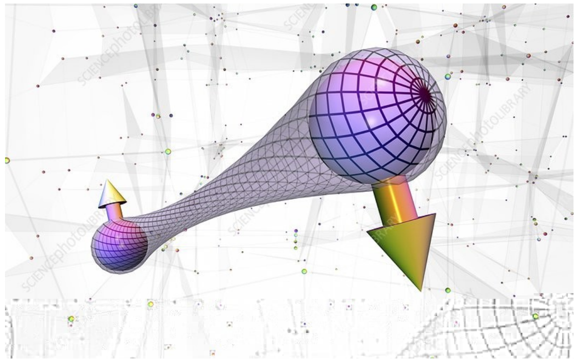
Bell Inequality is violated in $B^0 \to J/\Psi\ K^{0*}(892)$ decay



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Quantum Observables for Collider Physics

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based on: 2305.04982 [hep-ph]

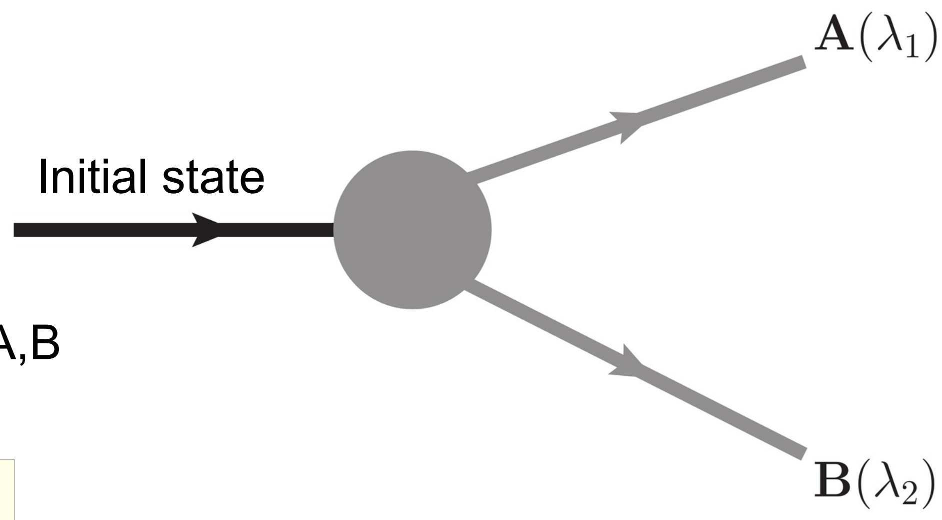
Outlook

- Entanglement as well as Bell inequality expected at high energy
- no test of Bell inequality has been performed at high energy so far
- We focus on **resonant processes** of scalar/pseudoscalars decays
- B meson decays in **two spin-1** particles allow to test Bell inequality
- polarization measurements available for many channels (LHCb, Belle)
- we focus on final states of two spin-1 mesons: K^*, Φ, J/Ψ, ρ, ..
- use experimental results on polarization amplitudes to compute the correlator of Bell inequality for qutrits
- **From LHCb analysis** → first time observation of Bell violation at high energy in B meson decays and in the presence of WEAK and STRONG interactions !

Entanglement and Bell inequalities in spin systems

- Requires the knowledge of the **polarization density matrix** of two-particles **A,B** in production
- it can be fully reconstructed from the angular distributions of the single **A,B decay products**
- **but it can also be computed analytically**

Quantum polarized amplitude $\mathcal{M}(\lambda_1, \lambda_2) \;=$



helicities of A,B

Polarization density matrix ρ

$$\rho(\lambda_1, \lambda_1', \lambda_2, \lambda_2') \;=\; \frac{\mathcal{M}(\lambda_1, \lambda_2)\mathcal{M}^\dagger(\lambda_1', \lambda_2')}{|\overline{\mathcal{M}}|^2}$$

unpolarized

- knowledge of the full polarization density matrix allows to quantify (where possible) entanglement and Bell inequality violations

Quantifying entanglement

in general is a difficult task → simplifies in the case of pure states

- Pure states (or pure ensambles): the system is composed only by one quantum state $|\psi >$

Density matrix $\rho = |\psi\rangle\langle\psi|$ for pure states

- in general we do not have pure states → but mixture of states $|\psi_n\rangle$ *n*=1,2,,...

