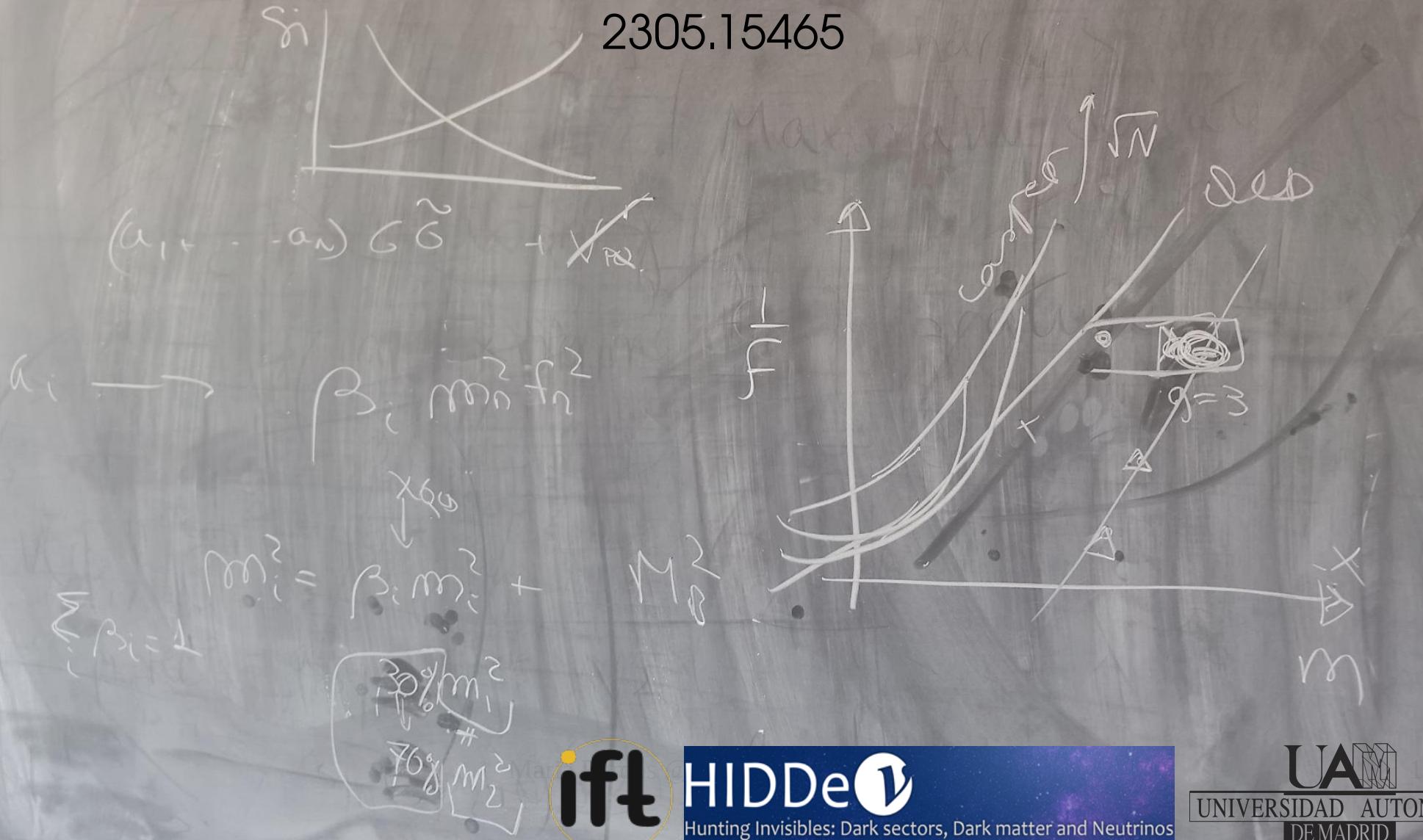


The QCD axion sum rule

Belén Gavela, Pablo Quílez, María Ramos

2305.15465



The standard QCD axion

In the Standard Model: $\mathcal{L} \supset \frac{\alpha_s}{8\pi} \bar{\theta} G\tilde{G}$ $d = u \overset{\text{---}}{=} d \lesssim 10^{-10}$

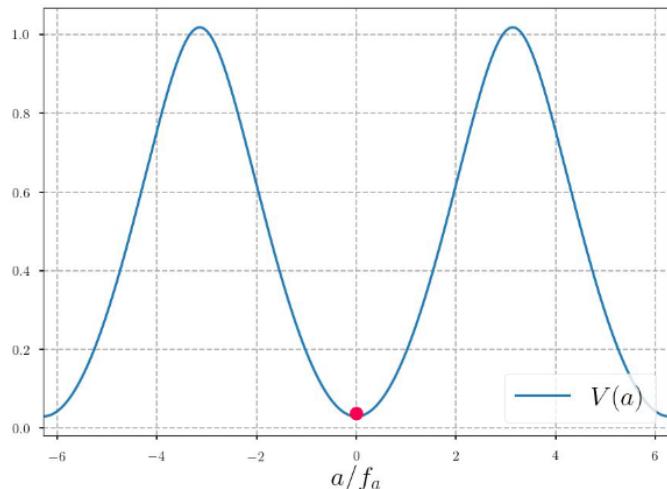
A dynamical $U(1)_A$ solution:

$$\mathcal{L} = \frac{\alpha_s}{8\pi} \underbrace{\left(\frac{a_{G\tilde{G}}}{f_a} - \bar{\theta} \right)}_{a/f_a} G\tilde{G}$$

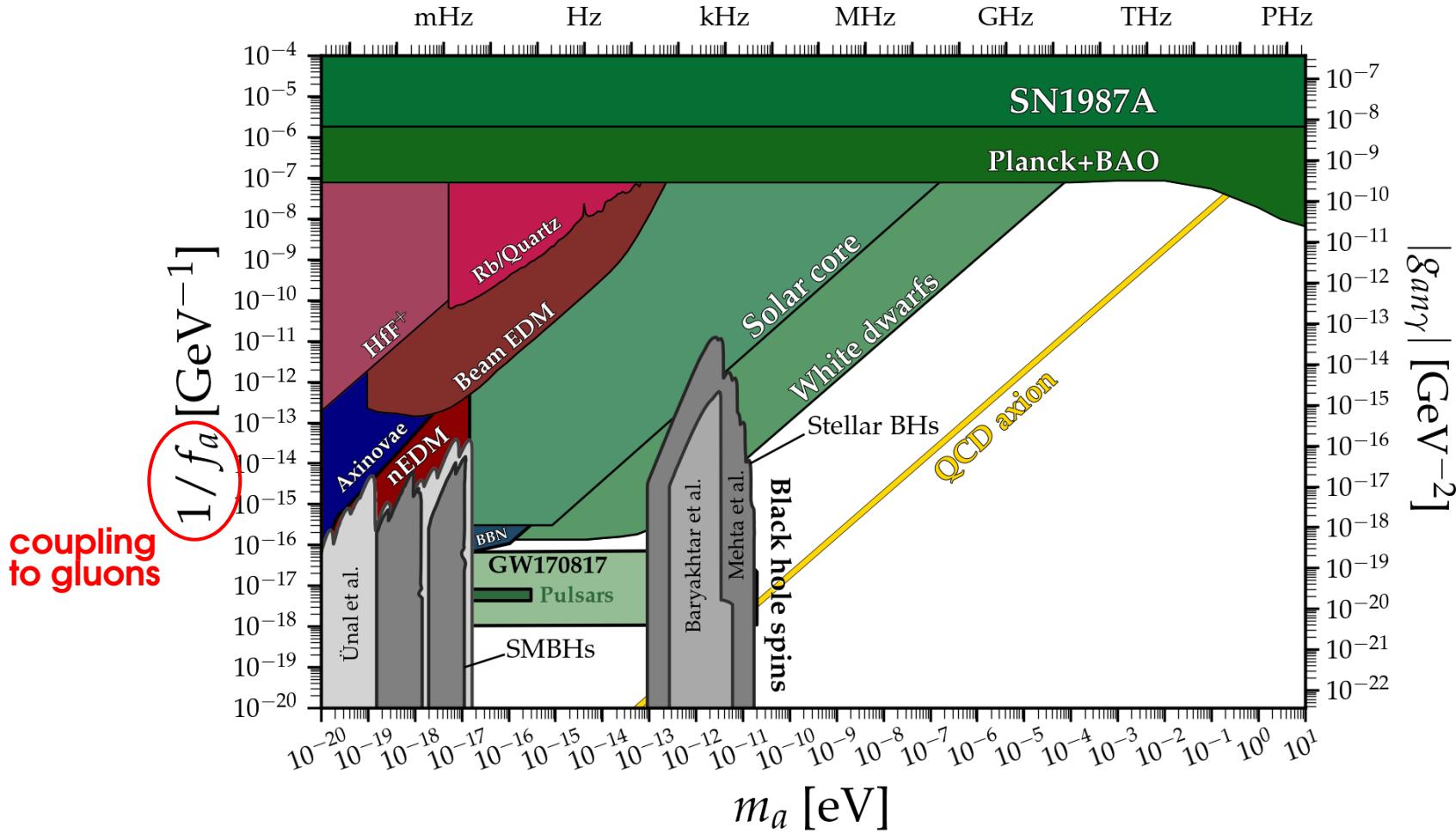
$$\Rightarrow V(a) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4m_u m_d}{(m_u + m_d)^2} \sin^2 \left(\frac{1}{2} \frac{a}{f_a} \right)}$$

standard QCD axion

$$m_a^2 f_a^2 = \chi_{\text{QCD}} \simeq m_\pi^2 f_\pi^2 \frac{m_u m_d}{(m_u + m_d)^2}$$

