Entanglement and Entropy in Multipartite Systems:
a Useful Approach

Collab. with
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ift
Madrid

Based on: ArXiv [2307.05205]

This is not a HEP talk

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But the notion of multipartite entanglement is relevant for systems with many particles and, maybe, QFT.

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Also relevant in:

- Spin chains
- Q-Cryptography
- ⋮
- Biological systems

2-partite system

$$\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$$

$|\psi\rangle$ is entangled if $|\psi\rangle \neq \underbrace{|\varphi_1\rangle}_{\in \mathscr{H}_A} \otimes \underbrace{|\varphi_2\rangle}_{\in \mathscr{H}_B}$

2-partite system

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$|\psi\rangle$ is entangled if $|\psi\rangle \neq |\varphi_1\rangle \otimes |\varphi_2\rangle$

$\in \mathscr{H}_A \qquad \in \mathscr{H}_B$

☆ *Criterion:* $|\psi\rangle$ is entangled *iff* $\mathrm{tr}\,\rho_A^2 \neq 1$

with $\rho_A = \mathrm{tr}_B\,\rho = \mathrm{tr}_B |\psi\rangle\langle\psi|$

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☆ ***Concurrence****: a practical measure of entanglement*

$$C^2_{A|B} = 2(1 - \mathrm{tr}\,\rho_A^2)$$

$$\boxed{|\psi\rangle \text{ is entangled } \textit{iff } C^2_{A|B} \neq 0}$$

Genuine Entanglement

N-partite system

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_N$$

$|\psi\rangle$ is genuinely entangled if it is entangled with respect to any bipartition of the system

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$|\psi\rangle$ is genuinely entangled if it is entangled with respect to any bipartition of the system

→ The state of any part of the system cannot be described without referring to the other parts of the system

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \mathscr{H}_3 \otimes \cdots \otimes \mathscr{H}_N$$

Bases: $\{\,|i\rangle\}$ $\{\,|j\rangle\}$ $\{\,|k\rangle\}$ $\cdots$

A bipartition $\mathscr{H}_A \otimes \mathscr{H}_{\widehat{A}}$ is defined by a set of subsystems, A, and the complementary, $\widehat{A}$:

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Concurrence of the bipartition:

$$C^2_{I|\widehat{I}} = 2(1 - \mathrm{tr}\ \rho_A^2) \qquad \text{with } \rho_A = \mathrm{tr}_{\widehat{I}}\ \rho$$

The bipartition $I\,|\,\widehat{I}$ is entangled *iff* $C^2_{I|\widehat{I}} \neq 0$

Note: In order to assess whether a state is genuinely entangled we need to evaluate $2^{N-1} - 1$ concurrences, $C^2_{I|\widehat{I}}$, $I \in \{i, j, k, \cdots\}$ and check that all of them are $\neq 0$

Computationally a formidable problem

see Latorre's lectures

The entanglements of bipartitions are not independent each other

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For a tripartition:

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \mathscr{H}_3$$

☆ Triangular inequality:

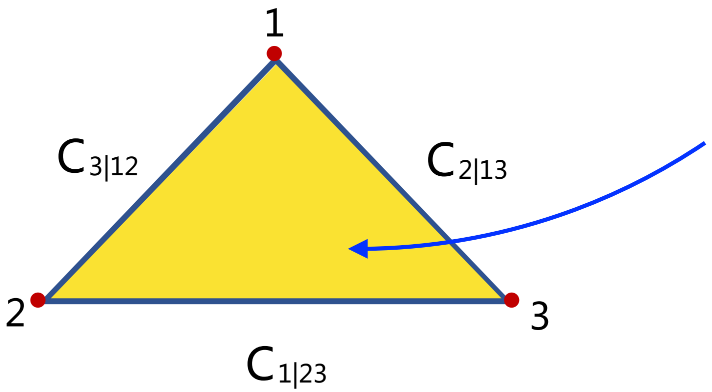
$$C_{1|23} \leq C_{2|13} + C_{3|12}$$

☆ Triangular inequality (for squares):