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n| \; , \; \sum_n p_n = 1 \;\Rightarrow\; \mathrm{Tr}[\rho] = 1$$

$$\langle A\rangle = \mathrm{Tr}[\rho A]$$

→ expectation value of operators (A) → can be computed in any base since Trace is base independent

- For pure states

$$\rho \doteq \begin{pmatrix} 0 & & & & & & & & 0 \\ & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & & & & & \\ & & & & 1 & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & \ddots \\ 0 & & & & & & & & 0 \end{pmatrix}$$

(diagonal form) ⇒

$$\rho^2 = \rho$$

$$\mathrm{Tr}[\rho^2] = 1$$

quantifying entanglement in pure states

2302.00683 [hep-ph]

- If the bipartite (A,B) system is a pure state it is possible to **quantify** its entanglement via

Entropy of entanglement

$$\mathscr{E}[\rho] = -\mathrm{Tr}\left[\rho_A \log \rho_A\right] = -\mathrm{Tr}\left[\rho_B \log \rho_B\right]$$

density matrix of reduced A or B system

equality holds if and only if the bipartite is separable

$$0 \leq \mathscr{E}[\rho] \leq \ln 3$$

(Von Neumann Entropy)

separable states

Maximum entanglement

- Very useful in the case of bipartite (A,B) system of **qutrit** for which there is not a solution for the general case

Two-QUTRITS systems

consider two massive spin-1 particles

Useful to decompose **density matrix** on the basis of tensor products of **Gell-Mann matrices** T^a

9x9 matrix

Unpolarized

$$[A \otimes B]_{ii'jj'} = A_{ii'} B_{jj'}$$

$$\rho(\lambda_1, \lambda_1', \lambda_2, \lambda_2') = \left(\frac{1}{9} [\mathbb{1} \otimes \mathbb{1}] + \sum_a f_a [\mathbb{1} \otimes T^a] + \sum_a g_a [T^a \otimes \mathbb{1}] + \sum_{ab} h_{ab} \left[T^a \otimes T^b \right] \right)_{\lambda_1 \lambda_1', \lambda_2 \lambda_2'}$$

helicities

$$f_a = \frac{1}{6} \mathrm{Tr} \left[\rho \left(\mathbb{1} \otimes T^a \right) \right], \quad g_a = \frac{1}{6} \mathrm{Tr} \left[\rho \left(T^a \otimes \mathbb{1} \right) \right], \quad h_{ab} = \frac{1}{4} \mathrm{Tr} \left[\rho \left(T^a \otimes T^b \right) \right]$$

Spin polarization of particle 1

Spin polarization of particle 2

Spin correlations of particles 1 and 2

- we can also extract them from data, using the angular distributions of their decay products

Bell inequalities for Two-QUTRITS

- take a bi-partite system with components A,B
- perform two measurements on A,B $\longrightarrow$ $(A_1\ ,\ A_2)$ $(B_1\ ,\ B_2)$ each can take values $\rightarrow \{0,1,2\}$
- consider the following correlator $\mathcal{I}_3$ for probability measurements

$$\begin{aligned}\mathcal{I}_3 \;&=\; P(A_1=B_1)+P(B_1=A_2+1)+P(A_2=B_2)+P(B_2=A_1)\\ &\quad -P(A_1=B_1-1)-P(A_1=B_2)-P(A_2=B_2-1)-P(B_2=A_1-1)\end{aligned}$$

$\mathcal{I}_3$ can be expressed as

$$\mathcal{I}_3 = \mathrm{Tr}\left[\rho\,\mathcal{B}\right]$$

with $\mathcal{B}$ a suitable Bell operator

Collins, Gisin, Linden, Massar, Popescu, PRL 88 (2002)