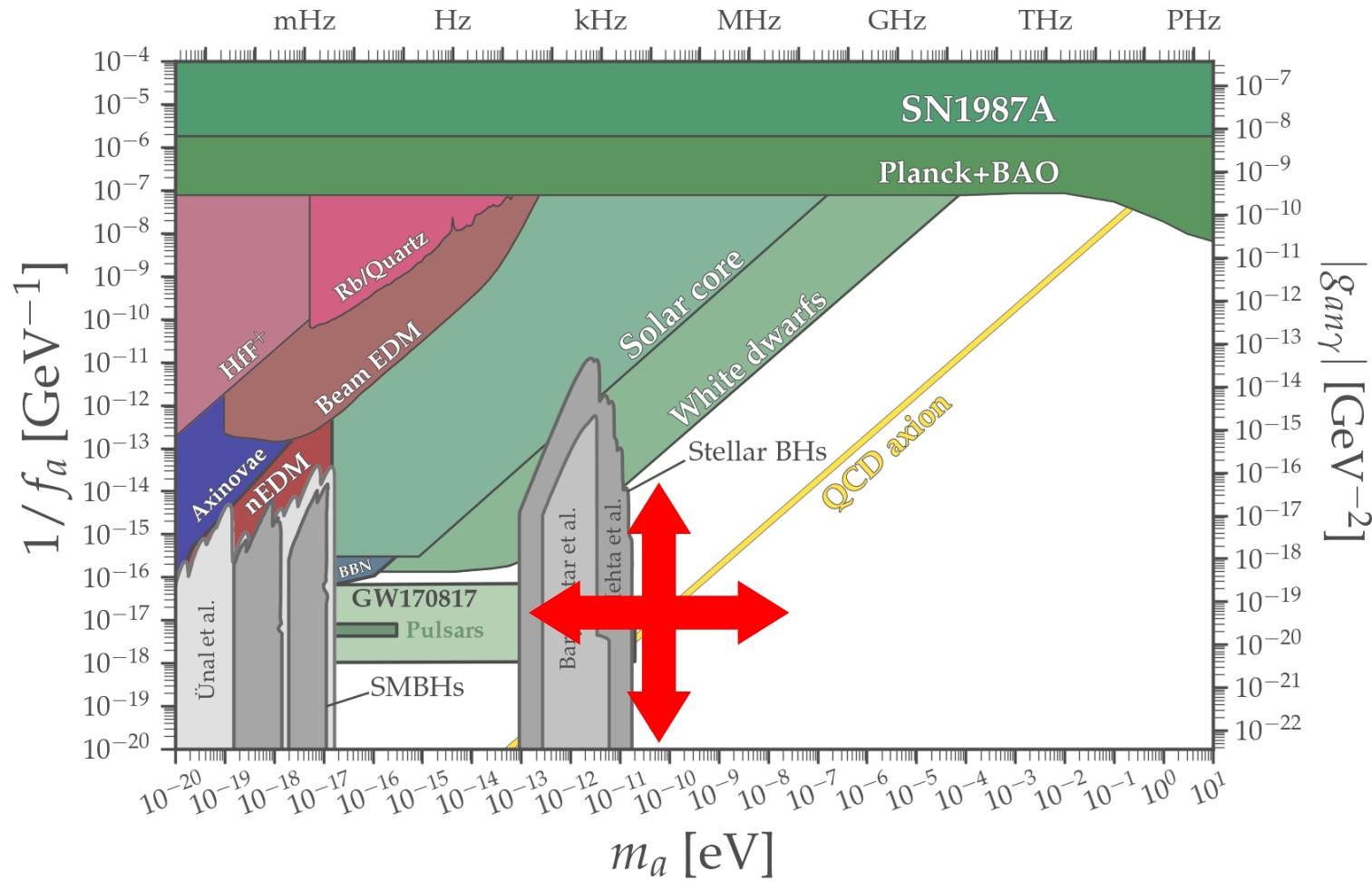


The standard QCD axion



The non-standard ~~-QCD-~~ axion

up to now



See Refs. in Pablo's review talk in Patras 21!

Challenging the standard assumptions

$$\mathcal{L} = \frac{\alpha_s}{8\pi} \left(\frac{a_{G\tilde{G}}}{f_a} - \bar{\theta} \right) G\tilde{G}$$

$$\Rightarrow m_a^2 f_a^2 = \chi_{\text{QCD}} \simeq m_\pi^2 f_\pi^2 \frac{m_u m_d}{(m_u + m_d)^2}$$

interaction basis = mass basis

But the axion may not be the only singlet scalar in Nature.

e.g. “String axiverse”

A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Emanja Kaloper, J. March-Russell 09

Challenging the standard assumptions

$$\mathcal{L} = \frac{\alpha_s}{8\pi} \left(\frac{a_{G\tilde{G}}}{f_a} - \bar{\theta} \right) G\tilde{G}$$

$$\Rightarrow m_a^2 f_a^2 = \chi_{\text{QCD}} \simeq m_\pi^2 f_\pi^2 \frac{m_u m_d}{(m_u + m_d)^2}$$

interaction basis \neq mass basis

So instead, one can have:

$$\mathcal{L} = \frac{\alpha_s}{8\pi} \left(\frac{\hat{a}_{G\tilde{G}}}{F} - \bar{\theta} \right) G\tilde{G} - V'(\hat{a}_{G\tilde{G}}, \dots, \hat{a}_N)$$

$$\Rightarrow m_i^2 f_i^2 = g_i \chi_{\text{QCD}} \quad \text{within QCD}$$

Toy model: 2 axions

$$\mathcal{L}_{N=2} = \frac{\alpha_s}{8\pi} \left(\frac{\hat{a}_1}{\hat{f}_1} + \frac{\hat{a}_2}{\hat{f}_2} + \bar{\theta} \right) G\tilde{G} - \hat{m}_2^2 \hat{a}_2^2$$

Below confinement, and using the PQ symmetry to remove the QCD angle:

$$V_{N=2} \supset \frac{\chi_{\text{QCD}}}{2} \left(\frac{\hat{a}_1}{\hat{f}_1} + \frac{\hat{a}_2}{\hat{f}_2} \right)^2 - V(\hat{a}_2)$$

Toy model: 2 axions

$$\mathcal{L}_{N=2} = \frac{\alpha_s}{8\pi} \left(\frac{\hat{a}_1}{\hat{f}_1} + \frac{\hat{a}_2}{\hat{f}_2} + \bar{\theta} \right) G\tilde{G} - \hat{m}_2^2 \hat{a}_2^2$$

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Minima

$$\frac{\chi_{\text{QCD}}}{\hat{f}_1} \sin \left(\frac{\hat{a}_1}{\hat{f}_1} + \frac{\hat{a}_2}{\hat{f}_2} \right) = 0 \quad \text{and} \quad \underbrace{\frac{\chi_{\text{QCD}}}{\hat{f}_2} \sin \left(\underbrace{\frac{\hat{a}_1}{\hat{f}_1} + \frac{\hat{a}_2}{\hat{f}_2}}_{v_1 + v_2 = 0} \right)}_{v_2 = 0} - \underbrace{\frac{\partial V(\hat{a}_2)}{\partial \hat{a}_2}}_{v_2 = 0} = 0$$

Toy model: 2 axions

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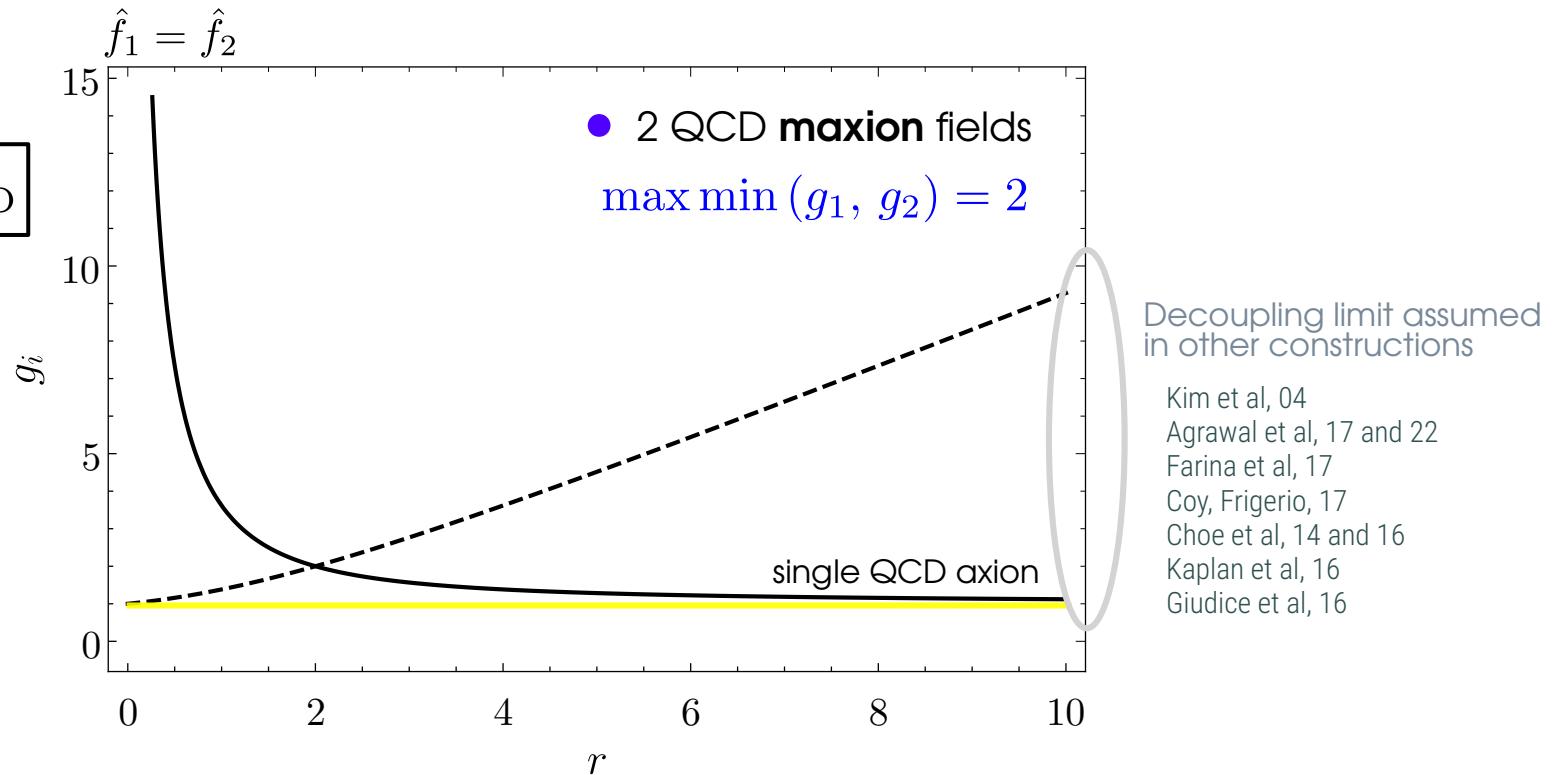
Minima

$$\frac{\chi_{\text{QCD}}}{\hat{f}_1} \sin \left(\frac{\hat{a}_1}{\hat{f}_1} + \frac{\hat{a}_2}{\hat{f}_2} \right) = 0 \quad \text{and} \quad \underbrace{\frac{\chi_{\text{QCD}}}{\hat{f}_2} \sin \left(\underbrace{\frac{\hat{a}_1}{\hat{f}_1} + \frac{\hat{a}_2}{\hat{f}_2}}_{v_1 + v_2 = 0} \right)}_{v_2 = 0} - \underbrace{\frac{\partial V(\hat{a}_2)}{\partial \hat{a}_2}}_{v_2 = 0} = 0$$

$$r \equiv \hat{m}_2^2 \frac{\hat{f}_2^2}{\chi_{\text{QCD}}} \quad \mathbf{M^2} = \chi_{\text{QCD}} \begin{pmatrix} 1/\hat{f}_1^2 & 1/(\hat{f}_1 \hat{f}_2) \\ 1/(\hat{f}_1 \hat{f}_2) & (1+r)/\hat{f}_2^2 \end{pmatrix}$$