$$C_{1|23}^2 \leq C_{2|13}^2 + C_{3|12}^2$$

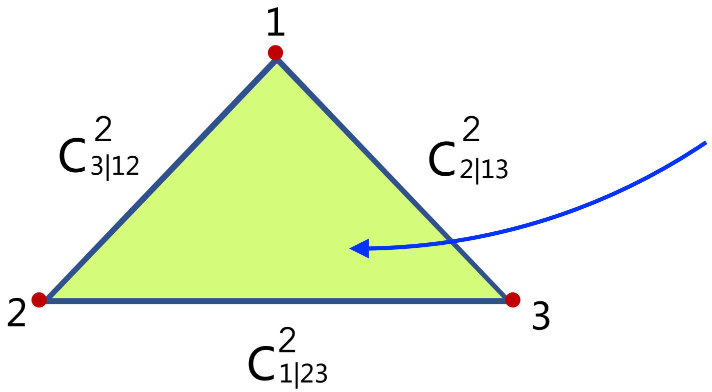
Coffman, Kundu, Wootters, 1999

Triangle measure of tri-partite entanglement



Area of the triangle of concurrences

Jin et al. 2022



Area of the triangle of squared concurrences

Xie, Eberly 2021

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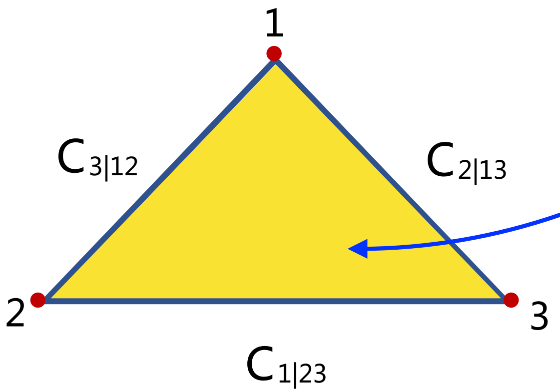

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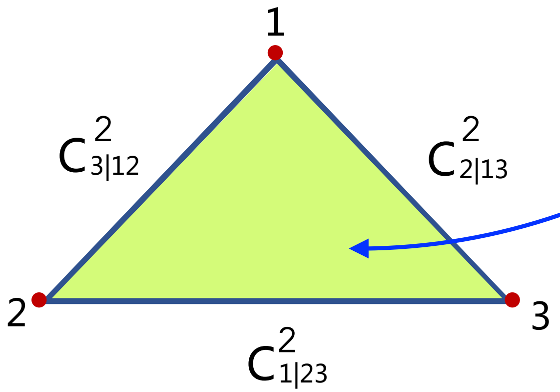

Area of the triangle of squared concurrences

Xie, Eberly 2021

Only if

$$C^2_{ij|\widehat{ij}} = C^2_{i|\widehat{i}} + C^2_{j|\widehat{j}} \quad \Longleftrightarrow \quad C_{i|\widehat{i}} = 0 \text{ or } C_{j|\widehat{j}} = 0$$

<u>proven for 3 qubits</u> Xie, Eberly 2021

Concurrence Vector

Concurrence Vector

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \mathscr{H}_3 \otimes \cdots \otimes \mathscr{H}_N$$

$$\{\,|i\rangle\} \quad \{\,|j\rangle\} \quad \{\,|k\rangle\} \quad \bullet \bullet \bullet$$

Consider a bipartition $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_{\widehat{A}}$

$A \longrightarrow$ subset of indices $I \in \{i, j, k, \cdots\}$

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A state $|\psi\rangle = \sum_{i,j,k\ldots} a_{ijk\ldots} |i\rangle |j\rangle |k\rangle \cdots$

is separable *iff* $|\psi\rangle = \left(\sum_I \alpha_I |I\rangle \right) \otimes \left(\sum_{\widehat{I}} \beta_{\widehat{I}} |I\rangle \right)$

$$\Longrightarrow \quad a_{ijk\ldots} = a_{I\widehat{I}} = \alpha_I \beta_{\widehat{I}} \quad \Longrightarrow \quad \mathrm{rank}\ \{a_{I\,\widehat{I}}\} = 1$$

Concurrence Vector

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$\Longrightarrow$ All minors $[a]_{\{I_1 I_2\}\{\widehat{I}_1\widehat{I}_2\}} = a_{I_1\widehat{I}_1}a_{I_2\widehat{I}_2} - a_{I_1\widehat{I}_2}a_{I_2\widehat{I}_1} = 0$

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Define the ***concurrence vector:***

$$\boxed{\vec{C}_{I|\widehat{I}} = \{[a]_{\{I_1 I_2\}\{\widehat{I}_1 \widehat{I}_2\}}\} = \{a_{I_1 \widehat{I}_1} a_{I_2 \widehat{I}_2} - a_{I_1 \widehat{I}_2} a_{I_2 \widehat{I}_1}\}}$$

Badziag, Horodecki, Horodecki, Horodecki
Audenaert, Verstraete, Moor
Akhtarshenas
Lie, Zhu

$$\dim \ \vec{C}_{I|\widehat{I}} = D^2 \ , \text{ with } \ D = \dim \mathscr{H}$$

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$$\boxed{\|\vec{C}_{I|\widehat{I}}\|^2 = C^2_{I|\widehat{I}}}$$

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Why use long concurrence vectors, $\overrightarrow{C}_{I|\widehat{I}}$, instead of the simple concurrences, $C^2_{I|\widehat{I}}$?

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Certainly, if the only merit of $\overrightarrow{C}_{I|\widehat{I}}$ were that $\|\overrightarrow{C}_{I|\widehat{I}}\|^2 = C^2_{I|\widehat{I}}$, it would make no sense.