Bell inequalities for two-qutrits

For deterministic local models

$$\mathcal{I}_3 \leq 2$$

QM for qutrits can violate this inequality **with upper bound = 4**

- problem of finding optimal choice has been found

for the maximal entangled state $|\Psi_+\rangle = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} |i\rangle \otimes |i\rangle$ $\rho = |\Psi_+\rangle\langle\Psi_+|$

given by the Bell operator $\mathcal{B}$ (see backup slides)

- Still freedom from Bob and Alice to choose directions where to measure the spin in such a way to maximize $\mathcal{I}_3$

- this can be achieved by performing unitary transformations on the Bell operator as

$$\mathcal{B} \to (U \otimes V)^\dagger \cdot \mathcal{B} \cdot (U \otimes V)$$

→ U,V are unitary 3x3 matrices
(depend on the kinematic of the process)

scalar/pseudoscalar decay in two vectors S $\rightarrow$ $V_1 V_2$

- V_1V_2 comes out as a pure state

Fabbrichesi, Floreanini, EG, Marzola
2304.02403 [hep-ph]
2305.04982 [hep-ph]

Approach based on helicity amplitudes

generic scalar meson (for instance B meson

helicity amplitude

$$h_\lambda = \langle V_1(\lambda) V_2(-\lambda) | \mathcal{H} | B \rangle \quad \text{with} \quad \lambda = (+,\, 0,\, -)$$

From angular momentum conservation

there are only 3 helicity amplitude combination

helicities defined with respect to
z-direction (spin quantization axis)
in rest frame of one of the two Vectors

$$|\Psi\rangle = \frac{1}{\sqrt{|H|^2}} \left[h_+ |\mathbf{V_1}(+)\mathbf{V_2}(-)\rangle + h_0 |\mathbf{V_1}(\mathbf{0})\mathbf{V_2}(\mathbf{0})\rangle + h_- |\mathbf{V_1}(-)\mathbf{V_2}(+)\rangle \right]$$

$$|H|^2 = |h_0|^2 + |h_+|^2 + |h_-|^2$$

Relative weight of |0 0> with respect to |+ - > or |- +> controlled by angular momentum conservation

Density matrix for a pure $V_1 V_2$ state can be written as

Fabbrichesi, Floreanini, EG, Marzola
2304.02403 [hep-ph]
2305.04982 [hep-ph]

$$\rho = |\Psi\rangle\langle\Psi|$$

Master formula

in helicity basis

$$\rho = \frac{1}{|H|^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_+h_+^* & 0 & h_+h_0^* & 0 & h_+h_-^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_0h_+^* & 0 & h_0h_0^* & 0 & h_0h_-^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_-h_+^* & 0 & h_-h_0^* & 0 & h_-h_-^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proved also using most general Lorentz invariant structure for amplitude decay of $S \to V_1 V_2$

$$\frac{h_0}{|H|} = A_0\,, \quad \frac{h_+}{|H|} = \frac{A_\parallel + A_\perp}{\sqrt{2}} \quad \text{and} \quad \frac{h_-}{|H|} = \frac{A_\parallel - A_\perp}{\sqrt{2}}$$

Polarization amplitudes directly measured in experimental analysis of B meson decays !

- One can also compute the polarization density matrix from the most general Lorentz invariant structure of the amplitude $\mathcal{M}$ for **S → V$_1$ V$_2$**

$$\mathcal{M}(\lambda_1,\lambda_2) = M_{\mu\nu}\varepsilon^{\mu\star}(k_1,\lambda_1)\varepsilon^{\nu\star}(k_2,\lambda_2)$$

$$M^{\mu\nu} = A_1\, g^{\mu\nu} + A_2\, k_1^{\nu} k_2^{\mu} + A_3\, \varepsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}$$

$$\rho = \frac{\mathcal{M}_{\mu\nu}\mathcal{M}^{\dagger}_{\mu'\nu'}}{|\overline{\mathcal{M}}|^2}\left[\mathscr{P}^{\mu\mu'}(k_1)\otimes\mathscr{P}^{\nu\nu'}(k_2)\right]$$ omitting helicity indices

where $\mathscr{P}^{\mu\nu}_{\lambda\lambda'}(p)$ is the covariant projector (see backup slides for details)

right-handed basis for spin

$$\mathscr{P}^{\mu\nu}_{\lambda\lambda'}(p) = \frac{1}{3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{M_V^2}\right)\delta_{\lambda\lambda'} - \frac{i}{2M_V}\epsilon^{\mu\nu\alpha\beta}p_\alpha n^i_\beta\,(S_i)_{\lambda\lambda'} - \frac{1}{2}n_i^\mu n_j^\nu\,(S_{ij})_{\lambda\lambda'}$$