Toy model: 2 axions

$$m_i^2 f_i^2 = g_i \chi_{\text{QCD}}$$

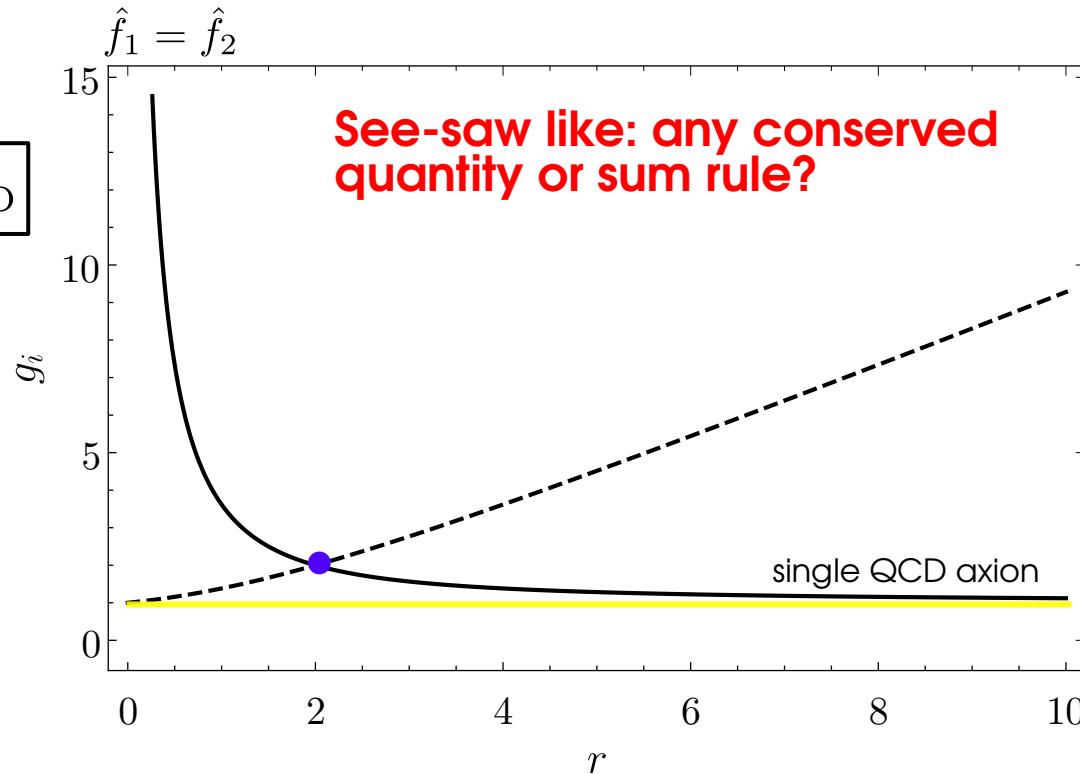


$$r \equiv \hat{m}_2^2 \frac{\hat{f}_2^2}{\chi_{\text{QCD}}}$$

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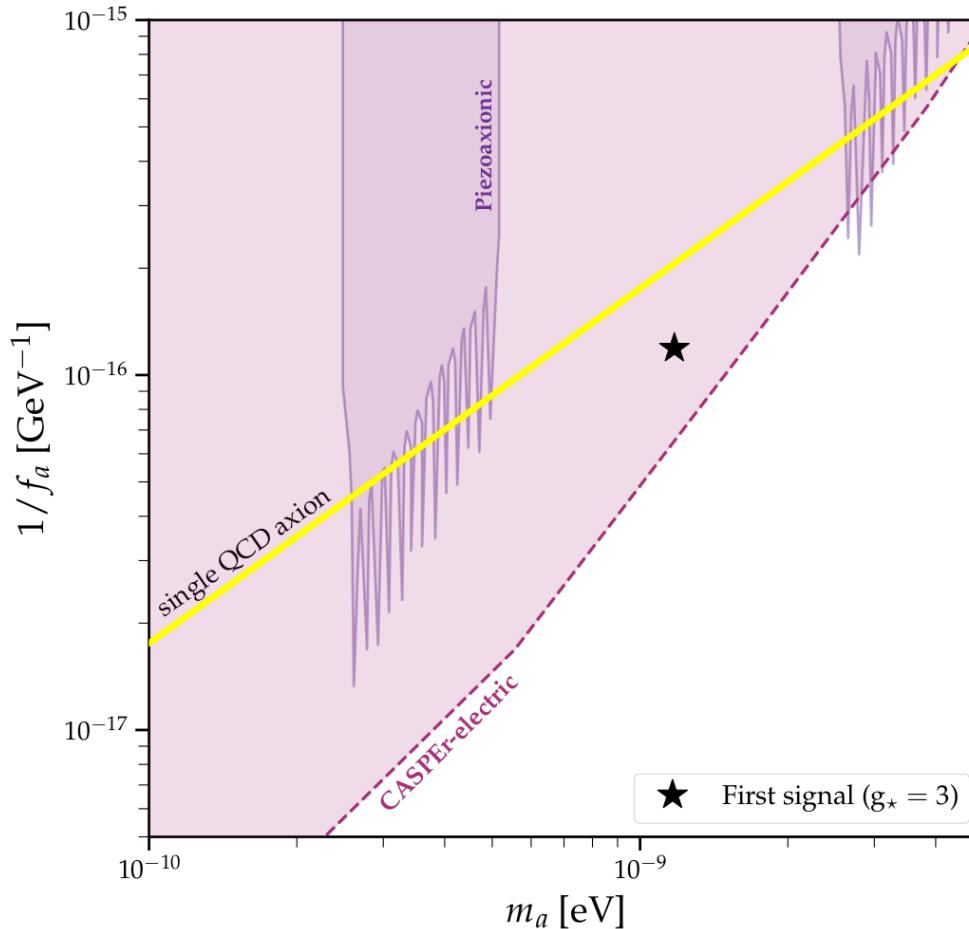


$$\left. \begin{aligned} g_1 &= \frac{2\hat{f}_2^2 h(r)}{\hat{f}_2^2 h(r) - \hat{f}_2^2 + \hat{f}_1^2 (r-1)} \\ g_2 &= \frac{2\hat{f}_2^2 h(r)}{\hat{f}_2^2 h(r) + \hat{f}_2^2 - \hat{f}_1^2 (r-1)} \end{aligned} \right\} \quad \begin{aligned} \beta_1 + \beta_2 &= 1 \\ \beta_i &\equiv 1/g_i \end{aligned}$$

Strong constraint on the system!

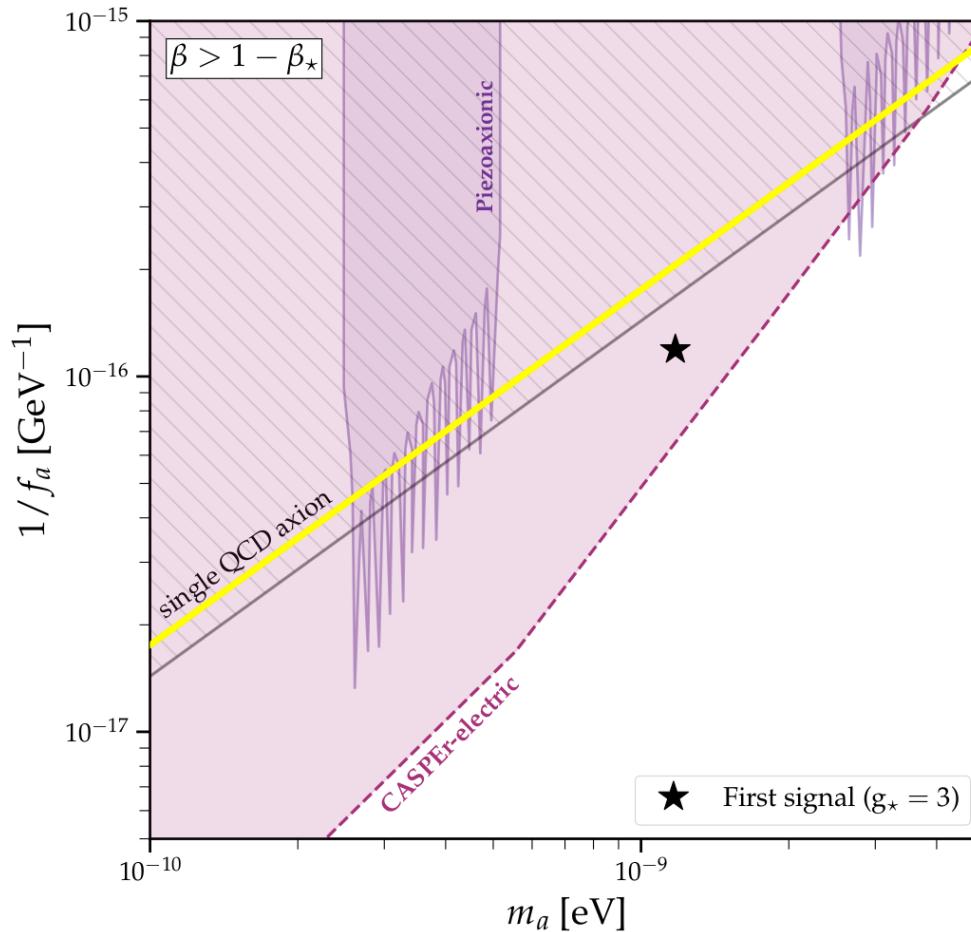
$$h(r) \equiv \sqrt{1 - 2\frac{\hat{f}_1^2}{\hat{f}_2^2} (r-1) + \frac{\hat{f}_1^4}{\hat{f}_2^4} (r+1)^2}$$

An ALP or a true QCD axion?