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Certainly, if the only merit of $\overrightarrow{C}_{I|\widehat{I}}$ were that $\|\overrightarrow{C}_{I|\widehat{I}}\|^2 = C^2_{I|\widehat{I}}$, it would make no sense.

The concurrence-vectors are much more powerful than the concurrence-values to explore the connection between entanglements.

A useful expression for $\vec{C}_{I|\widehat{I}}$

Bernal, JAC, Moreno 2023

For a state $|\psi\rangle = \sum_{i,j,k...} a_{ijk...} |i\rangle |j\rangle |k\rangle \cdots$

define $\overrightarrow{A}$ *as the vector of coefficients of* $|\psi\rangle \otimes |\psi\rangle \in \mathscr{H} \otimes \mathscr{H}$

$$A_{i_1 j_1 k_1 ...;\ i_2 j_2 k_2 ...} = (a_{i_1 j_1 k_1 ...})\,(a_{i_2 j_2 k_2 ...}) \qquad \dim \overrightarrow{A} = D^2$$

Under permutations of the index i, $\overrightarrow{A}$ changes as

$$P_i \overrightarrow{A} = \{(a_{i_2 j_1 k_1 ...})\ (a_{i_1 j_2 k_2 ...})\}$$

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☆ For the elementary bipartition $\quad \mathscr{H} = \mathscr{H}_1 \otimes (\mathscr{H}_2 \otimes \mathscr{H}_3 \cdots \mathscr{H}_N)$

$$\boxed{\vec{C}_{i|\widehat{i}} = (\mathbb{1} - P_i)\vec{A}}$$

☆ For a generic bipartition, $I|\widehat{I}$, with $I = i, j, \ldots, m$

$$\boxed{\vec{C}_{I|\widehat{I}} = (\mathbb{1} - P_I)\vec{A}} \qquad P_I = P_i P_j \cdots P_m$$

A useful expression for $\vec{C}_{I|\widehat{I}}$

$$\boxed{\vec{C}_{I|\widehat{I}} = (\mathbb{1} - P_I)\vec{A}}$$

$\overrightarrow{A} \equiv$ *vector of coefficients of* $|\psi\rangle \otimes |\psi\rangle = \left\{ (a_{i_1 j_1 k_1 \ldots})(a_{i_2 j_2 k_2 \ldots}) \right\}$

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$\{P_I\}$ have a friendly algebra (commutative Z_2^N group):

$$P_I^2 = 1$$

$$[P_I, P_J] = 0$$

$$P_I \overrightarrow{A} = P_{\widehat{I}} \overrightarrow{A}$$

Note also that $1 - P_I$ is essentially a projector:

$$(1 - P_I)^2 = 2(1 - P_I)$$

Proof of previous results

- Triangular inequality, $C_{ij|\widehat{ij}} \leq C_{i|\widehat{i}} + C_{j|\widehat{j}}$:

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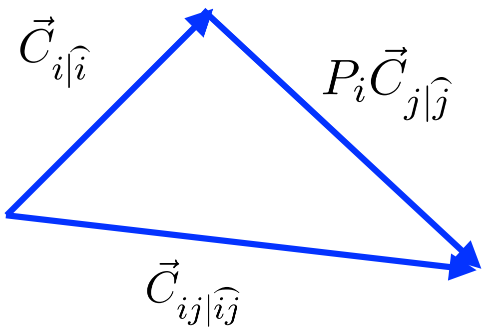
$$\vec{C}_{ij|\widehat{ij}} = (\mathbb{1} - P_i P_j)\vec{A} = \big((\mathbb{1} - P_i) + P_i(\mathbb{1} - P_j)\big)\vec{A} = \vec{C}_{i|\widehat{i}} + P_i \vec{C}_{j|\widehat{j}}$$

⇨ $\vec{C}_{ij|\widehat{ij}}$, $\vec{C}_{i|\widehat{i}}$, $P_i\vec{C}_{j|\widehat{j}}$ form a triangle

⇨ Their norms satisfy the triangular inequality



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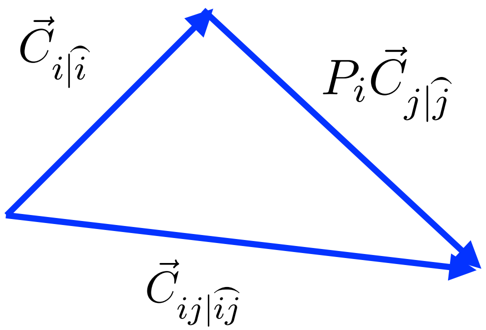

☆ Triangular inequality for the squares, $C^2_{ij|\widehat{ij}} \leq C^2_{i|\widehat{i}} + C^2_{j|\widehat{j}}$:

$$C^2_{ij|\widehat{ij}} = C^2_{i|\widehat{i}} + C^2_{j|\widehat{j}} - \frac{1}{2}\|(\mathbb{1} - P_i)(\mathbb{1} - P_j)\vec{A}\|^2$$