Spin-1 matrices

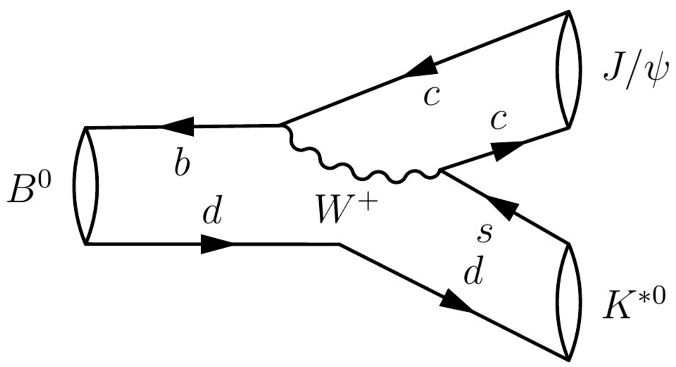
$$S_{ij} = S_iS_j + S_jS_i - \tfrac{4}{3}\mathbb{1}\,\delta_{ij} \qquad i\in\{1,2,3\}$$

- using the relation between the spin matrices S_i , S_{ij} and the Gell-Mann matrices T^a, one can extract the polarization coefficients f_a, g_a, h_{ab} in the Gell-Mann basis

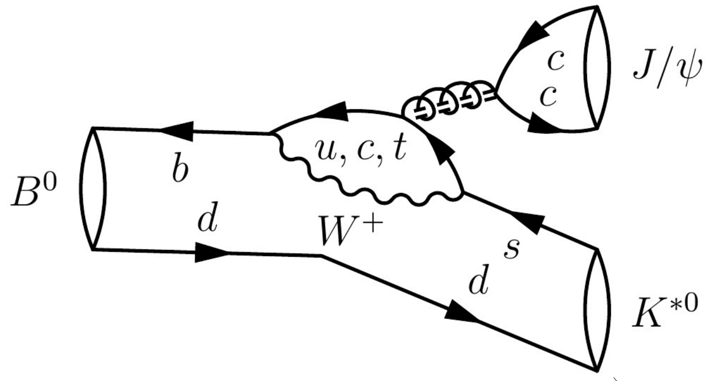
Non-vanishing matrix elements of the density matrix for **S → V$_1$ V$_2$** in the basis of **helicity amplitudes** and in the **Gell-Mann** basis

$$
\begin{aligned}
\hat{h}_-\hat{h}_-^* &= \frac{1}{9}\left[1+3\sqrt{3}\,\left(f_8-2g_8-2h_{38}\right)+9f_3-6h_{88}\right], \\
\hat{h}_0\hat{h}_-^* &= h_{16}+i\,\left(h_{17}-h_{26}\right)+h_{27}\,, \\
\hat{h}_+\hat{h}_-^* &= h_{44}+i\,\left(h_{45}-h_{54}\right)+h_{55}\,, \\
\hat{h}_0\hat{h}_0^* &= \frac{1}{9}\left[1-9\left(f_3+g_3-h_{33}\right)+3\sqrt{3}\,\left(f_8+g_8-h_{38}-h_{83}\right)+3h_{88}\right] \\
\hat{h}_+\hat{h}_0^* &= h_{61}+i\,\left(h_{62}-h_{71}\right)+h_{72}\,, \\
\hat{h}_+\hat{h}_+^* &= \frac{1}{9}\left[1+3\sqrt{3}\,\left(g_8-2f_8-2h_{83}\right)+9g_3-6h_{88}\right],
\end{aligned}
$$