Assume a first signal is measured in the star location

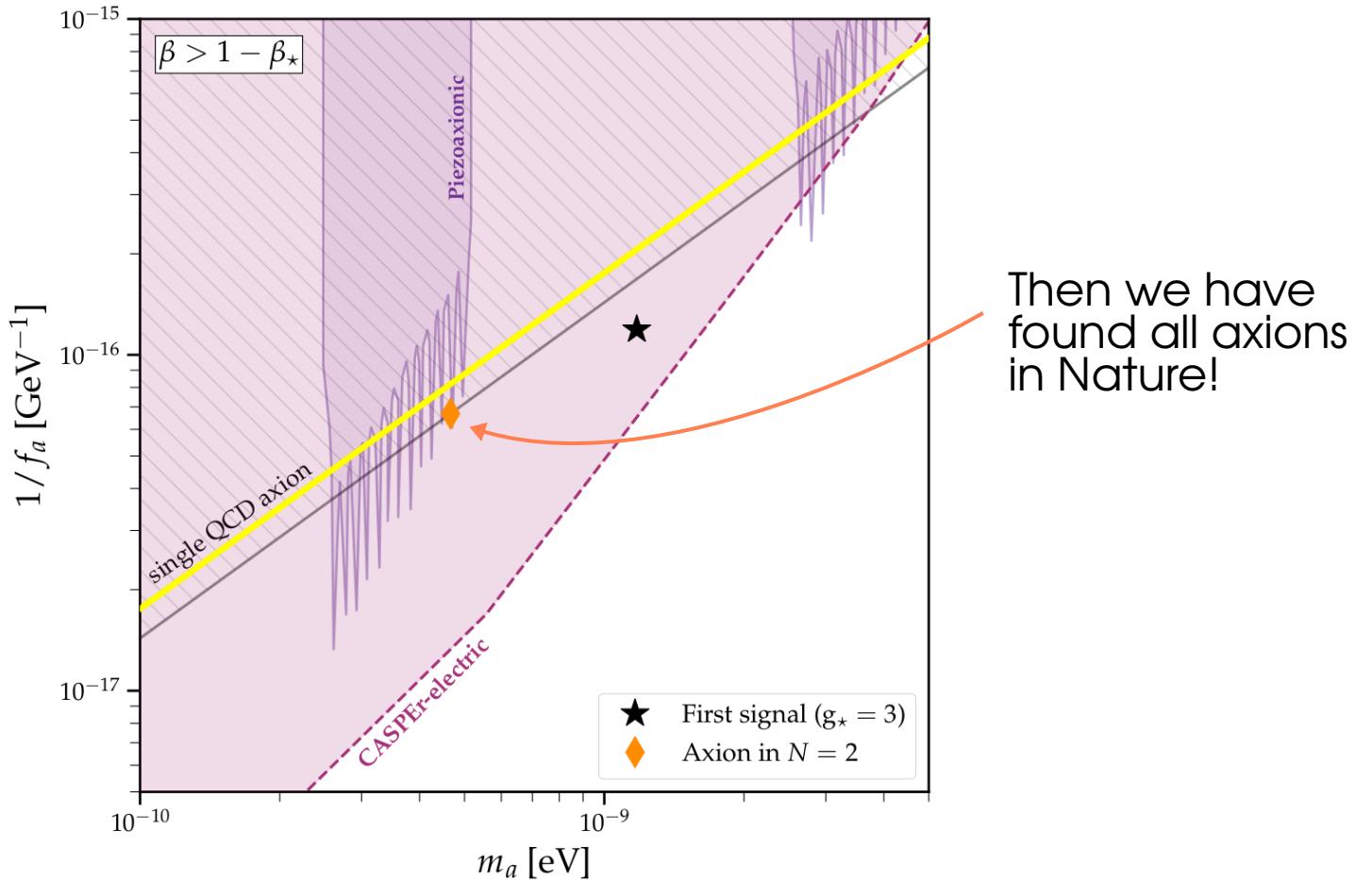
An ALP or a true QCD axion?



Have to wait for another signal outside the dashed region

An ALP or a true QCD axion?

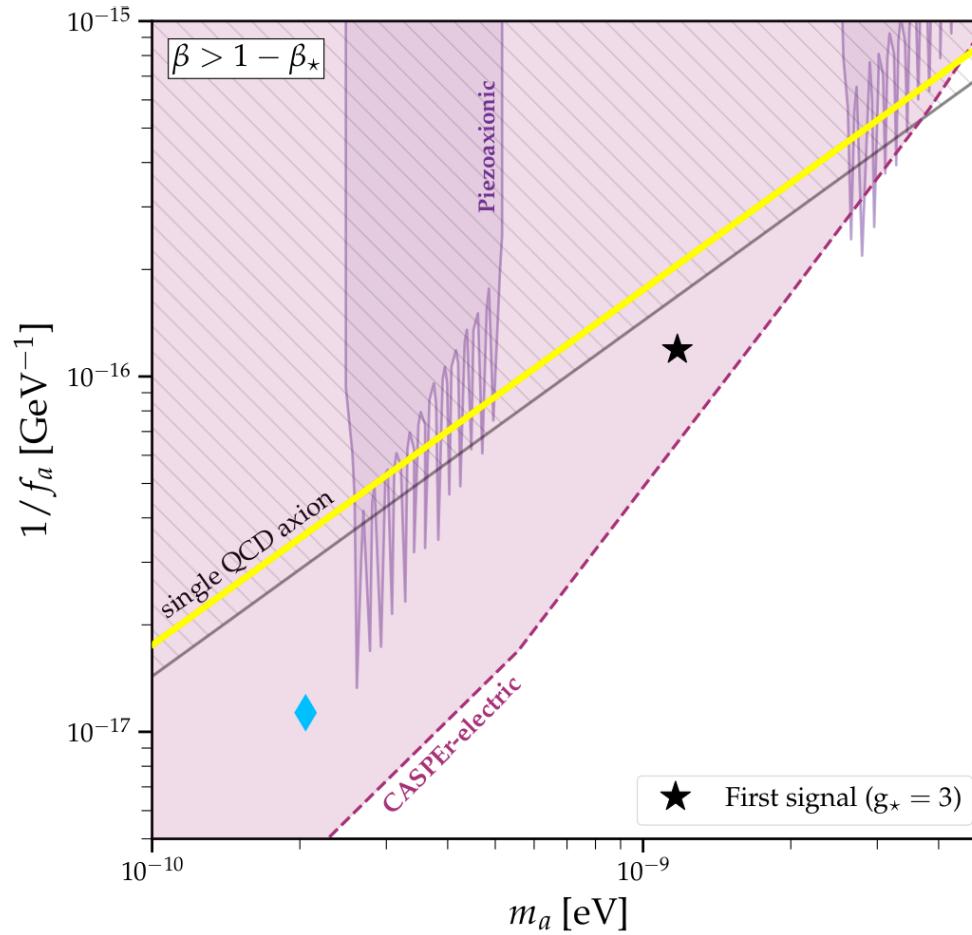
$$\sum_{i=1}^2 \beta_i = 1$$



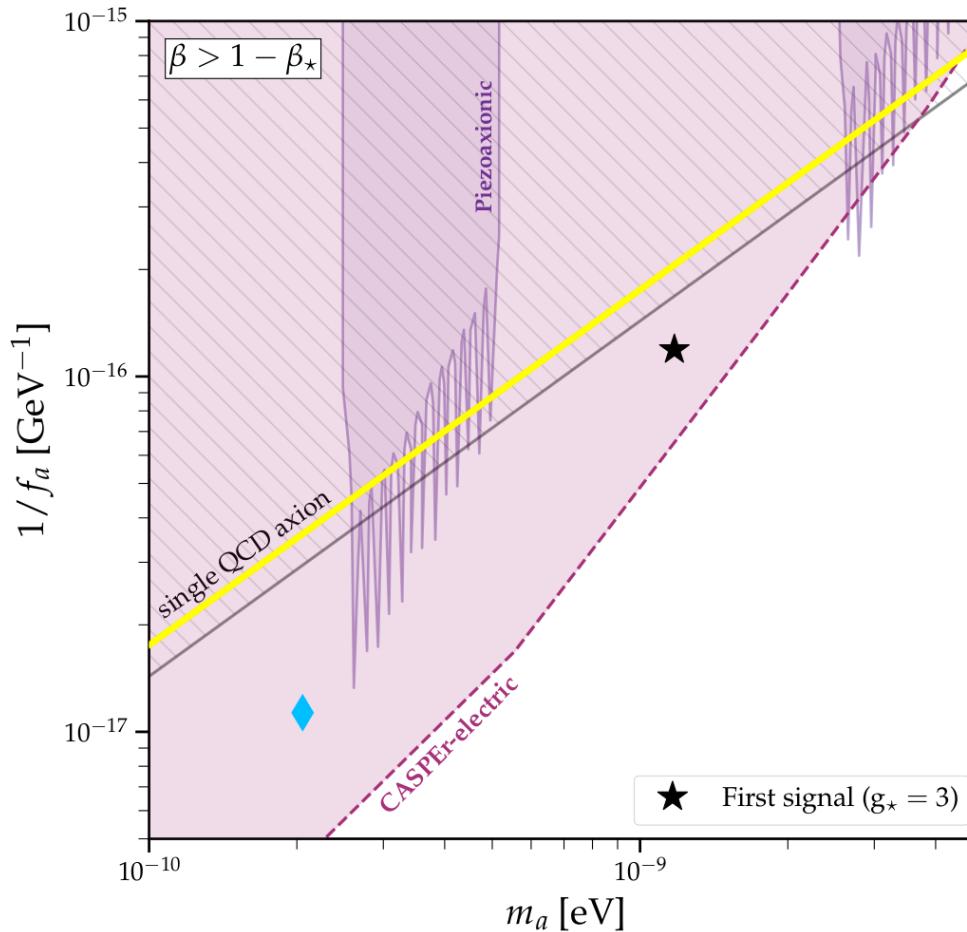
Have to wait for another signal outside the dashed region

What if?

$$\sum_{i=1}^2 \beta_i < 1$$



General potential for arbitrary N scalars



Exact results and sum rules

PQ condition

$$\mathcal{L} = \frac{\alpha_s}{8\pi} \left(\sum_{k=1}^N \frac{\hat{a}_k}{\hat{f}_k} - \bar{\theta} \right) G\tilde{G} - V_B(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N) \rightarrow \frac{\alpha_s}{8\pi} \left(\frac{\hat{a}_{G\tilde{G}}}{F} - \bar{\theta} \right) G\tilde{G} - V_B^R(\hat{a}_{G\tilde{G}}, \dots)$$

A preferred basis.

$$\boxed{\mathbf{M}^2 \equiv \mathbf{R} \hat{\mathbf{M}}^2 \mathbf{R}^T}$$

$$\frac{1}{F^2} = \sum_{k=1}^N \frac{1}{\hat{f}_k^2}$$

$$\mathbf{M}^2 = \mathbf{M}_A^2 + \mathbf{M}_B^2 = \begin{pmatrix} b_{11} & \mathbf{X}^\dagger \\ \mathbf{X} & \mathbf{M}_1^2 \end{pmatrix} = \frac{\chi_{\text{QCD}}}{F^2} \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{0} \end{pmatrix} + \begin{pmatrix} b_{11} - \frac{\chi_{\text{QCD}}}{F^2} & \mathbf{X}^\dagger \\ \mathbf{X} & \mathbf{M}_1^2 \end{pmatrix},$$

PQ condition

$$\mathcal{L} = \frac{\alpha_s}{8\pi} \left(\sum_{k=1}^N \frac{\hat{a}_k}{\hat{f}_k} - \bar{\theta} \right) G\tilde{G} - V_B(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N) \rightarrow \frac{\alpha_s}{8\pi} \left(\frac{\hat{a}_{G\tilde{G}}}{F} - \bar{\theta} \right) G\tilde{G} - V_B^R(\hat{a}_{G\tilde{G}}, \dots)$$

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$$\boxed{\exists U(1)_{PQ} \implies \lim_{\chi_{\text{QCD}} \rightarrow 0} \det \mathbf{M}^2 = 0 \implies \det \mathbf{M}_B^2 = 0 \quad \langle \hat{a}_0 | a_{G\tilde{G}} \rangle \neq 0}$$

