IBFFM and Transfer Reactions

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in collaboration with R. Biker and R. Magana

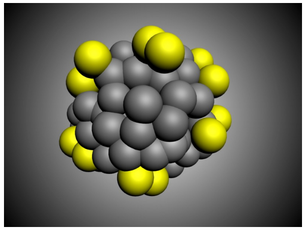
IBFFM and Transfer Reactions

based on R. Magana, E. Santopinto, R. Bijker, PRC106**,**044307 (2022)

- Interacting Boson Model (IBM) and its extensions
- IBFFM-2
- One body transition density for ^{116}Cd - ^{116}In and ^{116}In- ^{116}Sn

  for single charge exchange reactions for NUMEN
- Future plan (Hugo Garcia-Tecocoatzi, R. Magana, R. Biker, E. Santopinto)

Interacting Boson Model



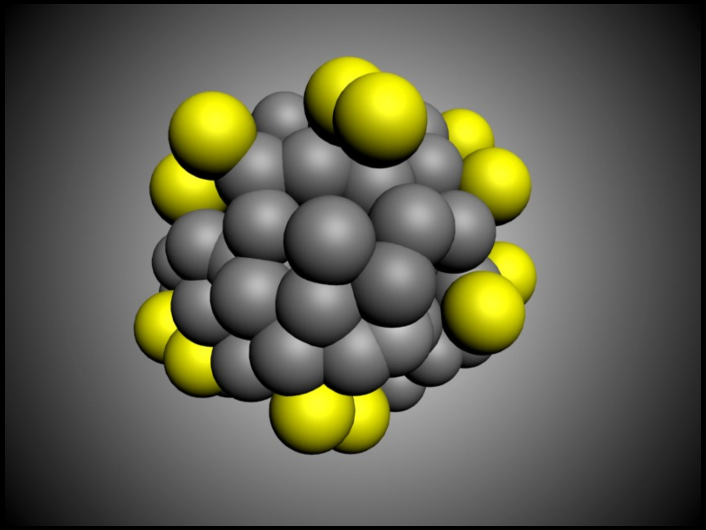
- The IBM describes collective excitations in even- even nuclei in terms of a system of correlated pairs of nucleons with angular momentum L=0 and L=2 which are treated as bosons (s and d bosons) (Arima and Iachello, 1974)
- The number of bosons N is half the number of valence nucleons
- Introduce boson creation and annilation operators

$$b_i^\dagger,\ b_i, \quad i = l, m \quad (l = 0, 2 \quad -l \leq m \leq l)$$

which satisfy the commutation relations

$$\left[b_i, b_j^\dagger\right] = \delta_{ij}, \quad \left[b_i^\dagger, b_j^\dagger\right] = [b_i, b_j] = 0$$

even-even



$$b_i^\dagger,\ b_i, \quad i = l, m \quad (l = 0, 2 \quad -l \leq m \leq l)$$

$$\left[b_i, b_j^\dagger\right] = \delta_{ij}, \quad \left[b_i^\dagger, b_j^\dagger\right] = [b_i, b_j] = 0$$

$$G_i^{\ j} = b_i^\dagger b_j$$

Gij are the Lie algebra U (6) algebra

$i,j=1,\ldots,6$

$$\left[G_i^{\ j}, G_l^{\ k}\right] = G_i^{\ j}\delta_{jk} - G_l^{\ k}\delta_{il}$$

$i,j,k,l=1,\ldots,6$

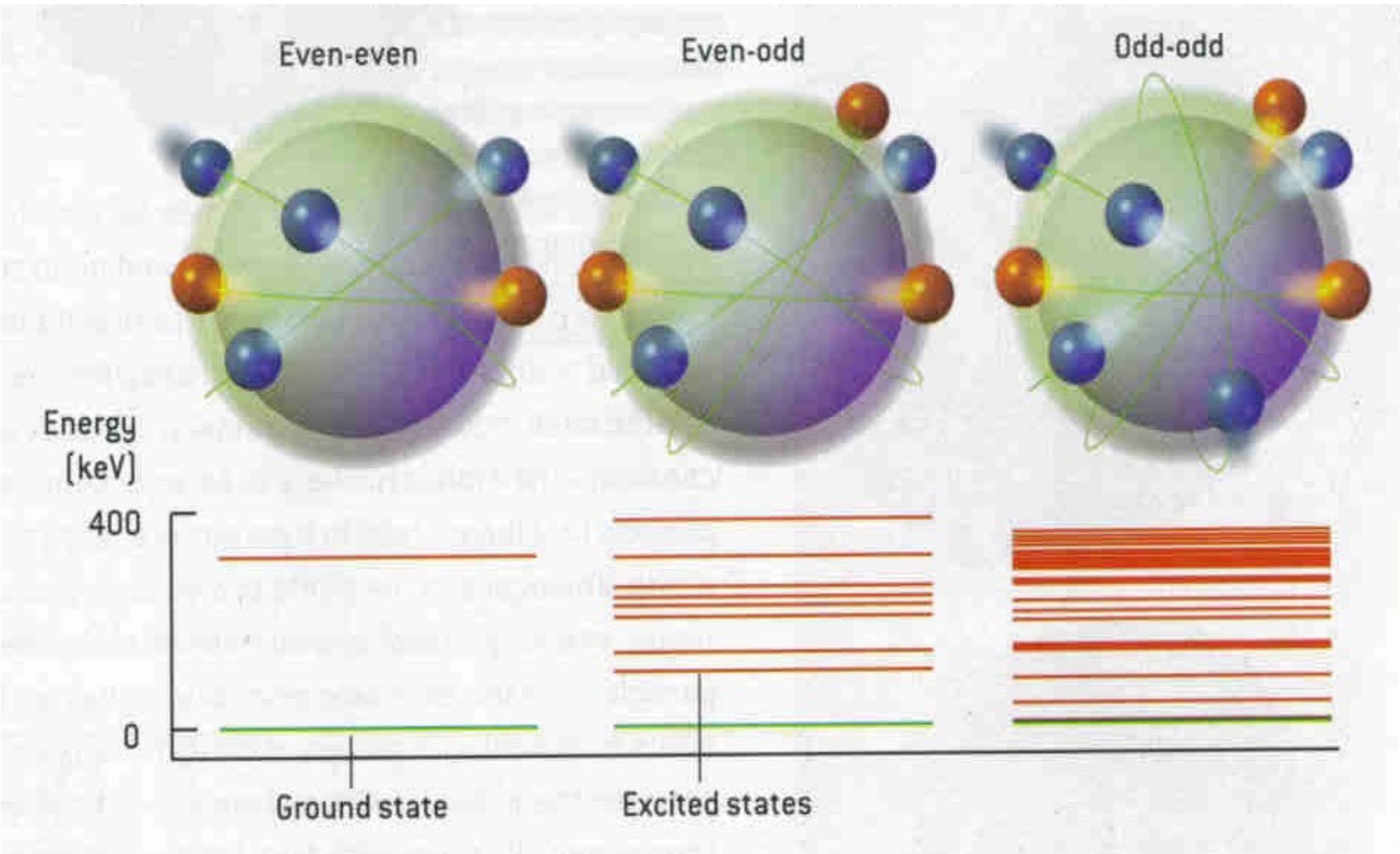
A. Arima y F. Iachello Phys. Rev. Lett. ***35*** *1069* ***(1975)***