Bell inequality violation in B meson decay $B \to J/\Psi\ K^*$



(a) Tree



(b) Penguin

Smaller contribution, not relevant for Bell inequality, but crucial for FCNC, CP violation tests

Measured by several collaborations, most recent one

R.Aaij et al **[LHCb]**, PRD, 88 (2013) 052002, 1307.2782 [hep-exp]

From each decay

$J/\Psi \to \mu+\ \ \mu-$

$K^* \to K^+\ \pi^-$

We can measure polarization amplitudes

- from polarization amplitudes we can reconstruct the density matrix and compute the $\mathcal{I}_3$ observable for qutrits

- 4 independent parameters: 3 amplitudes modulus and 2 relative phases

Polarization amplitudes

$$A_{\parallel} = |A_{\parallel}|\; e^{i\delta_{\parallel}}$$

$$A_{\perp} = |A_{\perp}|\; e^{i\delta_{\perp}}$$

$$A_0 = |A_0|\; e^{i\delta_0}$$

We choose $\delta_0 = 0$

Phases due to final state interactions

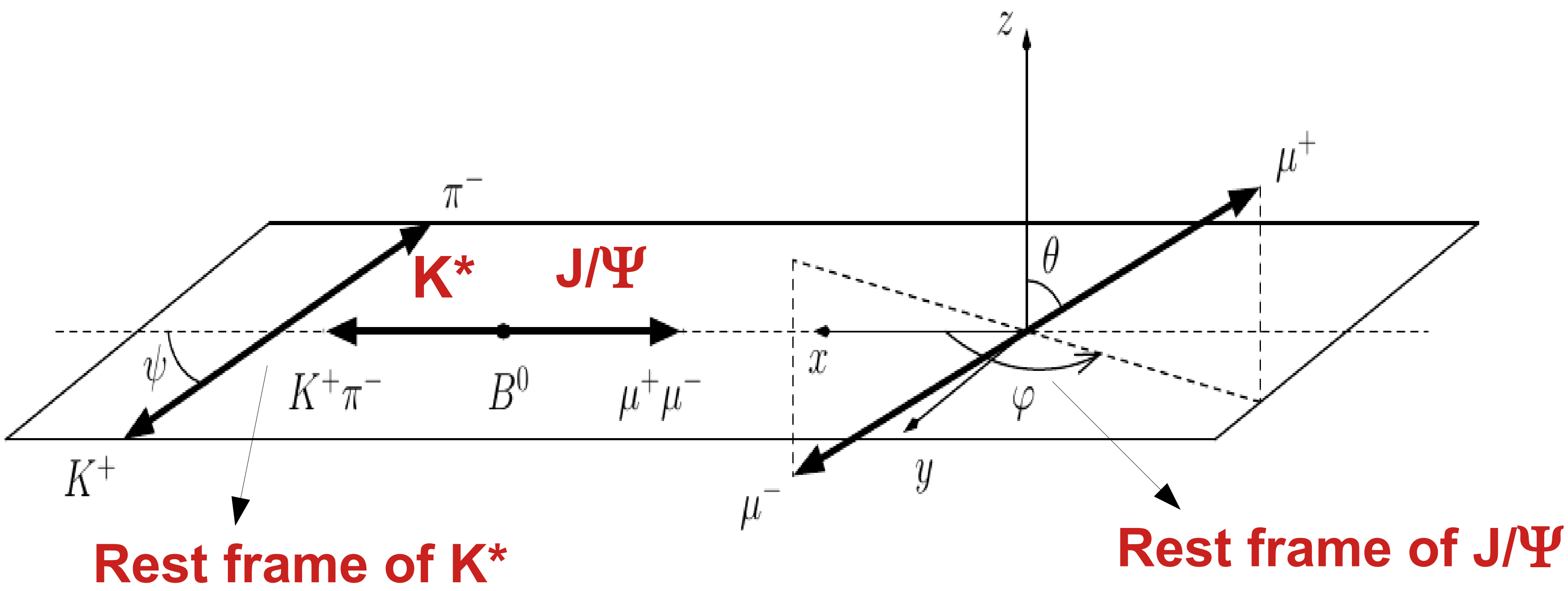
induced by strong interactions due to rescattering