PQ condition

$$\mathcal{L} = \frac{\alpha_s}{8\pi} \left(\sum_{k=1}^N \frac{\hat{a}_k}{\hat{f}_k} - \bar{\theta} \right) G\tilde{G} - V_B(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N) \rightarrow \frac{\alpha_s}{8\pi} \left(\frac{\hat{a}_{G\tilde{G}}}{F} - \bar{\theta} \right) G\tilde{G} - V_B^R(\hat{a}_{G\tilde{G}}, \dots)$$

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Applying Schur's formula.

$$\begin{aligned} \det \mathbf{M}_1^2 \left(b_{11} - \frac{\chi_{\text{QCD}}}{F^2} - \mathbf{X}^\dagger \mathbf{M}_1^{-2} \mathbf{X} \right) &= 0 \\ \Rightarrow \frac{\det \mathbf{M}^2}{\det \mathbf{M}_1^2} &= \left(b_{11} - \mathbf{X}^\dagger \mathbf{M}_1^{-2} \mathbf{X} \right) = \frac{\chi_{\text{QCD}}}{F^2} \end{aligned}$$

Generalized sum-rule

$$\Rightarrow \frac{\det \mathbf{M}^2}{\det \mathbf{M}_1^2} = (b_{11} - \mathbf{X}^\dagger \mathbf{M}_1^{-2} \mathbf{X}) = \frac{\chi_{\text{QCD}}}{F^2}$$

Moving to the physical basis.

$$\mathcal{L} \supset \frac{\alpha_s}{8\pi} \frac{a_i}{f_i} G \tilde{G} \quad \text{with} \quad \frac{1}{f_i} = \frac{\langle \hat{a}_{G\tilde{G}} | a_i \rangle}{F} \equiv \frac{v_{i1}}{F} \implies \sum_{i=1}^N \frac{1}{f_i^2} = \frac{1}{F^2}$$

Generalized sum-rule

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PETER B. DENTON, STEPHEN J. PARKE, TERENCE TAO, AND XINING ZHANG

ABSTRACT. If A is an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$ and $i, j = 1, \dots, n$, then the j^{th} component $v_{i,j}$ of a unit eigenvector v_i associated to the eigenvalue $\lambda_i(A)$ is related to the eigenvalues $\lambda_1(M_j), \dots, \lambda_{n-1}(M_j)$ of the minor M_j of A formed by removing the j^{th} row and column by the formula

$$|v_{i,j}|^2 \prod_{k=1; k \neq i}^n (\lambda_i(A) - \lambda_k(A)) = \prod_{k=1}^{n-1} (\lambda_i(A) - \lambda_k(M_j)).$$

We refer to this identity as the *eigenvector-eigenvalue identity*

Generalized sum-rule

$$\Rightarrow \frac{\det \mathbf{M}^2}{\det \mathbf{M}_1^2} = (b_{11} - \mathbf{X}^\dagger \mathbf{M}_1^{-2} \mathbf{X}) = \frac{\chi_{\text{QCD}}}{F^2}$$

Moving to the physical basis.

$$\mathcal{L} \supset \frac{\alpha_s}{8\pi} \frac{a_i}{f_i} G \tilde{G} \quad \text{with} \quad \frac{1}{f_i} = \frac{\langle \hat{a}_{G\tilde{G}} | a_i \rangle}{F} \equiv \frac{v_{i1}}{F} \implies \sum_{i=1}^N \frac{1}{f_i^2} = \frac{1}{F^2}$$

A corollary.
(generic A matrix)

$$\frac{\det(\lambda \mathbb{I}_{N-1} - M_j)}{\det(\lambda \mathbb{I}_N - A)} = \sum_{i=1}^N \frac{|v_{ij}|^2}{\lambda(A) - \lambda_i(A)}$$

$$\frac{\det \mathbf{M}_1^2}{\det \mathbf{M}^2} = \sum_{i=1}^N \frac{|v_{1i}|^2}{m_i^2} = \frac{F^2}{\chi_{\text{QCD}}} \sum_{i=1}^N \frac{1}{g_i} \xrightarrow{\exists U(1)_{\text{PQ}}}$$

axionness is shared

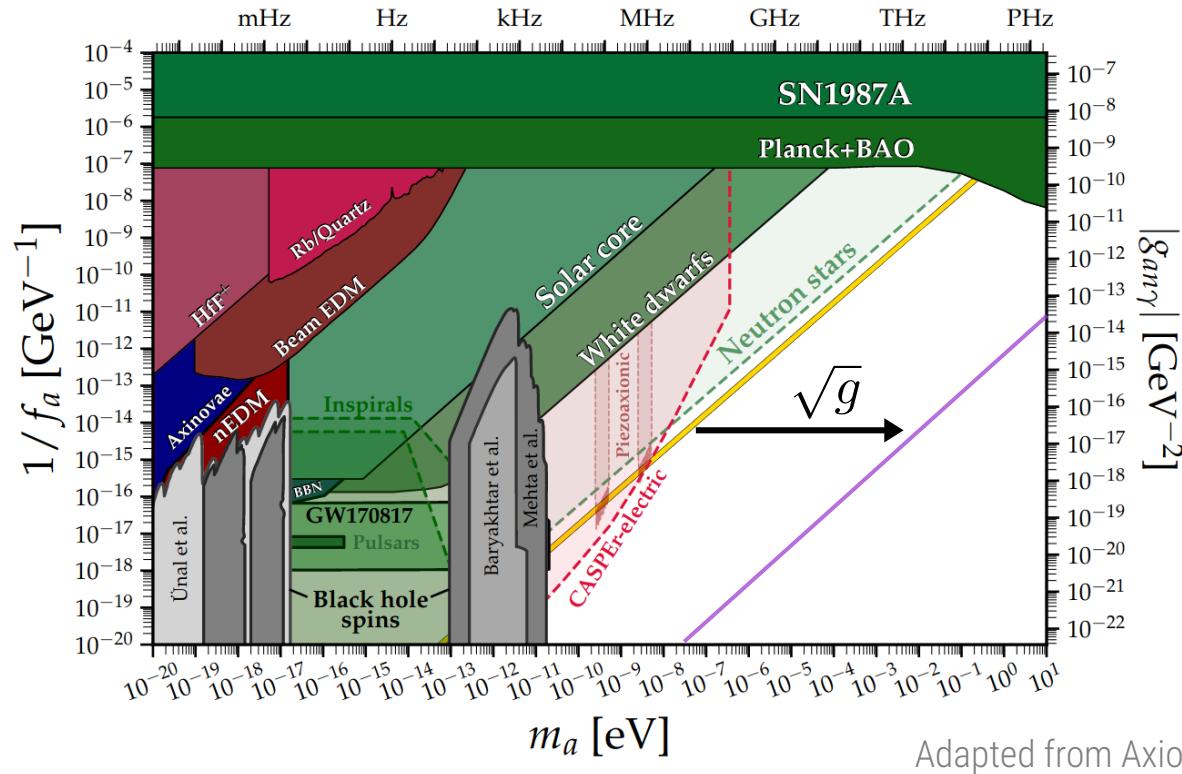
$$\sum_{i=1}^N \beta_i = 1, \quad \beta_i \equiv \frac{1}{g_i}$$

$$\beta_i = \frac{\langle \hat{a}_{\text{PQ}} | a_i \rangle \langle a_i | \hat{a}_{G\tilde{G}} \rangle}{\langle \hat{a}_{\text{PQ}} | \hat{a}_{G\tilde{G}} \rangle}$$

Experimental consequences

$$1. \ g_i \geq 1 \quad \left(= 1 + \frac{F^2}{\chi_{\text{QCD}}} \frac{\langle a_i | \mathbf{M}_B^2 | a_i \rangle}{\left| \langle a_i | a_{G\tilde{G}} \rangle \right|^2} \right)$$

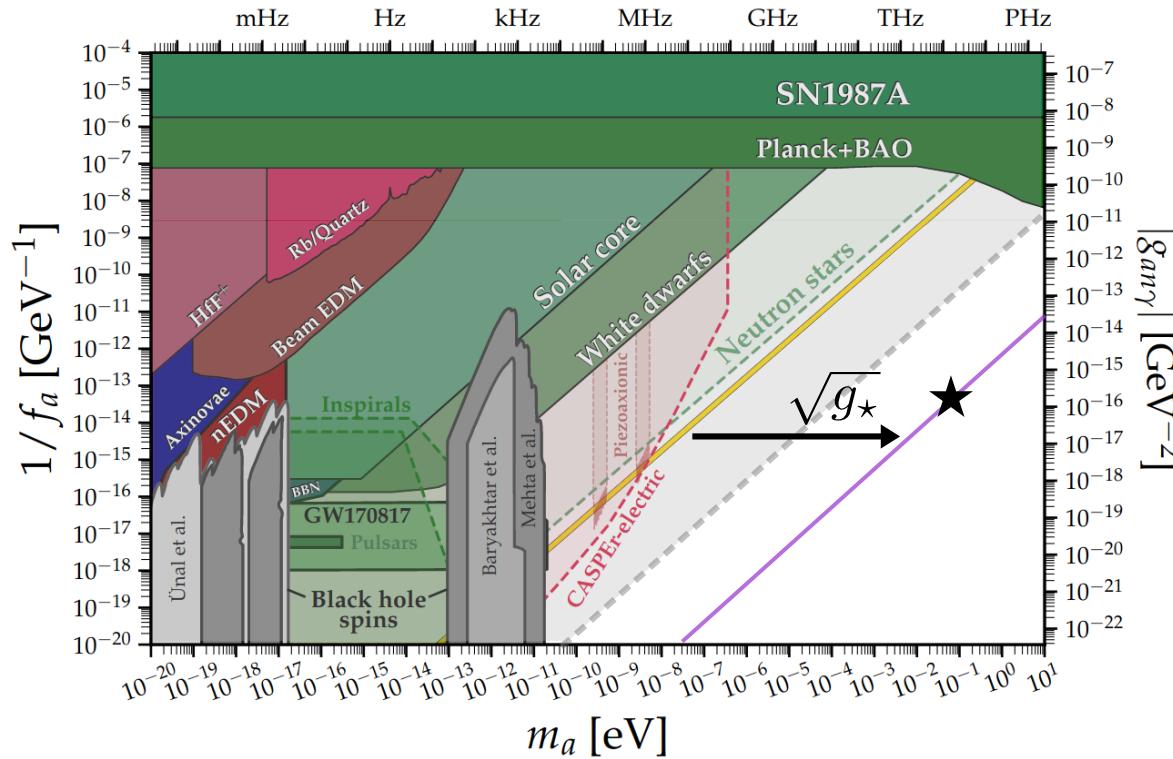
$$g_i = \frac{m_i^2 f_i^2}{\chi_{\text{QCD}}}$$