IBFM

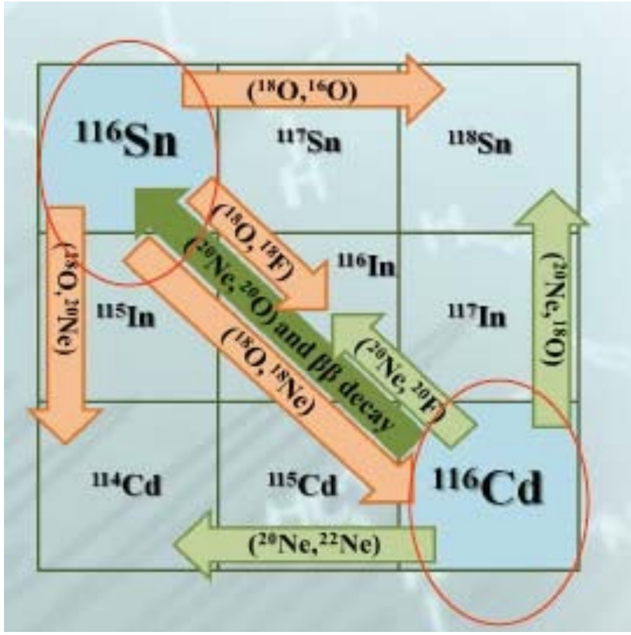
- Consider an extension of the IBM which includes, in addition to the collective degrees of freedom (bosons), single- particle degrees of freedom of an extra unpaired proton or neutron (fermion with angular momentum $j=j_1, j_2, \ldots$)
- For the extra nucleon, introduce fermion creation and annihilation operators satisfy anticommutation relations

$$\{a_\mu, a_\nu^\dagger\} = \delta_{\mu\nu}, \quad \{a_\mu^\dagger, a_\nu^\dagger\} = \{a_\mu, a_\nu\} = 0$$

IBM IBFM IBFFM



IBFFM-2 Model



$^{117}_{49}\mathrm{In}_{68} \; : \; ^{118}_{50}\mathrm{Sn}_{68} \otimes \pi^{-1}$

$^{117}_{50}\mathrm{Sn}_{67} \; : \; ^{118}_{50}\mathrm{Sn}_{68} \otimes \nu^{-1}$

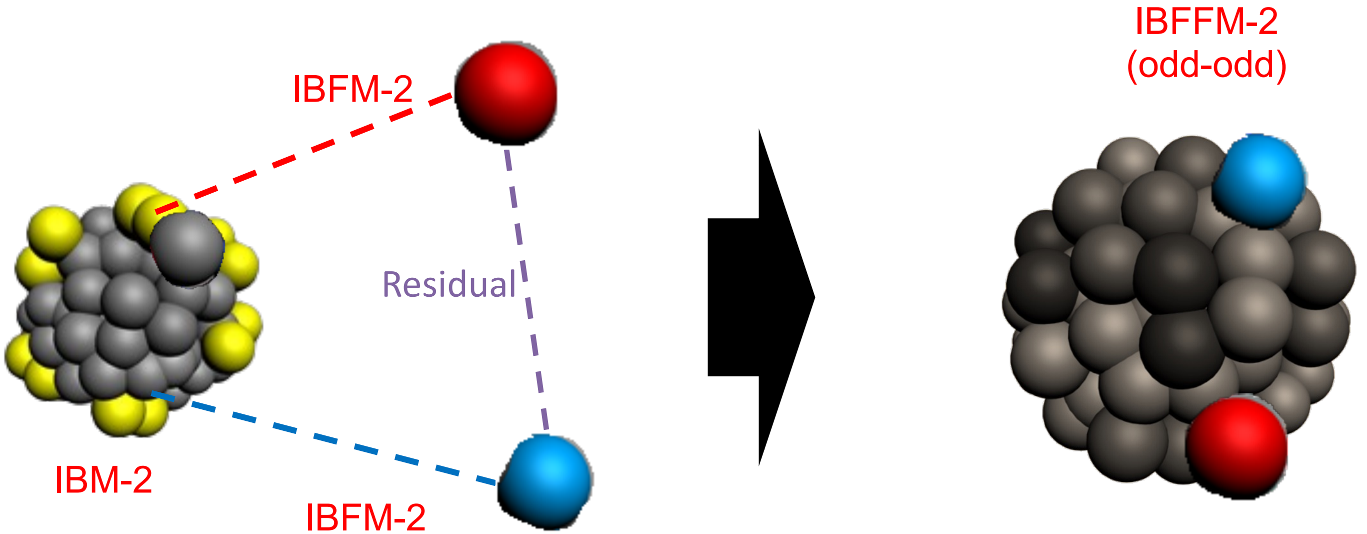
$^{116}_{49}\mathrm{In}_{67} \; : \; ^{118}_{50}\mathrm{Sn}_{68} \otimes \nu^{-1}\pi^{-1}$

Hamiltonian

$$\begin{aligned} H &= H^B + H^F_\pi + V^{BF}_\pi \\ &\quad + H^F_\nu + V^{BF}_\nu + V_{\mathrm{res}} \end{aligned}$$

H^B	even-even	$^{118}\mathrm{Sn}$
$H^F_\pi + V^{BF}_\pi$	odd-proton	$^{117}\mathrm{In}$
$H^F_\nu + V^{BF}_\nu$	odd-neutron	$^{117}\mathrm{Sn}$
V_{res}	odd-odd	$^{116}\mathrm{In}$

Interacting Boson Fermion Fermion Model- 2



excitations of nuclei are described by bosons and two fermions.