Distance when both have decayed can be estimated to be ~ 1100 fm

Decays take place outside the range of strong interaction at the time of their production ($3\text{x}10^{-5}$ fm gluon exchange) as well as of final state interactions (1.5 fm exchange of pions)

From angular distribution one can extract the polarization amplitudes and phases by a fit of coefficients h_k (see next slide)

$$\frac{d^4\Gamma(B^0 \to J/\psi K^{*0})}{dt\, d\Omega} \propto e^{-\Gamma_d t} \sum_{k=1}^{10} h_k f_k(\Omega) \qquad \Omega = \{\cos\theta, \cos\psi, \varphi\}$$



FIT of coefficients h_k

$$\frac{d^4\Gamma(B^0 \to J/\psi K^{*0})}{dt\, d\Omega} \propto e^{-\Gamma_d t} \sum_{k=1}^{10} h_k f_k(\Omega)$$

$$\Omega = \{\cos\theta, \cos\psi, \varphi\}$$

1307.2782 [hep-exp]

Dighe, Dunietz, Lipkin, Rosner, PLB 369 144 (1996), **9511363 [hep-ph]**

k	h_k		$f_k(\Omega)$
1	$\|A_0\|^2$	→	$\frac{9}{32\pi} 2\cos^2\psi(1-\sin^2\theta\cos^2\varphi)$
2	$\|A_\parallel\|^2$	→	$\frac{9}{32\pi}\sin^2\psi(1-\sin^2\theta\sin^2\varphi)$
3	$\|A_\perp\|^2$	→	$\frac{9}{32\pi}\sin^2\psi\sin^2\theta$
4	$\|A_\parallel\|\|A_\perp\|\sin(\delta_\perp-\delta_\parallel)$	→	$-\frac{9}{32\pi}\sin^2\psi\sin 2\theta\sin\varphi$
5	$\|A_0\|\|A_\parallel\|\cos(\delta_\parallel)$	→	$\frac{9}{32\pi\sqrt{2}}\sin 2\psi\sin^2\theta\sin 2\varphi$
6	$\|A_0\|\|A_\perp\|\sin(\delta_\perp)$	→	$\frac{9}{32\pi\sqrt{2}}\sin 2\psi\sin 2\theta\cos\varphi$
7	$\|A_S\|^2$		$\frac{3}{32\pi}2(1-\sin^2\theta\cos^2\varphi)$
8	$\|A_\parallel\|\|A_S\|\cos(\delta_\parallel-\delta_S)$		$\frac{3}{32\pi}\sqrt{6}\sin\psi\sin^2\theta\sin 2\varphi$
9	$\|A_\perp\|\|A_S\|\sin(\delta_\perp-\delta_S)$		$\frac{3}{32\pi}\sqrt{6}\sin\psi\sin 2\theta\cos\varphi$
10	$\|A_0\|\|A_S\|\cos(\delta_S)$		$\frac{3}{32\pi}4\sqrt{3}\cos\psi(1-\sin^2\theta\cos^2\varphi)$

(A_S contribution from non-resonant J/Y K* amplitude)

Using experimental analysis from LHCb

1307.2782 [hep-exp]

$$|A_{\parallel}|^2 = 0.227 \pm 0.004\ (\text{stat}) \pm 0.011\ (\text{sys})$$
$$|A_{\perp}|^2 = 0.201 \pm 0.004\ (\text{stat}) \pm 0.008\ (\text{sys})$$
$$\delta_{\parallel}\ [\text{rad}] = -2.94 \pm 0.02\ (\text{stat}) \pm 0.03\ (\text{sys})$$
$$\delta_{\perp}\ [\text{rad}] = 2.94 \pm 0.02\ (\text{stat}) \pm 0.02\ (\text{sys})\,,$$

+ error matrix correlation (included)

$$|A_0|^2 + |A_{\perp}|^2 + |A_{\parallel}|^2 = 1$$

$$\mathcal{I}_3 = \text{Tr}[\rho\,\mathcal{B}]$$

maximized using $\mathcal{B} \to (U \otimes V)^{\dagger} \cdot \mathcal{B} \cdot (U \otimes V)$

