\vartheta)}\right\}_+^{1/\xi(x;\vartheta)}$$

  to the unit Fréchet scale for inclusion in joint model, with
  - splines for space-varying location, scale and shape parameters,
  - and often constant shape, $\xi(x;\vartheta) \equiv \xi$.
- ☐ Similar transformation for threshold exceedances, with EDF/GPD below/above threshold.
- ☐ Either
  - estimate ϑ first, using **independence likelihood**, and then treat marginal transformation as fixed, or
  - perform **joint estimation** of margins and dependence structure.
- ☐ Balancing good marginal and joint fits can be tricky (easier in some Bayesian formulations).

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Likelihood for maxima

- ☐ Given independent annual maxima observed at $\mathcal{D} = \{x_1, \ldots, x_D\}$ for n years, the maxima for each year have joint distribution

$$\mathrm{P}\{Z(x_1) \le z_1, \ldots, Z(x_D) \le z_D\} = \exp\{-V(z_1, \ldots, z_D)\}, \quad z_1, \ldots, z_D > 0.$$

- ☐ The form of the CDF means that to compute the likelihood we must differentiate e^{-V} with respect to $z_1, \ldots, z_D$, leading to combinatorial explosion:

$$-V_1 e^{-V}, \quad (V_1V_2 - V_{12})e^{-V}, \quad (-V_1V_2V_3 + V_{12}V_3[3] - V_{123})e^{-V}, \quad \ldots,$$

  with about 10^5 terms for $D = 10$. Clearly this is infeasible for realistic applications, so we try to avoid this, by
  - using a composite (usually a pairwise) likelihood;
  - using event timings to determine the required term, e.g., with $D = 3$,

$$(-V_1V_2V_3 \quad + \quad V_{12}V_3 + V_{13}V_2 + V_{23}V_1 \quad - V_{123})e^{-V};$$

  - or using threshold exceedances.
- ☐ We need V and its derivatives, or Poisson process intensity $\dot{\mu}$ and its integrals ...

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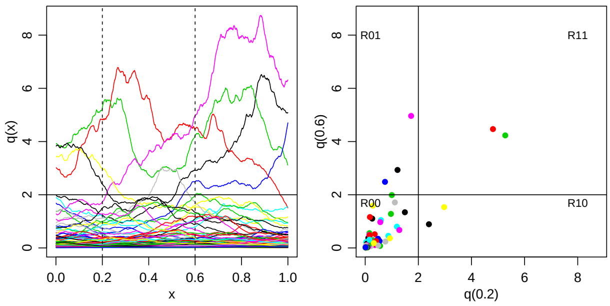
Likelihood for events

- Base extremal modelling on those individual events $q(x)$ falling into $\mathcal{E}' = \{q : \rho(q) > 1\}$, where ρ only uses $q(x)$ for $x \in \mathcal{D}$:
  - allows more detailed modelling and may include more data,
  - if $\mu(\mathcal{E}')$ is readily computed, in principle has simpler likelihood,

$$\exp\left\{-\mu(\mathcal{E}')\right\} \times \prod_{q \in \mathcal{E}'} \dot{\mu}(q), \quad \dot{\mu}(q) = -\frac{\partial^D V(z_1, \ldots, z_D)}{\partial z_1 \cdots \partial z_D},$$

  - but components of some q may be non-extreme, so use a **censored likelihood**.



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Brown–Resnick likelihood

- If $z_d > u$ for $d = 1, \ldots, C$ and $z_d < u$ for $d \in \mathcal{C}' = \{C+1, \ldots, D\}$, and $\mathcal{C} = \{2, \ldots, C\}$, the censored likelihood contribution has form

$$\frac{1}{z_1^2 z_2 \cdots z_C} \times \phi_{C-1}(\log \tilde{z}_{\mathcal{C}}; \tilde{\Omega}_{\mathcal{C},\mathcal{C}}) \times \Phi_{D-C}\left(\tilde{\mu}_{\mathcal{C}'|\mathcal{C}}; \tilde{\Omega}_{\mathcal{C}'|\mathcal{C}}\right),$$

  where ϕ_k and Φ_k denote the k-dimensional normal density and distribution functions, Ω is defined in terms of the variogram γ, and

$$\begin{aligned}
\log \tilde{z}_d &= \log z_d - \log z_1 + \Omega_{d,1}/2, \quad d = 2, \ldots, C, \\
\tilde{\Omega}_{c,d} &= \tfrac{1}{2}\{\Omega_{c,1} + \Omega_{1,d} - \Omega_{c,d}\}, \quad c, d \in \{2, \ldots, D\}, \\
\mu_{\mathcal{C}'|\mathcal{C}} &= (\log u - \log z_1 + \tfrac{1}{2}\Omega_{1,\mathcal{C}'}) - \tilde{\Omega}_{\mathcal{C}',\mathcal{C}}\tilde{\Omega}_{\mathcal{C},\mathcal{C}}^{-1} \log \tilde{z}_{\mathcal{C}}, \\
\tilde{\Omega}_{\mathcal{C}'|\mathcal{C}} &= \tilde{\Omega}_{\mathcal{C}',\mathcal{C}'} - \tilde{\Omega}_{\mathcal{C}',\mathcal{C}}\tilde{\Omega}_{\mathcal{C},\mathcal{C}}^{-1}\tilde{\Omega}_{\mathcal{C},\mathcal{C}'}.
\end{aligned}$$

- Feasible for $D \le 100$, with modified R function for Φ (de Fondeville and Davison, 2018).
- Gradient score needed for higher D:
  - differentiate with respect to data, so normalising constants not needed;
  - use weight function to downweight effects of observations near thresholds.
- Similar computations are possible for extremal-t processes.

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Closing

- ☐ Basic ideas on maxima and point processes extend to spatial and space-time settings.
- ☐ Max-stable processes give asymptotic dependence models—asymptotic independence also seen.
- ☐ Can fit such models using
  - composite (especially pairwise) likelihood,
  - full likelihood (needs additional information, difficult with large D),
  - Bayesian methods, or
  - gradient score methods.
- ☐ Model-checking possible, using simulation from fitted models and other techniques—but difficult to validate far into tails, because of lack of data.
- ☐ Currently much research in area (e.g., threshold models, non-stationarity, downscaling, semiparametric inference, networks, , . . .).

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Some reading

- ☐ Coles (2001), *Introduction to the Statistical Modeling of Extreme Values*, Springer
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