- A detailed knowledge of the even even core and odd mass neighbours is required.

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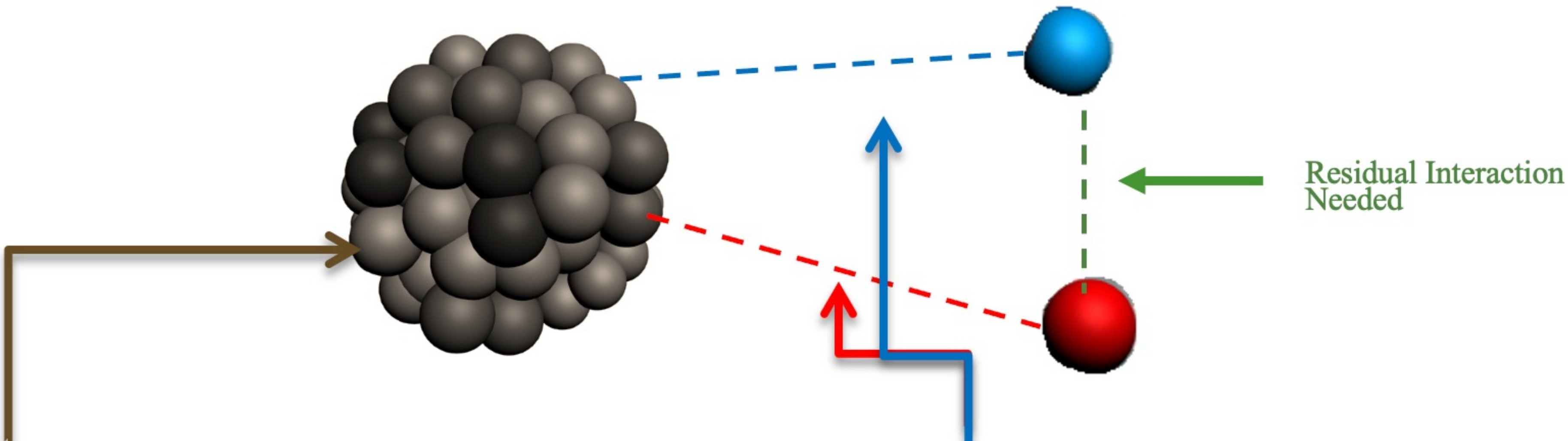
odd-odd nuclei



$$H_B = E_0 + \epsilon_\pi \hat{n}_{d_\pi} + \epsilon_\nu \hat{n}_{d_\nu} + \kappa \hat{Q}^\chi_\pi \cdot \hat{Q}^\chi_\nu + \lambda' \hat{M}_{\pi\nu} + V_{\pi\pi} + V_{\nu\nu}$$

- And the extra fermion is given by the standard IBFM Hamiltonian

$$\begin{aligned} H_F &= E_0 + \sum_{j_\pi} \epsilon_{j_\pi} \hat{n}_{j_\pi} + \sum_{j_\nu} \epsilon_{j_\nu} \hat{n}_{j_\nu} \\ V_{BF} &= \sum_{j_\pi} A_{j\pi} (\hat{n}_{d_\pi} \hat{n}_{j_\pi}) + \sum_{j_\nu} A_{j\nu} (\hat{n}_{d_\nu} \hat{n}_{j_\nu}) \\ &+ \Gamma_{\pi\nu} \hat{Q}^\chi_\nu \cdot \hat{q}_\pi + + \Gamma_{\nu\pi} \hat{Q}^\chi_\pi \cdot \hat{q}_\nu + \Gamma_{\nu\nu} \hat{Q}^\chi_\nu \cdot \hat{q}_\nu + \Gamma_{\pi\pi} \hat{Q}^\chi_\pi \cdot \hat{q}_\pi \\ &+ \Lambda_{\nu\pi} F_{\pi\nu} + \Lambda_{\pi\nu} F_{\nu\pi} \end{aligned}$$

V_{res}

$$V_{\rm res} = H_\delta + H_{\sigma\sigma\delta} + H_{\sigma\sigma} + H_T,$$

S. Brant and V. Paar,
Z. Phys., 329, 151 (1988).

$$H_\delta = 4\pi V_\delta \delta(\vec{r}_\pi - \vec{r}_\nu)\delta(r_\pi - R_0)\delta(r_\nu - R_0),$$

$$H_{\sigma\sigma\delta} = 4\pi V_{\sigma\sigma\delta}\delta(\vec{r}_\pi - \vec{r}_\nu)(\vec{\sigma}_\pi \cdot \vec{\sigma}_\nu) \times\delta(r_\pi - R_0)\delta(r_\nu - R_0),$$

$$H_{\sigma\sigma} = -\sqrt{3}V_{\sigma\sigma}\vec{\sigma}_\pi \cdot \vec{\sigma}_\nu,$$

$$H_T = V_T\left[\frac{3(\vec{\sigma}_\pi \cdot \vec{r}_{\pi\nu})(\vec{\sigma}_\nu \cdot \vec{r}_{\pi\nu})_{\pi\nu}}{r^2} - \vec{\sigma}_\pi \cdot \vec{\sigma}_\nu\right]$$

TABLE IV. Parameters of proton-neutron residual interaction in MeV.