(see backup slides for U,V)

classical locality requires $\mathcal{I}_3 < 2$

2305.04982 [hep-ph]

Bell inequality → $$\mathcal{I}_3 = 2.548 \pm 0.015$$ ⇨ $B^0 \to J/\psi\, K^*(892)^0$

$\mathcal{I}_3 < 2$ is violated with a significance of 36σ

First time violation of Bell inequalities in qutrits and in the presence of weak and strong interactions

Quantum entanglement and Bell violation in other B decays

2305.04982 [hep-ph]

0< von Neumann entropy < ln[3] ~ 1.1

$$\mathcal{I}_3 = \mathrm{Tr}\left[\rho\, \mathcal{B}\right]$$

	$\mathscr{E}$	$\mathcal{I}_3$	
• $B^0 \to J/\psi\, K^*(892)^0$ [5]	0.756 ± 0.009	2.548 ± 0.015	
• $B^0 \to \phi\, K^*(892)^0$ [20]	$0.707 \pm 0.133^*$	$2.417 \pm 0.368^*$	→ 1.1σ
• $B^0 \to \rho\, K^*(892)^0$ [21]	$0.450 \pm 0.067^*$	$2.208 \pm 0.129^*$	→ 1.6σ
• $B_s \to \phi\,\phi$ [22]	$0.734 \pm 0.050^*$	$2.525 \pm 0.084^*$	→ 6.2σ
• $B_s \to J/\psi\, \phi$ [23]	0.731 ± 0.032	2.462 ± 0.080	→ 5.8σ

- all these decays show a high level of entanglement
- values with * refer to lower bound on errors (because correlation matrix is not available)

significance on Bell inequality should increase after including error correlations

[20] K. F. Chen *et al.* [Belle], Phys. Rev. Lett. **94**, 221804 (2005) [arXiv:hep-ex/0503013 [hep-ex]].

[21] R. Aaij *et al.* [LHCb], JHEP **05**, 026 (2019) [arXiv:1812.07008 [hep-ex]].

[22] R. Aaij *et al.* [LHCb], [arXiv:2304.06198 [hep-ex]].

[23] G. Aad *et al.* [ATLAS], Eur. Phys. J. C **81**, no.4, 342 (2021) [arXiv:2001.07115 [hep-ex]].

[5] R. Aaij *et al.* [LHCb], Phys. Rev. D **88**, 052002 (2013) [arXiv:1307.2782 [hep-ex]].

Closing the locality loophole in B $\to \Phi\Phi$

- One must consider the case where the decay products are the same particles (same width)
- B $\to \Phi\Phi$ decay offers this possibility

events satisfying the space-like condition ⇒

$$\frac{|t_1 - t_2|\, c}{(t_1 + t_2)\, v} < 1$$

$t_{1,2}$ → time of decays

Using the fact that decay times follow the PDF distribution in time (t) → $P(t) \sim \exp[- t\, \Gamma / \gamma\,]$

Γ → the particle width at rest
γ → the Lorentz factor

- we find that almost 90% of events satisfies this condition
- Locality loophole can be closed

Summary

- Bell inequality is violated in B meson decay $B \rightarrow J/\Psi K^*$

- First time observation (with 36σ level significance) in Weak and Strong interactions and at high energies (~ GeV) and in qutrit systems

- Observed also in other B mesons decays (with less significance)

- $B \rightarrow \phi \phi$ violates Bell inequality at 6σ level and it is also locality loophole free $\rightarrow$ higher significance could be achieved if error correlations are included

- Final remark: LHCb experiment was not designed to test the Bell inequalities and quantum entanglement

Backup slides

- Bell operator on the basis of S_3 spin operator

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 2 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 2 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Maximization of $\mathcal{I}_3$ achieved by $\mathcal{B} \to (U \otimes V)^\dagger \cdot \mathcal{B} \cdot (U \otimes V)$