New results

☆ Relationships between entanglements of non-disjoint subsystems

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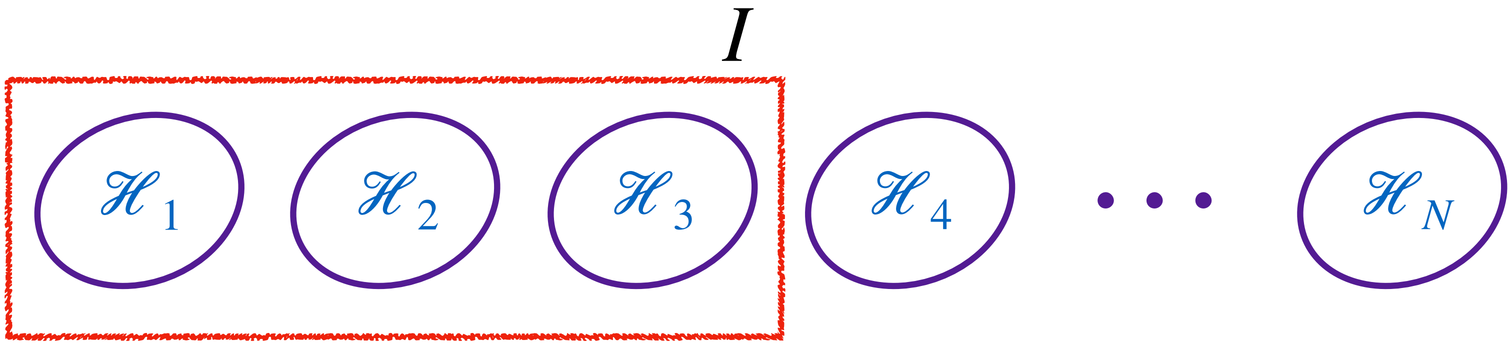

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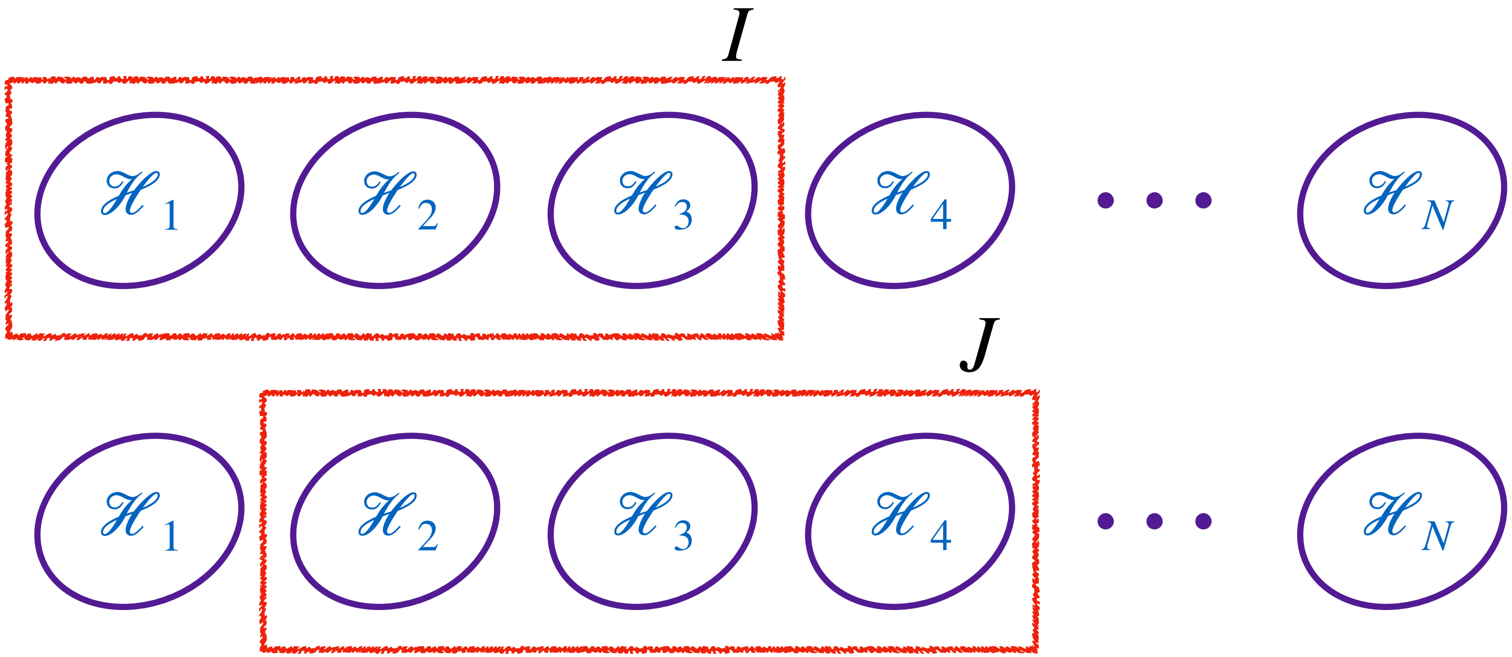