Parameter	Value
V_δ	-0.40
$V_{\sigma\sigma\delta}$	0.00
$V_{\sigma\sigma}$	0.00
V_T	0.80

R. Magana, E. Santopinto, R. Bijker, PRC106, 044307 (2022)

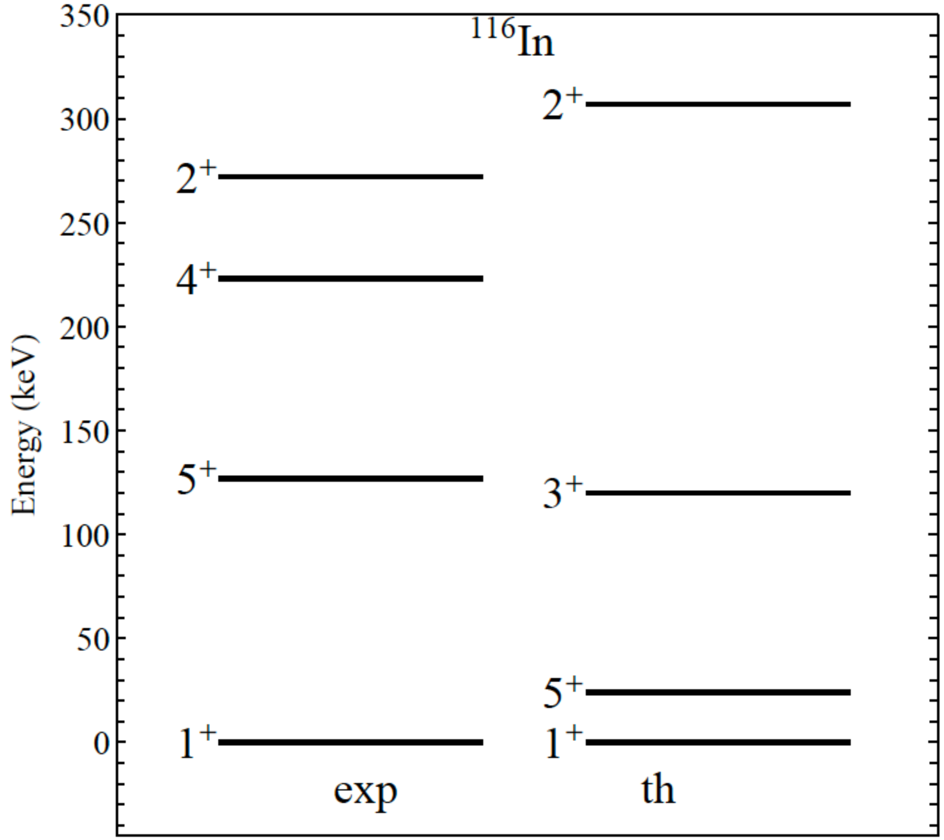


TABLE V. Comparison of theoretical and experimental energy levels in keV of the odd-odd nucleus ^{116}In [24].

Exp.		Th.	
J^P	E	J^P	E
1^+	0.0	1^+	0.0
5^+	127.3	5^+	23.9
4^+	223.3	3^+	120.3
2^+	273.0	2^+	307.9
$4^+, 5^+$	313.5	6^+	359.8
4^+	425.9	4^+	373.9
$4^+, 5^+$	460.0	3^+	401.7
3^+	508.2	5^+	489.8
		4^+	491.1

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Single-Charge Exchange and OBTD

	N_π	N_ν	
$^{116}_{48}\mathrm{Cd}_{68}$	1	7	
$^{116}_{49}\mathrm{In}_{67}$	0	7	$\nu^{-1}\pi^{-1}$
$^{116}_{50}\mathrm{Sn}_{66}$	0	8	

$$c^\dagger_{j_\rho} \quad \rightarrow \quad P^\dagger_{j_\rho} = \xi_{j_\rho} a^\dagger_{j_\rho} + \sum_{j'_\rho} \xi_{j_\rho j'_\rho} [[s^\dagger_\rho \times \tilde{d}_\rho]^{(2)} \times a^\dagger_{j'_\rho}]^{(j_\rho)}$$

$$c^\dagger_{j_\rho} \quad \rightarrow \quad Q^\dagger_{j_\rho} = \theta_{j_\rho} (s^\dagger_\rho \times \tilde{a}_{j_\rho})^{(j_\rho)} + \sum_{j'_\rho} \theta_{j_\rho j'_\rho} [d^\dagger_\rho \times \tilde{a}_{j'_\rho}]^{(j_\rho)}$$

N.Yoshida, F. Iachello, Prog. Theor. Exp. Phys. 2013, 043D01

$$\mathrm{OBTD}_1 = \frac{\mathrm{SA}_1}{\sqrt{2\lambda+1}} = \frac{\left\langle \Psi(^{116}\mathrm{In})_J \left| \left[P^\dagger_{j_\nu} \times \tilde{Q}_{j'_\pi} \right]^{(\lambda)} \right| \Psi(^{116}\mathrm{Cd})_{gs} \right\rangle}{\sqrt{2\lambda+1}}$$

$$\mathrm{OBTD}_2 = \frac{\mathrm{SA}_2}{\sqrt{2\lambda+1}} = \frac{\left\langle \Psi(^{116}\mathrm{Sn})_{gs} \left| \left[Q^\dagger_{j_\nu} \times \tilde{P}_{j'_\pi} \right]^{(\lambda)} \right| \Psi(^{116}\mathrm{In})_J \right\rangle}{\sqrt{2\lambda+1}}$$

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TABLE VI. Spectrocopic amplitudes (SA) and one-body transition densities (OBTD) for transitions from the 0_1^+ ground state of ^{116}Sn and ^{116}Cd to the positive parities J_i^+ states of ^{116}In.