Experimental consequences

$$2. \ g_j \geq \frac{1}{1 - 1/g_\star}$$

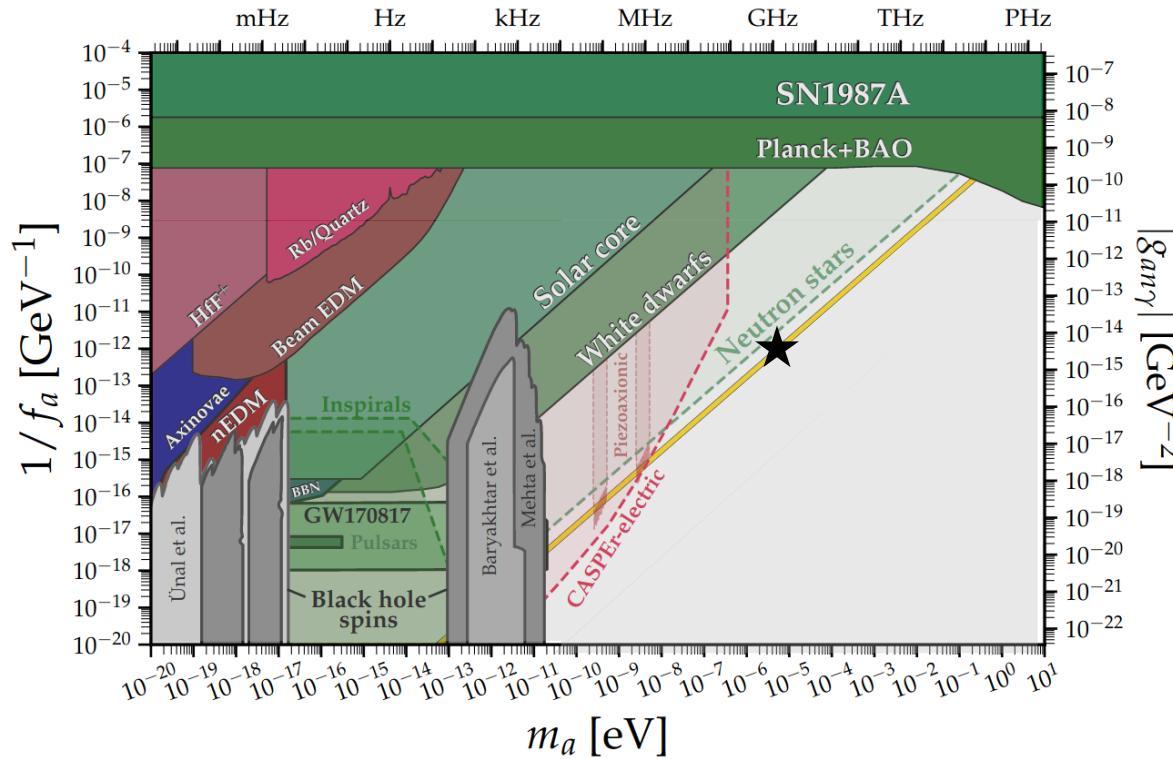
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Experimental consequences

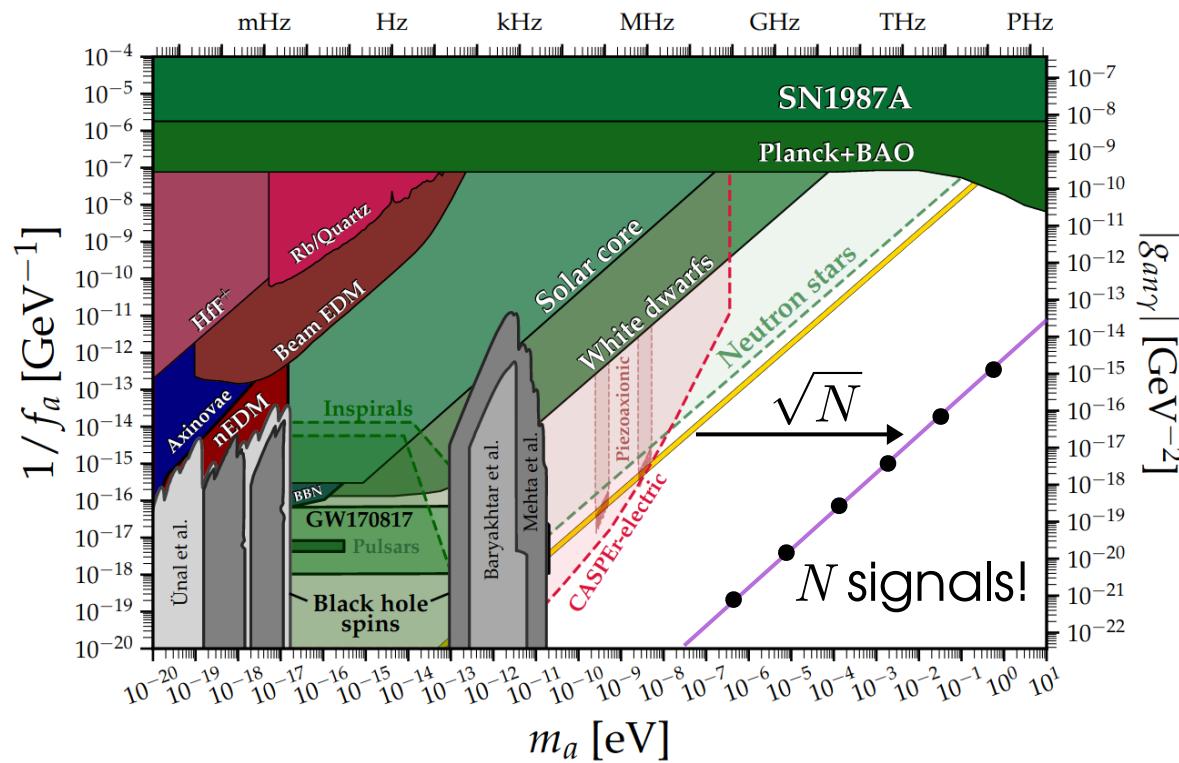
$$2. \ g_j \geq \frac{1}{1 - 1/g_\star}$$

$$g_i = \frac{m_i^2 f_i^2}{\chi_{\text{QCD}}}$$



Experimental consequences

3. $\max_i \left\{ \min_i \{g_i\} \right\} = N \implies g_i = N, \forall i$



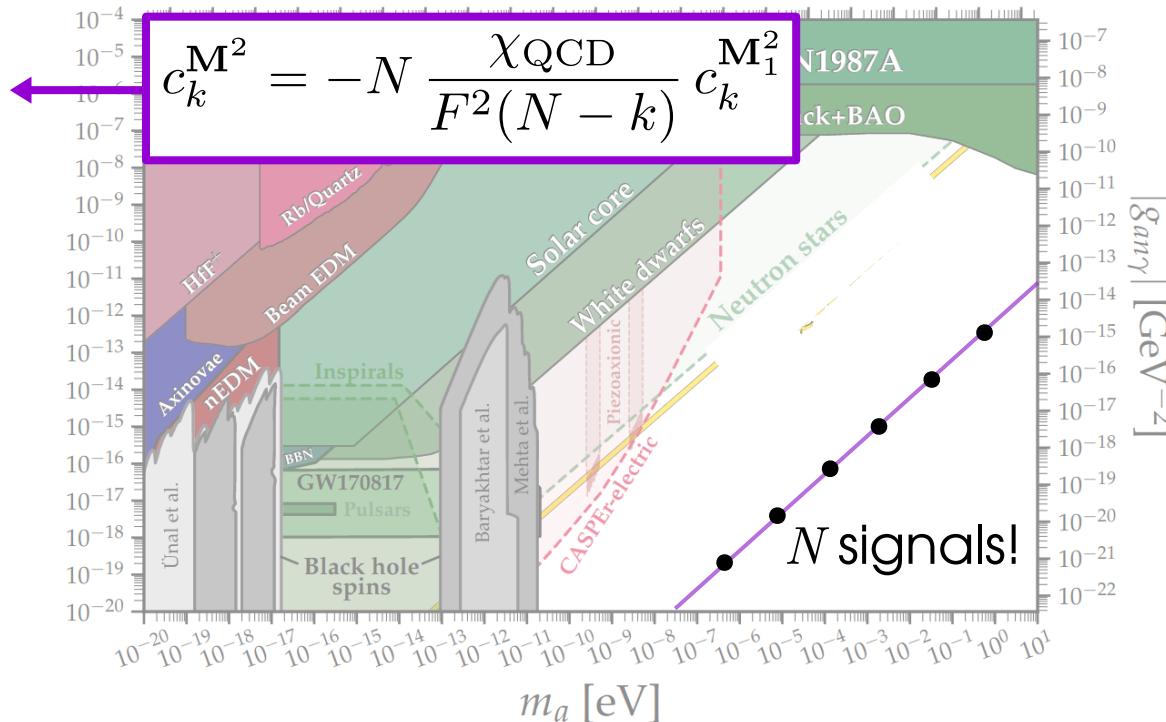
QCD Maxions

$$3. \max_i \left\{ \min_i \{g_i\} \right\} = N \implies g_i = N, \forall i$$

m-parameter family of maxions: $m = N(N + 1)/2$

$$\text{tr } \mathbf{M}^2 = N \frac{\chi_{\text{QCD}}}{F^2}$$

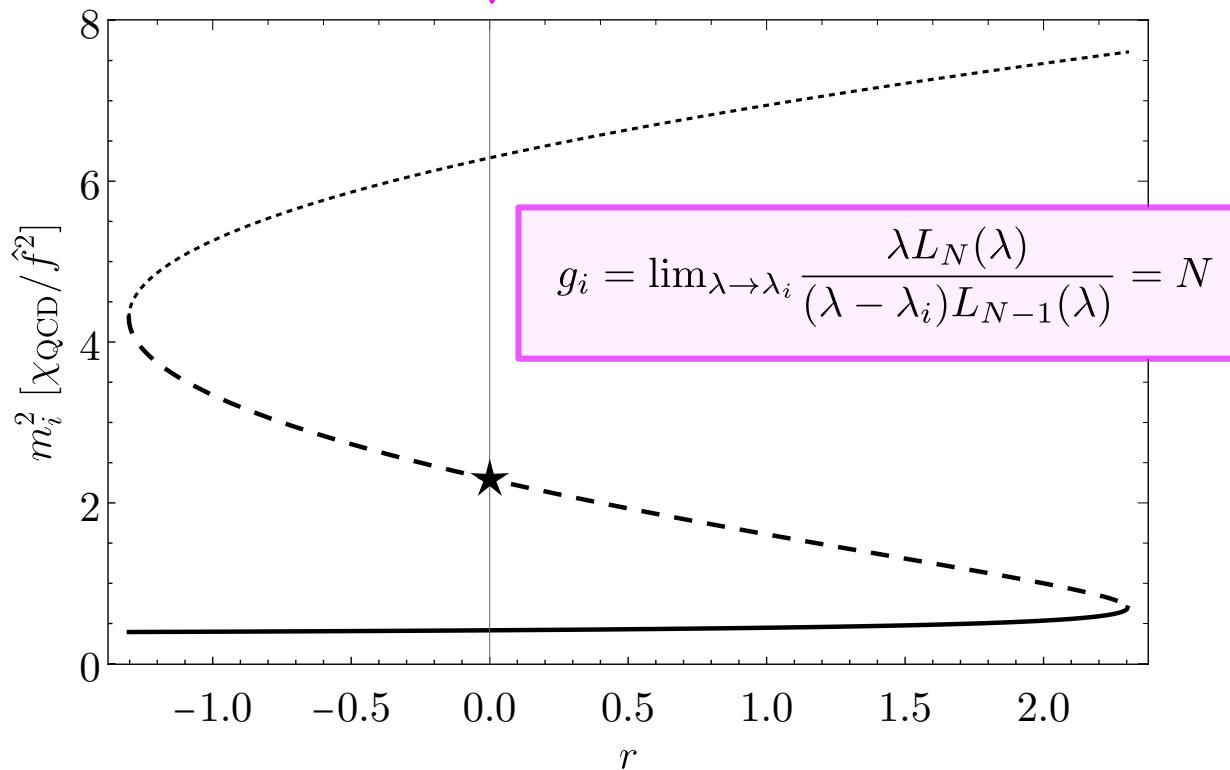
$$\frac{\det \mathbf{M}^2}{\det \mathbf{M}_1^2} = \frac{\chi_{\text{QCD}}}{F^2}$$