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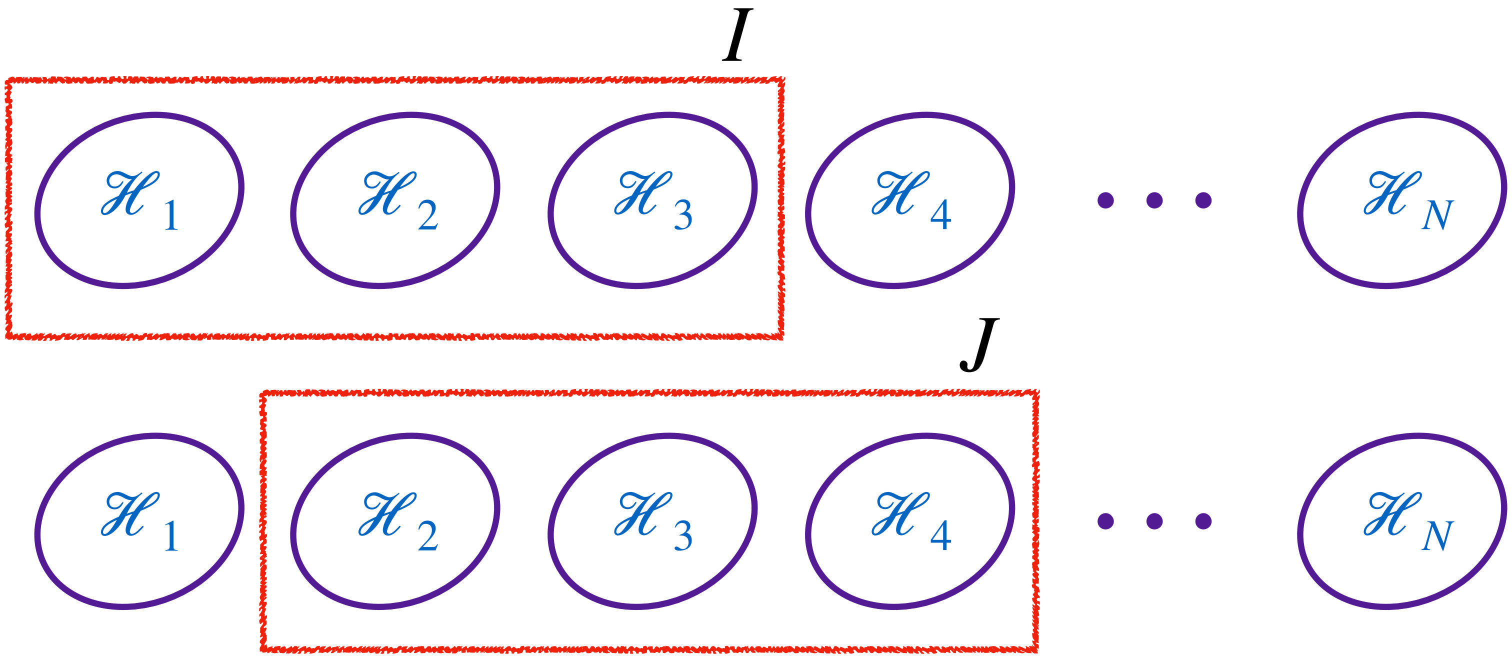

Define $\quad I \triangle J = I \cup J \setminus I \cap J$ (set of indices in I or J, but in both)