^{116}In			^{116}Sn(0_1^+)		^{116}Cd(0_1^+)	
J_i^P	j_ν	j_π	SA	OBTD	SA	OBTD
1_1^+	$1g_{7/2}$	$1g_{9/2}$	0.0650	0.0375	−0.0788	−0.0455
5_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0300	−0.0090	0.0055	0.0016
	$1g_{7/2}$	$1g_{9/2}$	−0.0270	−0.0081	0.0187	0.0056
	$3s_{1/2}$	$1g_{9/2}$	−0.0677	−0.0204	0.2289	0.0690
	$2d_{3/2}$	$1g_{9/2}$	−0.0338	−0.0102	0.1576	0.0475
	$1h_{11/2}$	$2p_{1/2}$	0.0134	0.0040	−0.1451	−0.0438
	$1h_{11/2}$	$2p_{3/2}$	−0.0045	−0.0014	0.0666	0.0201
	$1h_{11/2}$	$1f_{5/2}$	0.0042	0.0013	−0.0554	−0.0167
4_1^+	$2d_{5/2}$	$1g_{9/2}$	0.0083	0.0028	−0.0226	−0.0075
	$1g_{7/2}$	$1g_{9/2}$	0.0195	0.0065	0.0015	0.0005
	$3s_{1/2}$	$1g_{9/2}$	0.0271	0.0090	−0.1058	−0.0353
	$2d_{3/2}$	$1g_{9/2}$	0.0323	0.0108	−0.2125	−0.0708
	$1h_{11/2}$	$2p_{3/2}$	0.0006	0.0002	−0.0050	−0.0017
	$1h_{11/2}$	$1f_{5/2}$	0.0032	0.0011	−0.0523	−0.0174
2_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0036	−0.0016	0.0055	0.0025
	$1g_{7/2}$	$1g_{9/2}$	0.0453	0.0203	−0.0667	−0.0298
4_2^+	$2d_{5/2}$	$1g_{9/2}$	0.0014	0.0005	0.0064	0.0021
	$1g_{7/2}$	$1g_{9/2}$	−0.0066	−0.0022	−0.0053	−0.0018
	$3s_{1/2}$	$1g_{9/2}$	0.0205	0.0068	−0.0884	−0.0295
	$2d_{3/2}$	$1g_{9/2}$	0.0138	0.0046	−0.0651	−0.0217
	$1h_{11/2}$	$2p_{3/2}$	−0.0001	−0.0000	0.0035	0.0012
	$1h_{11/2}$	$1f_{5/2}$	0.0007	0.0002	−0.0003	−0.0001
3_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0196	−0.0074	0.0262	0.0099
	$1g_{7/2}$	$1g_{9/2}$	−0.0359	−0.0136	0.0301	0.0114
	$2d_{3/2}$	$1g_{9/2}$	−0.0437	−0.0165	0.2567	0.0970
	$1h_{11/2}$	$1f_{5/2}$	0.0110	0.0042	−0.1473	−0.0557

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Future Work:

Contruction of the complete IBFFM-2 odd-odd code
considering also the negative parity states
(Hugo Garcia-Tecocoatzi, R. Magana, R. Biker, E. Santopinto)

Our Goal for NUMEN:
Calculation of the form factors
to be inserted into the Lenske-Colonna reaction code
(Hugo Garcia-Tecocoatzi, R. Magana, R. Biker, E. Santopinto)

Other possible applications :
$0\nu\beta\beta$ decay without closure approximation

The matrix elements of the residual interaction were calculated in the quasi particle basis, which is linked to the particle basis by :

$$\begin{aligned}\langle j'_\nu j'_\pi; J|V_{\rm res}|j_\nu j_\pi; J\rangle_{qp} = {} & (u_{j'_\nu} u_{j'_\pi} u_{j_\nu} u_{j_\pi} + v_{j'_\nu} v_{j'_\pi} v_{j_\nu} v_{j_\pi}) \\ & \times \langle j'_\nu j'_\pi; J|V_{\rm res}|j_\nu j_\pi; J\rangle \\ & - (u_{j'_\nu} v_{j'_\pi} u_{j_\nu} v_{j_\pi} + v_{j'_\nu} u_{j'_\pi} v_{j_\nu} u_{j_\pi}) \\ & \times \sum_{J'} (2J'+1) \begin{Bmatrix} j'_\nu & j_\pi & J' \\ j_\nu & j'_\pi & J \end{Bmatrix} \\ & \times \langle j'_\nu j_\pi; J'|V_{RES}|j_\nu j'_\pi; J'\rangle, \qquad (11)\end{aligned}$$

where $v_j^2 = 1 - u_j^2$ denotes the occupation probability.

TABLE I. IBM-2 parameters in MeV taken from Ref. [22]. χ_ν and χ_π are dimensionless.

Nucleus	N_ν	N_π	ϵ_d	κ	χ_ν	χ_π	$\xi_1 = \xi_3$	ξ_2	$c_\nu^{(0)}$	$c_\nu^{(2)}$	$c_\nu^{(4)}$	v_0^ν	v_2^ν
^{116}Sn	8	0	1.32						−0.50	−0.22	−0.07	−0.06	0.04
^{118}Sn	7	0	1.31						−0.50	−0.23	−0.08	−0.11	0.09
^{116}Cd	7	1	0.85	−0.27	−0.58	0.00	−0.18	0.24	−0.15	−0.06			

$$H = H^B + H_\pi^F + V_\pi^{BF} + H_\nu^F + V_\nu^{BF} + V_{\text{res}}. \tag{1}$$

H^B is the IBM-2 Hamiltonian. H_π^F and H_ν^F are the proton and neutron single-particle terms

$$H_\rho^F = \sum_{j_\rho} \epsilon_{j_\rho} \hat{n}_{j_\rho}, \tag{2}$$

where ϵ_{j_ρ} is the quasiparticle energy of the extra nucleon and $\hat{n}$ is the number operator. The quasiparticle energies ϵ_{j_ρ} are obtained within the BCS approximation with a gap $\Delta = 12/\sqrt{A}$ MeV, where A is the mass number of the even-even core nucleus. In the BCS approximation, the quasi-particle energies are related to the single-particle level E_j, the occupation probabilities v_j, and the Fermi level λ as follows:

$$\begin{aligned} \epsilon_j &= \sqrt{(E_j - \lambda)^2 + \Delta^2}, \\ v_j^2 &= \frac{1}{2}\left(1 - \frac{E_j - \lambda}{\epsilon_j}\right), \\ u_j^2 &= 1 - v_j^2. \end{aligned} \tag{3}$$