If you are interested in a UV completion, please ask!

e.g. Laguerre maxions

$$\hat{\mathbf{M}}^2 = \frac{\chi_{\text{QCD}}}{\hat{f}^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 - \sqrt{3 + r - r^2} & 1 + r \\ 1 & 1 + r & 4 + \sqrt{3 + r - r^2} \end{pmatrix}$$



Coupling to photons

Assuming universal anomaly factors,

$$\mathcal{L} \supset \frac{\alpha_{em}}{8\pi} \sum_{k=1}^N \frac{E_k}{\mathcal{N}_k} \frac{\hat{a}_k}{\hat{f}_k} F \tilde{F} \implies \frac{\alpha_{em}}{8\pi} \frac{E}{\mathcal{N}} \frac{a_{G\tilde{G}}}{F} F \tilde{F}$$

Making an axion-dependent rotation, $q = \begin{pmatrix} u \\ d \end{pmatrix} \rightarrow e^{i\gamma_5 a_{G\tilde{G}}/(2F)Q_a} \begin{pmatrix} u \\ d \end{pmatrix}$:

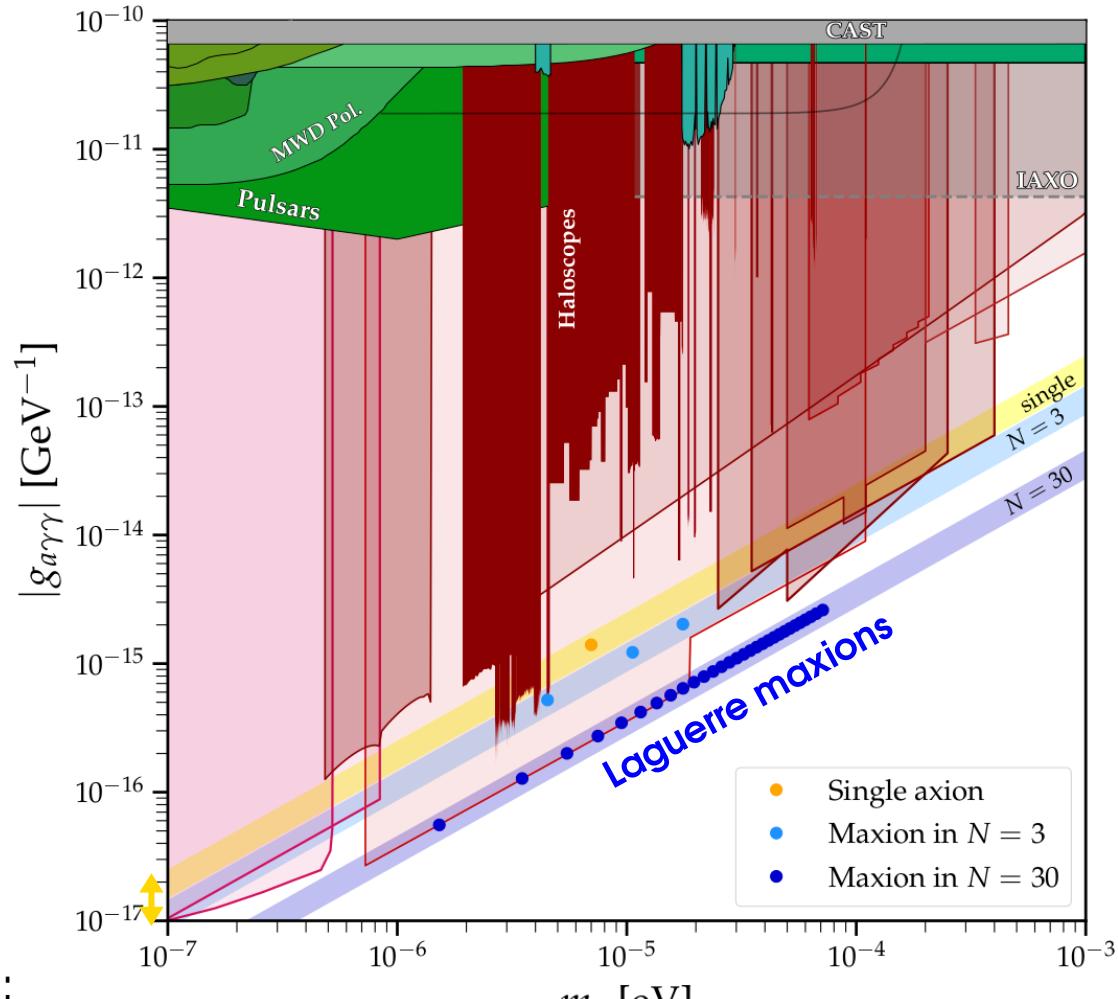
G. di Cortona, E. Hardy, J. Vega, G. Villadoro 15

$$\mathcal{L} \supset \frac{\alpha_{em}}{2\pi} \left[\frac{E}{\mathcal{N}} - 1.92 \right] \sum_i \frac{a_i}{f_i} F \tilde{F}$$

$$\boxed{\frac{m_i^2}{g_{a_i\gamma\gamma}^2} = \frac{m_a^2}{g_{a\gamma\gamma}^2} \Big|_{\text{single QCD axion}} \times g_i}$$

$$\frac{(2\pi)^2}{\alpha_{em}^2} \left[\frac{E}{\mathcal{N}} - 1.92 \right]^{-2} \sum_{i=1}^N \frac{g_{a_i\gamma\gamma}^2}{m_i^2} = 1$$

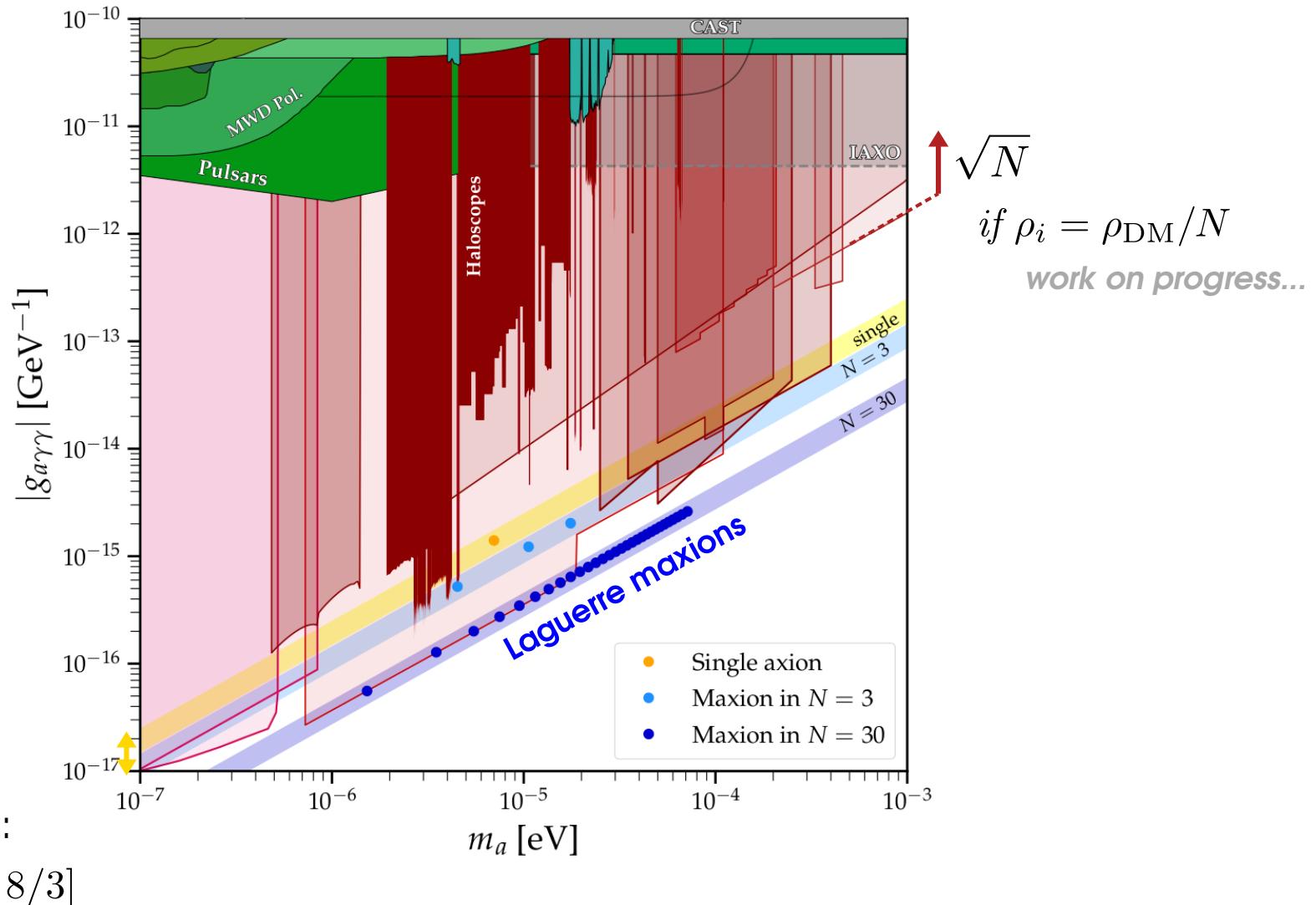
Coupling to photons



Band width spans:

$$E/\mathcal{N} \in [2/3, 8/3]$$

Coupling to photons



In summary

1. Any signal to the right of the canonical axion band can indicate a multiple QCD axion solution to the strong CP problem!
2. Our sum rule links the possible mass-scale values of the different axions, and allows us to count how many axions may exist in Nature
3. All axions can be maximally deviated from the QCD line, by a factor of \sqrt{N} .
4. Sizable effects require that the contribution from the extra potential is of the order of the QCD contribution

backup

UV completion

$$\mathcal{L}_{\text{UV}} = |\partial_\mu S_1|^2 + |\partial_\mu S_2|^2 + \bar{\Psi}_1 i D \Psi_1 + \bar{\Psi}_2 i D \Psi_2 - [y_1 \bar{\Psi}_1 \Psi_1 S_1 + y_2 \bar{\Psi}_2 \Psi_2 S_2 + \text{h.c.}] - V(S_{1,2})$$