- approximated matrices up to 1% for the central value of $\mathcal{I}_3$ for $B^0 \to J/\psi\, K^*$

$$U = \begin{pmatrix} \frac{5}{14} - \frac{8i}{13} & \frac{1}{20} - \frac{4i}{7} & \frac{1}{4} - \frac{i}{3} \\ 0 & \frac{1}{18} - \frac{7i}{12} & -\frac{7}{15} + \frac{11i}{17} \\ -\frac{4}{11} + \frac{3i}{5} & \frac{1}{17} - \frac{4i}{7} & \frac{1}{4} - \frac{i}{13} \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{5}{12} - \frac{i}{99} & -\frac{2}{9} + \frac{8i}{15} & \frac{1}{51} + \frac{5i}{7} \\ -\frac{4}{5} - \frac{i}{38} & \frac{3}{13} - \frac{6i}{11} & 0 \\ -\frac{5}{12} - \frac{i}{59} & -\frac{3}{13} + \frac{9i}{17} & -\frac{5i}{7} \end{pmatrix}$$

covariant polarization vector of spin-1

$$\varepsilon^\mu(p,\lambda) = -\frac{1}{\sqrt{2}}|\lambda|\,(\lambda\, n_1^\mu + i\, n_2^\mu) + \Big(1-|\lambda|\Big) n_3^\mu$$

helicity $\lambda = \pm 1, 0$

rest frame limit

$$\varepsilon^\mu(p,\lambda) \xrightarrow[(\beta\to 0)]{} \psi_\pm\,,\ \psi_0$$

β is the velocity

boosted base

$$n_1^\mu = (0,\,\hat{\mathbf{n}})\,,\ \ n_2^\mu = (0,\,\hat{\mathbf{r}})\,,\ \ n_3^\mu = \frac{E}{M}(\beta,\,\hat{\mathbf{k}})$$

Covariant Projector

$$\mathscr{P}^{\mu\nu}_{\lambda\lambda'}(p) \;=\; \varepsilon^\mu(p,\lambda)^\star \varepsilon^\nu(p,\lambda')$$

$$\varepsilon^{0123} = 1$$

master formula

$$= \frac{1}{3}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}\right)\delta_{\lambda\lambda'} - \frac{i}{2M}\epsilon^{\mu\nu\alpha\beta}p_\alpha n^i_\beta\,(S_i)_{\lambda\lambda'} - \frac{1}{2}n_i^\mu n_j^\nu\,(S_{ij})_{\lambda\lambda'}$$

$S_i,\ i\in\{1,2,3\}$ → rotation matrices for spin-1 particle

H.S. Song, Lett. Nuovo Cim. 25 (1979)
S.Y. Choi, T. Lee, H.S. Song, PRD 40 (1989)
Fabbrichesi, Floreanini, EG, Marzola, 2302.00683 [hep-ph]

(see backup slides)

$$S_{ij} = S_iS_j + S_jS_i - \frac{4}{3}\mathbb{1}\,\delta_{ij}$$

basis correspondence for $(S_i)_{\lambda\lambda'}$

$$|+\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix},\quad |0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix},\quad |-\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

corresponding to eigenevalues $\lambda = \pm 1, 0$

Spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Expressed as a function of Gell-Mann matrices

$$S_1 = \frac{1}{\sqrt{2}}\left(T^1 + T^6\right), \quad S_2 = \frac{1}{\sqrt{2}}\left(T^2 + T^7\right), \quad S_3 = \frac{1}{2} T^3 + \frac{\sqrt{3}}{2} T^8$$

$$\begin{aligned}
S_{31} = S_{13} &= \frac{1}{\sqrt{2}}\left(T^1 - T^6\right), \\
S_{12} = S_{21} &= T^5, \\
S_{23} = S_{32} &= \frac{1}{\sqrt{2}}\left(T^2 - T^7\right) \\
S_{11} &= \frac{1}{2\sqrt{3}} T^8 + T^4 - \frac{1}{2} T^3, \\
S_{22} &= \frac{1}{2\sqrt{3}} T^8 - T^4 - \frac{1}{2} T^3, \\
S_{33} &= T^3 - \frac{1}{\sqrt{3}} T^8,
\end{aligned}$$

Gell-Mann basis

$$T^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$T^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$