V_π^{BF} and V_ν^{BF} describe the core-particle coupling of the odd proton and odd neutron in the IBFM-2 model [19–21] as the sum of a quadrupole term (Γ_ρ), an exchange term (Λ_ρ), and a monopole term (A_ρ):

$$V_\rho^{BF} = \Gamma_\rho Q_{\rho'}^{(2)} \cdot q_\rho^{(2)} + \Lambda_\rho F_{\rho'\rho} + A_\rho \hat{n}_{d_{\rho'}} \cdot \hat{n}_\rho \tag{4}$$

where $\rho' \neq \rho$ and $\rho, \rho' = \nu, \pi$. The first term in Eq. (4) is a quadrupole-quadrupole interaction with

$$q_\rho^{(2)} = \sum_{j_\rho, j'_\rho} (u_{j_\rho} u_{j'_\rho} - v_{j_\rho} v_{j'_\rho}) Q_{j_\rho j'_\rho} (a^\dagger_{j_\rho} \times \tilde{a}_{j'_\rho})^{(2)},$$
$$Q_\rho^{(2)} = (s^\dagger_\rho \times \tilde{d}_\rho + d^\dagger_\rho \times \tilde{s}_\rho)^{(2)} + \chi_\rho (d^\dagger_\rho \times \tilde{d}_\rho)^{(2)}. \quad (5)$$

The second term is the exchange interaction

$$F_{\rho,\rho'} = - \sum_{j_\rho j'_\rho j''_\rho} \beta_{j_\rho j'_\rho} \beta_{j''_\rho j_\rho} \sqrt{\frac{10}{N_\rho (2 j_\rho + 1)}}$$
$$\times Q^{(2)}_{\rho'} \cdot : [(d_\rho \times \tilde{a}_{j''_\rho})^{(j_\rho)} \times (a^\dagger_{j'_\rho} \times \tilde{s}_\rho)^{(j'_\rho)}]^{(2)} : + \text{H.c.} \quad (6)$$

The coefficients $\beta_{j_\rho j'_\rho}$ are related to the single-particle matrix elements of the quadrupole operator $Q_{j_\rho j'_\rho}$ by

$$\beta_{j_\rho j'_\rho} = (u_{j_\rho} v_{j'_\rho} + v_{j_\rho} u_{j'_\rho}) Q_{j_\rho j'_\rho},$$
$$Q_{j_\rho j'_\rho} = \langle j_\rho \| Y^{(2)} \| j'_\rho \rangle. \quad (7)$$

The last term is the monopole-monopole interaction with

$$n_{d_\rho} = \sum_m d^\dagger_{\rho,m} d_{\rho,m},$$
$$\hat{n}_\rho = \sum_{j_\rho} \hat{n}_{j_\rho} = \sum_{j_\rho, m} a^\dagger_{j_\rho,m} a_{j_\rho,m}. \quad (8)$$

The residual interaction between the odd-proton and odd-neutron is defined as [13,14]

$$V_{\rm res} = H_\delta + H_{\sigma\sigma\delta} + H_{\sigma\sigma} + H_T, \tag{9}$$

with

$$\begin{aligned} H_\delta &= 4\pi V_\delta \delta(\vec{r}_\pi - \vec{r}_\nu)\delta(r_\pi - R_0)\delta(r_\nu - R_0), \\ H_{\sigma\sigma\delta} &= 4\pi V_{\sigma\sigma\delta}\delta(\vec{r}_\pi - \vec{r}_\nu)(\vec{\sigma}_\pi \cdot \vec{\sigma}_\nu) \\ &\quad \times \delta(r_\pi - R_0)\delta(r_\nu - R_0), \\ H_{\sigma\sigma} &= -\sqrt{3} V_{\sigma\sigma} \vec{\sigma}_\pi \cdot \vec{\sigma}_\nu, \\ H_T &= V_T \left[\frac{3(\vec{\sigma}_\pi \cdot \vec{r}_{\pi\nu})(\vec{\sigma}_\nu \cdot \vec{r}_{\pi\nu})_{\pi\nu}}{r^2} - \vec{\sigma}_\pi \cdot \vec{\sigma}_\nu \right], \end{aligned} \tag{10}$$

where $\vec{r}_{\pi\nu} = \vec{r}_\pi - \vec{r}_\nu$ and $R_0 = 1.2A^{1/3}$fm. The matrix elements of the residual interaction are calculated in the quasiparticle basis, which is related to the particle basis by [14]

$$\begin{aligned} \langle j'_\nu j'_\pi; J | V_{\rm res} | j_\nu j_\pi; J \rangle_{qp} &= (u_{j'_\nu} u_{j'_\pi} u_{j_\nu} u_{j_\pi} + v_{j'_\nu} v_{j'_\pi} v_{j_\nu} v_{j_\pi}) \\ &\quad \times \langle j'_\nu j'_\pi; J | V_{\rm res} | j_\nu j_\pi; J \rangle \\ &\quad - (u_{j'_\nu} v_{j'_\pi} u_{j_\nu} v_{j_\pi} + v_{j'_\nu} u_{j'_\pi} v_{j_\nu} u_{j_\pi}) \\ &\quad \times \sum_{J'} (2J' + 1) \begin{Bmatrix} j'_\nu & j_\pi & J' \\ j_\nu & j'_\pi & J \end{Bmatrix} \\ &\quad \times \langle j'_\nu j_\pi; J' | V_{RES} | j_\nu j'_\pi; J' \rangle, \end{aligned} \tag{11}$$

where $v_j^2 = 1 - u_j^2$ denotes the occupation probability.