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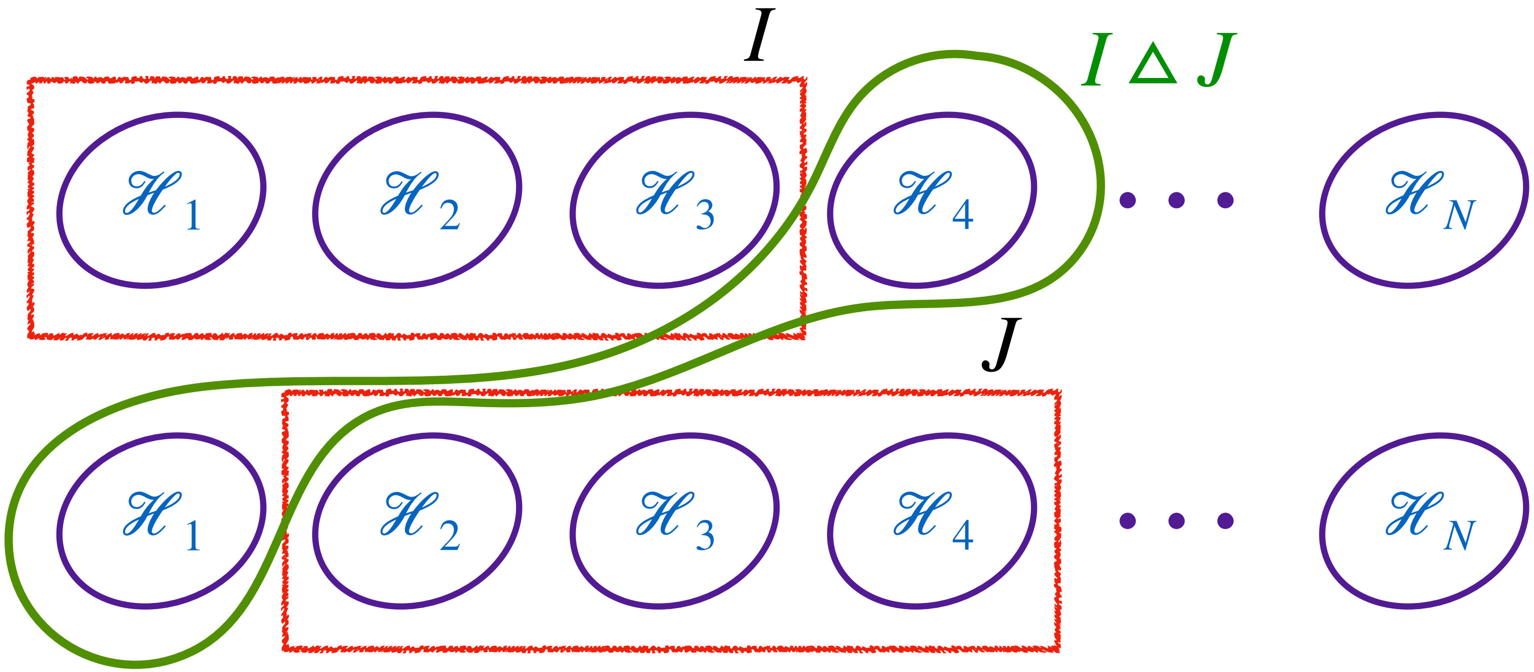

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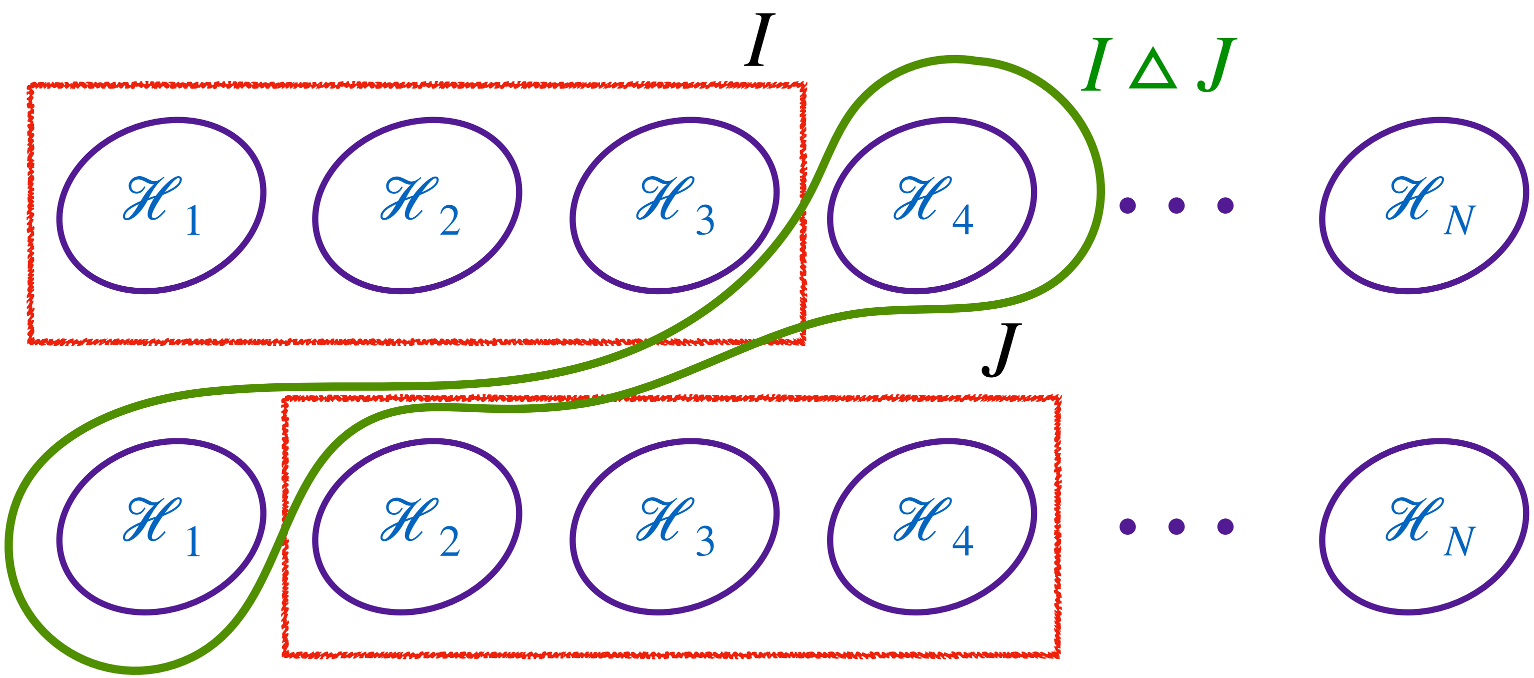