After SSB,

$$S_{1,2} = \frac{1}{\sqrt{2}} \left(\hat{f}_{1,2} + \rho_{1,2} \right) e^{i \hat{a}_{1,2} / \hat{f}_{1,2}}.$$

$$V(S_{1,2}) = \lambda S_1^3 S_2 + \text{h.c.} \implies U(1) \times U(1) \rightarrow U(1)$$

After QCD confinement,

$$V_{\text{eff}} = \frac{1}{2} \chi_{\text{QCD}} \left(\frac{\hat{a}_1 + \hat{a}_2}{\hat{f}} - \bar{\theta} \right)^2 + \frac{\lambda}{4} \hat{f}^4 \left(\frac{3\hat{a}_1 + \hat{a}_2}{\hat{f}} \right)^2$$

$$\mathbf{M}^2 = \frac{\chi_{\text{QCD}}}{\hat{f}^2} \begin{pmatrix} 2 + 8r & -4r \\ -4r & 2r \end{pmatrix}, \quad \text{with } 1/F^2 = 2/\hat{f}^2$$

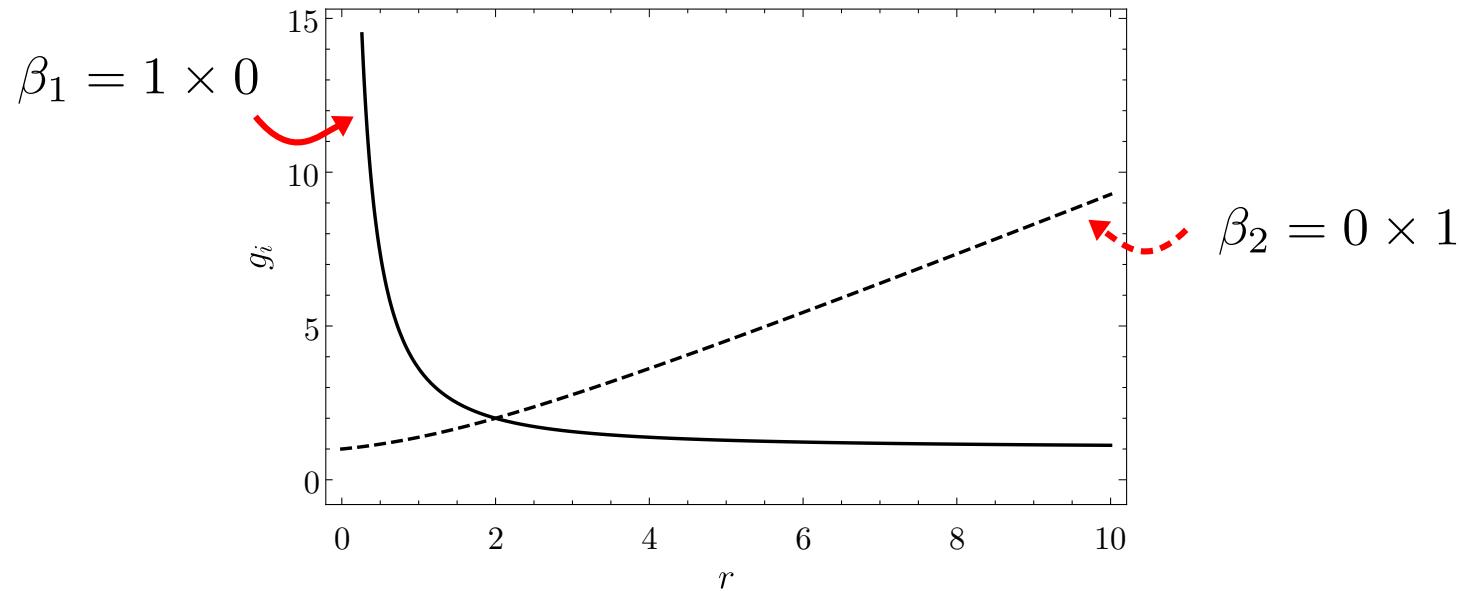
which contains **maxion** solutions ($r = 1/5$)!

backup Physical interpretation of the sum rule

$$\beta_i = \frac{\langle \hat{a}_{\text{PQ}} | a_i \rangle \langle a_i | \hat{a}_{G\tilde{G}} \rangle}{\langle \hat{a}_{\text{PQ}} | \hat{a}_{G\tilde{G}} \rangle}$$

Recall the [toy example](#):

$$\mathcal{L}_{N=2} = \frac{\alpha_s}{8\pi} \left(\frac{\hat{a}_1}{\hat{f}_1} + \frac{\hat{a}_2}{\hat{f}_2} + \bar{\theta} \right) G\tilde{G} - \hat{m}_2^2 \hat{a}_2^2 \implies \hat{a}_{GG} = \frac{1}{2} (\hat{a}_1 + \hat{a}_2) \quad \text{and} \quad \hat{a}_{\text{PQ}} = \hat{a}_1$$



backup

Potential scales

In the basis where the extra potential is diagonal, $\mathbf{M}_B^2 = \text{diag}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_N)$

$$g_i = \frac{m_i^2 F^2}{|\langle a_{G\tilde{G}}|a_i\rangle|^2 \chi_{\text{QCD}}} = \frac{m_i^2}{\left|\langle a_{\text{PQ}}|a_i\rangle/f_{\text{PQ}} + \sum_j^{N-1} \langle \tilde{a}_j|a_i\rangle/\tilde{f}_j\right|^2 \chi_{\text{QCD}}}$$

For $\tilde{\lambda}_j \gg \chi_{\text{QCD}}/F^2$:

$$\frac{1}{g_j} \sim \frac{|\langle a_{G\tilde{G}}|\tilde{a}_j\rangle|^2}{\tilde{\lambda}_j F^2} = \frac{(F/\tilde{f}_j)^2 \chi_{\text{QCD}}}{\tilde{\lambda}_j F^2} \leq \frac{\chi_{\text{QCD}}}{\tilde{\lambda}_j F^2} \rightarrow 0$$

For $\tilde{\lambda}_j \ll \chi_{\text{QCD}}/F^2$:

$$a_\varepsilon = \frac{a_{\text{PQ}}}{f_{\text{PQ}}} - \frac{\tilde{a}_j}{\tilde{f}_j} + \mathcal{O}(\varepsilon), \quad m_\varepsilon^2 \sim \tilde{\lambda}_j = \varepsilon \chi_{\text{QCD}}/F^2$$

$$\frac{1}{g_j} \sim \frac{|\langle a_{G\tilde{G}}|\tilde{a}_\varepsilon\rangle|^2 \chi_{\text{QCD}}}{\tilde{\lambda}_j F^2} \sim \frac{\varepsilon^2}{\varepsilon} \rightarrow 0$$

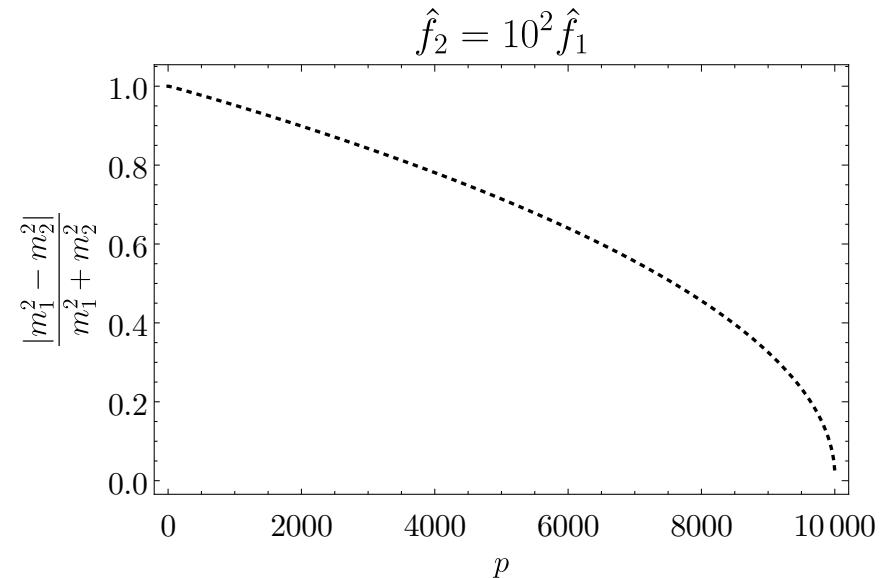
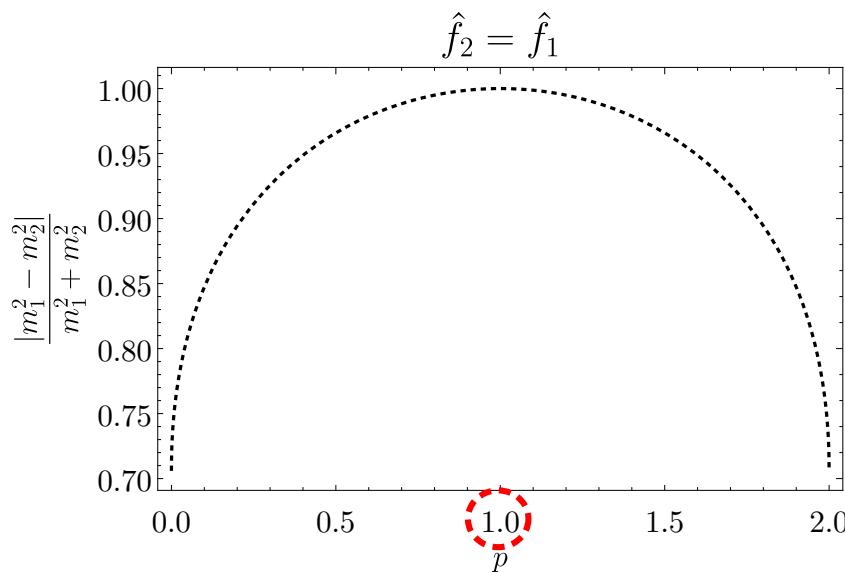
Whenever one scale is very different from the QCD induced mass, one state decouples.

backup

Eigenvalues dispersion

All families of maxions (with same scale) for N=2:

$$\mathbf{M}_{N=2}^2 = \frac{\chi_{\text{QCD}}}{\hat{f}^2} \begin{pmatrix} 2-p & 1 + \sqrt{p(2-p)} \\ 1 + \sqrt{p(2-p)} & 1+p \end{pmatrix}$$



Limiting case: Massless state has no mixing with gluons, the heavy one with mass $\sim 4 \frac{\chi_{\text{QCD}}}{\hat{f}^2}$