In the IBM and its extensions the boson and fermion degrees of freedom are counted with respect to the nearest closed shell. The initial even-even nucleus ${}^{116}_{48}\mathrm{Cd}_{68}$ has $N_\pi = 1$ proton bosons and $N_\nu = 7$ neutron bosons, and the final nucleus ${}^{116}_{50}\mathrm{Sn}_{66}$ has $N_\pi = 0$ and $N_\nu = 8$. The intermediate odd-odd nucleus ${}^{116}_{49}\mathrm{In}_{67}$ is described in the IBM as a proton hole and a neutron hole coupled to the core nucleus ${}^{118}_{50}\mathrm{Sn}_{68}$ with $N_\pi = 0$ and $N_\nu = 7$. The charge-exchange reaction from ${}^{116}\mathrm{Cd}$ to ${}^{116}\mathrm{In}$ involves a change in the number of proton bosons, and from ${}^{116}\mathrm{In}$ to ${}^{116}\mathrm{Sn}$ a change in the number of neutron bosons. The parameters in the boson-fermion Hamiltonians are determined in a study of the corresponding odd-even nucleus: V_ν^{BF} from the odd-neutron nucleus ${}^{117}\mathrm{Sn}$ and V_π^{BF} from the odd-proton nucleus ${}^{117}\mathrm{In}$.

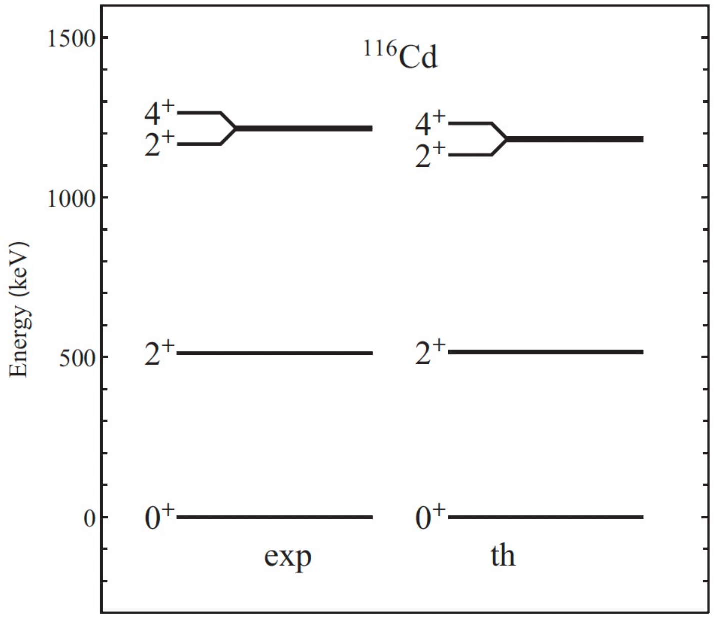

FIG. 1. The energy levels obtained in the calculation in comparison with the available experimental data for even-even nucleus ^{116}Cd [24].

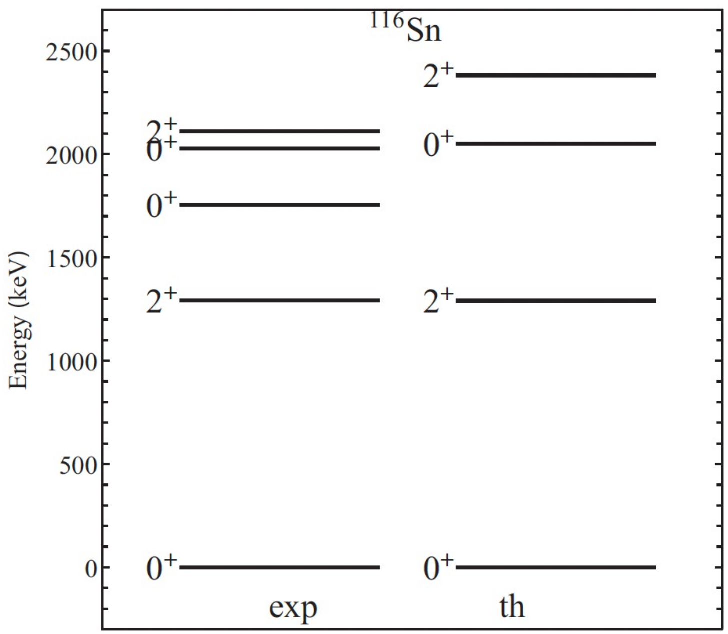

FIG. 2. The spectrum of ^{116}Sn in comparison with the experimental data [24].

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Nucleus	N_ν	N_π	ϵ_d	κ	χ_ν	χ_π	$\xi_1 = \xi_3$	ξ_2	$c_\nu^{(0)}$	$c_\nu^{(2)}$	$c_\nu^{(4)}$	v_0^ν	v_2^ν
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[22] J. Barea, J. Kotila, and F. Iachello, Phys. Rev. C **87**, 057301 (2013).

[24] J. Blachot, Nucl. Data Sheets **111**, 717 (2010).

TABLE II. Single-particle energies E_j (MeV), quasiparticle energies ϵ_j (MeV), and occupation probabilities v_j^2.

	E_j	ϵ_j	v_j^2
$2\nu d_{5/2}$	0.20	2.17	0.93
$1\nu g_{7/2}$	0.60	1.84	0.90
$3\nu s_{1/2}$	2.10	1.11	0.48
$1\nu h_{11/2}$	3.00	1.45	0.18
$2\nu d_{3/2}$	2.60	1.23	0.28
$2\pi p_{1/2}$	1.05	2.69	0.96
$2\pi p_{3/2}$	0.20	3.48	0.97
$1\pi f_{5/2}$	0.45	3.25	0.97
$1\pi g_{9/2}$	1.50	2.29	0.94

The second step is to construct the two odd-even associated nuclei, i.e., the core nucleus plus an extra neutron and the core nucleus plus an extra proton. It allows us to get a reliable set of parameters that we use to construct the odd-odd wave function. In this case, the odd-even nuclei are described in the context of the IBFM-2 [19,20,26], where the degrees of freedom of the extra nucleon are taken into account. In the IBFM-2, the Hamiltonian is given by

$$H = H^B + H^F_\rho + V^{BF}_\rho, \tag{12}$$

where H^B is the boson Hamiltonian that describes the core nucleus, and the label ρ refers to the π (extra proton) or ν (extra neutron) is added in the even-even core to form the odd-even nucleus.