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$$
\begin{aligned}
C_{I\triangle J|\widehat{I\triangle J}} &\leq C_{I|\widehat{I}} + C_{J|\widehat{J}} \\
C^2_{I\triangle J|\widehat{I\triangle J}} &\leq C^2_{I|\widehat{I}} + C^2_{J|\widehat{J}}
\end{aligned}
$$

New results

☆ More generally, using

$$\underbrace{\vec{A}^\dagger(\mathbb{1}-P_I)(\mathbb{1}\pm P_J)(\mathbb{1}\pm P_K)\cdots\ (\mathbb{1}\pm P_M)\vec{A}}_{\text{Combination of squared concurrences}} \geq 0$$

since
$(1 \pm P) = (1 \pm P)^2/2$

one gets many more inequalities.

New results

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one gets many more inequalities.

E.g.

$$\overrightarrow{A}^\dagger(1-P_iP_j)\,(1-P_jP_k)\,(1+P_iP_k)\overrightarrow{A} \geq 0$$

$$C^2_{ik|\widehat{ik}} \leq C^2_{ij|\widehat{ij}} + C^2_{jk|\widehat{jk}}$$

New results

☆ We have also proven that the result

$$C^2_{ij|\widehat{ij}} = C^2_{i|\widehat{i}} + C^2_{j|\widehat{j}} \iff C_{i|\widehat{i}} = 0 \text{ or } C_{j|\widehat{j}} = 0$$

holds for any dimension of the three Hilbert spaces of the tripartition, not just for three qubits.

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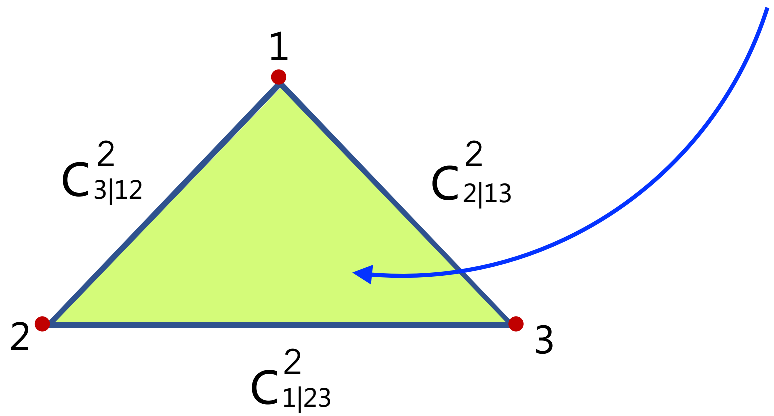
Hence, the area of the triangle of squared concurrences is a sound measure of genuine entanglement (for a tripartition)



Relation to (linear) entropy

From an entropy-of-entanglement perspective, the concurrence can be identified with the **Tsallis-2** ("linear") entropy

$$C^2_{I|\widehat{I}} = 2\left(1 - \mathrm{tr}\,\rho_A^2\right) \equiv 2\ S_2\left(\rho_A\right)$$

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We can use the relationships between concurrences to obtain relations between (linear) entropies for generic (pure or mixed) states

Relation to (linear) entropy

E.g. for a tri-partition of the Hilbert space:

$$\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_{\widehat{AB}}$$

$$\{\,|i\rangle\} \qquad \{\,|j\rangle\} \qquad \{\,|k\rangle\} \equiv \{\,|\widehat{ij}\rangle\}$$

Triangular inequalities (for squares):

$$C^2_{ij|\widehat{ij}} \leq C^2_{i|\widehat{i}} + C^2_{j|\widehat{j}} \quad \Longrightarrow \quad S_2(\rho_{AB}) \leq S_2(\rho_A) + S_2(\rho_B)$$

$$\left.\begin{aligned} C^2_{i|\widehat{i}} &\leq C^2_{j|\widehat{j}} + C^2_{ij|\widehat{ij}} \\ C^2_{j|\widehat{j}} &\leq C^2_{i|\widehat{i}} + C^2_{ij|\widehat{ij}} \end{aligned}\right\} \quad \Longrightarrow \quad |S_2(\rho_A) - S_2(\rho_B)| \leq S_2(\rho_{AB})$$

Relation to (linear) entropy

E.g. for a tri-partition of the Hilbert space:

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$$\{|i\rangle\} \qquad \{|j\rangle\} \qquad \{|k\rangle\} \equiv \{|\widehat{ij}\rangle\}$$

Triangular inequalities (for squares):

$$C^2_{ij|\widehat{ij}} \leq C^2_{i|\widehat{i}} + C^2_{j|\widehat{j}} \quad \Longrightarrow \quad S_2(\rho_{AB}) \leq S_2(\rho_A) + S_2(\rho_B)$$

$$\left.\begin{array}{l} C^2_{i|\widehat{i}} \leq C^2_{j|\widehat{j}} + C^2_{ij|\widehat{ij}} \\ C^2_{j|\widehat{j}} \leq C^2_{i|\widehat{i}} + C^2_{ij|\widehat{ij}} \end{array}\right\} \quad \Longrightarrow \quad |S_2(\rho_A) - S_2(\rho_B)| \leq S_2(\rho_{AB})$$

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Subadditivity Conditions

Relation to (linear) entropy

$$\boxed{|S_2(\rho_A) - S_2(\rho_B)| \leq S_2(\rho_{AB}) \leq S_2(\rho_A) + S_2(\rho_B)}$$

Subadditivity Conditions

Note: Even though the relation has been obtained starting with a pure global state,

$$|\psi\rangle \in \mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_{\widehat{AB}}$$

The subadditivity relation holds for pure or mixed states since any ρ_{AB} can be obtained from a pure state by appropriately choosing the "environment" $\mathscr{H}_{\widehat{AB}}$

$$\rho_{AB} = \mathrm{tr}_{\widehat{AB}} |\psi\rangle\langle\psi|$$

(purification theorem)

Relation to (linear) entropy

$$\boxed{|S_2(\rho_A) - S_2(\rho_B)| \leq S_2(\rho_{AB}) \leq S_2(\rho_A) + S_2(\rho_B)}$$

Subadditivity Conditions

Also: the equality in the subadditivity, i.e.

$$S_2(\rho_{AB}) = S_2(\rho_A) + S_2(\rho_B)$$

occurs *iff*

$$S_2(\rho_A) = 0 \quad \text{or} \quad S_2(\rho_B) = 0$$

(new)

Relation to (linear) entropy

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☆ All relationships found for concurrences can be translated to relationships between linear entropies, e.g.

$$S_2\left(\rho_{AC}\right)\leq S_2\left(\rho_{AB}\right)+S_2\left(\rho_{BC}\right)$$

Sufficient Conditions for Genuine Entanglement

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Recall:

In order to assess whether a state is genuinely entangled we need to evaluate $2^{N-1} - 1$ concurrences, $C^2_{I|\widehat{I}}$, $I \in \{i, j, k, \cdots\}$ and check that all of them are $\neq 0$

(a very hard problem):

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(a very hard problem):

Can our expression $\vec{C}_{I|\widehat{I}} = (\mathbb{1} - P_I)\vec{A}$ be of some help?

Sufficient Conditions for Genuine Entanglement

Suppose the Hilbert space is

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_N$$

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which is a linear combination of **all** the concurrence vectors:

$$\vec{V}_N = \sum_{I \subset \{1,2,\ldots N-1\}} (-1)^{\# I} (\mathbb{1} - P_I)\vec{A} \;\; = \sum_{I \subset \{1,2,\ldots N-1\}} (-1)^{\# I} \vec{C}_{I|\widehat{I}}$$

Sufficient Conditions for Genuine Entanglement

Suppose there is a certain odd bipartition, $\Sigma\,|\,\widehat{\Sigma}$, which is separable:

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☆ For the other cases, Σ even, N odd, we got also sufficient conditions for genuine entanglement, computable in pol. time.

Conclusions

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- For a given bipartition, $I|\widehat{I}$, a useful expression for the associated concurrence vector is

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with $\overrightarrow{A} \equiv$ vector of coefficients of $|\psi\rangle \otimes |\psi\rangle$, $\qquad \dim \overrightarrow{A} = D^2$

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- In this way, many new relationsships between the concurrences of the various bipartitions of a multipartite system can be obtained

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- The same holds for the Tsallis-2 (or “linear") entropy, e.g. a

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- This approach is also useful to obtain sufficient conditions for genuine entanglement, computable in polynomial time.

Prospects

Explore:

- pure state $\longrightarrow$ mixed state
- linear entropy $\longrightarrow$ other quantum entropies (e.g. von Neumann)
- necessary-and-sufficient conditions for genuine entanglement (pure states)
- relation of the formalism with entanglement invariants