The quasiparticle energies and occupation probabilities were obtained solving the BCS equations with the single-particle energies calculated and reported in Table II. The value of λ is constrained to the conservation of the number of particles as follows:

$$2N_\rho = \sum_{j_\rho} v^2_{j_\rho}(2j_\rho + 1). \tag{13}$$

The odd-odd ^{116}In nucleus uses the quasiparticle energies obtained from the single-particle energies for the odd-neutron nucleus ^{117}Sn and the odd-proton nucleus ^{117}In. Both odd-even nuclei have the same even-even core nucleus ^{118}Sn. The core-particle coupling parameters reported in Table III were obtained by fitting the experimental data for ^{117}Sn and ^{117}In.

For the odd-odd nucleus ^{116}In, the tensor and surface delta interaction (SDI) play an essential role. In this work, the parameters are obtained by minimization of the mean square

TABLE III. Parameters in boson-fermion interaction in MeV.

ρ	Γ_ρ	Λ_ρ	A_ρ
π	0.2	0.0	−0.8
ν	0.0	0.0	0.0

V. TRANSITION OPERATOR

The one-body transition density operator from even-even to odd-odd in the IBFFM formalism can be obtained considering the mapping from the fermionic space into the boson-fermion-fermion space. The operator for one-nucleon transfer in which the number of bosons is conserved is given by [21]

$$c_{j_\rho}^\dagger \rightarrow P_{j_\rho}^\dagger = \xi_{j_\rho} a_{j_\rho}^\dagger + \sum_{j'_\rho} \xi_{j_\rho j'_\rho} [[s_\rho^\dagger \times \tilde{d}_\rho]^{(2)} \times a_{j'_\rho}^\dagger]^{(j_\rho)}, \quad (15)$$

where $s^\dagger$ is the creation operator and $s = \tilde{s}$ the annihilation operator of the s boson, and $\tilde{d}$ is related to the d-boson annihilation operator by $\tilde{d}_\mu = (-1)^\mu d_{-\mu}$.

In case the number of bosons is changed by one unit,

$$c_{j_\rho}^\dagger \rightarrow Q_{j_\rho}^\dagger = \theta_{j_\rho} (s_\rho^\dagger \times \tilde{a}_{j_\rho})^{(j_\rho)} + \sum_{j'_\rho} \theta_{j_\rho j'_\rho} [d_\rho^\dagger \times \tilde{a}_{j'_\rho}]^{(j_\rho)}, \quad (16)$$

The coefficients ξ_{j_ρ}, $\xi_{j_\rho j'_\rho}$ and θ_{j_ρ}, $\theta_{j_\rho j'_\rho}$ are defined in Refs. [21,26].

$$\zeta_j = u_j \frac{1}{K'_j}, \tag{14}$$

$$\zeta_{jj'} = -v_j \beta_{j'j} \left(\frac{10}{N(2j+1)} \right)^{1/2} \frac{1}{K K'_j}, \tag{15}$$

$$\theta_j = \frac{v_j}{\sqrt{N}} \frac{1}{K''_j}, \tag{16}$$

$$\theta_{jj'} = u_j \beta_{j'j} \left(\frac{10}{2j+1} \right)^{1/2} \frac{1}{K K''_j}, \tag{17}$$

where u_j and v_j are Bardeen–Cooper–Schrieffer (BCS) unoccupation and occupation amplitudes, and the quantities K, K'_j, K''_j,

$$K = \left(\sum_{jj'} \beta^2_{jj'} \right)^{1/2}, \tag{18}$$

$$K'_j = \left(1 + 2 \left(\frac{v_j}{u_j} \right)^2 \frac{\langle \text{even}; 0_1^+ | (\hat{n}_s + 1)\hat{n}_d | \text{even}; 0_1^+ \rangle}{N(2j+1)} \frac{\sum_{j'} \beta^2_{j'j}}{K^2} \right)^{1/2}, \tag{19}$$

$$K''_j = \left(\frac{\langle \text{even}; 0_1^+ | \hat{n}_s | \text{even}; 0_1^+ \rangle}{N} + 2 \left(\frac{u_j}{v_j} \right)^2 \frac{\langle \text{even}; 0_1^+ | \hat{n}_d | \text{even}; 0_1^+ \rangle}{2j+1} \frac{\sum_{j'} \beta^2_{j'j}}{K^2} \right)^{1/2}, \tag{20}$$

are calculated from the expectation values of the s-boson and d-boson numbers, $\hat{n}_s$, $\hat{n}_d$, and

$$\beta_{j'j} = (u_{j'} v_j + v_{j'} u_j)\, Q_{j'j} \tag{21}$$

$$Q_{j'j} = \langle l' \tfrac{1}{2} j' \| Y^{(2)} \| l \tfrac{1}{2} j \rangle. \tag{22}$$

If the odd fermion is a hole, then u_j and v_j are interchanged, and the sign of $\beta_{j'j}$ is reversed in Eqs. (14) to (20).

The transitions between even-even nuclei and odd-odd nuclei can be computed by considering the tensorial product of the transfer operator of a proton and a neutron coupled to the angular momenta λ, which is the value of the spin of the final state of the odd-odd nucleus. For the transition between ^{116}Cd and ^{116}In, the one-body transition in the IBM and its extensions is given by

$$T^{(\lambda)}_{j_\nu j_\pi} = \left[P^\dagger_{j_\nu} \times \tilde{Q}_{j_\pi}\right]^{(\lambda)} \tag{17}$$

and for the transition between ^{116}In and ^{116}Sn by

$$T^{(\lambda)}_{j_\nu j_\pi} = \left[Q^\dagger_{j_\nu} \times \tilde{P}_{j_\pi}\right]^{(\lambda)} \tag{18}$$

We proceed to compute the matrix elements of the operators of Eqs. (17) and 18.

A. Spectroscopic amplitudes in IBFFM

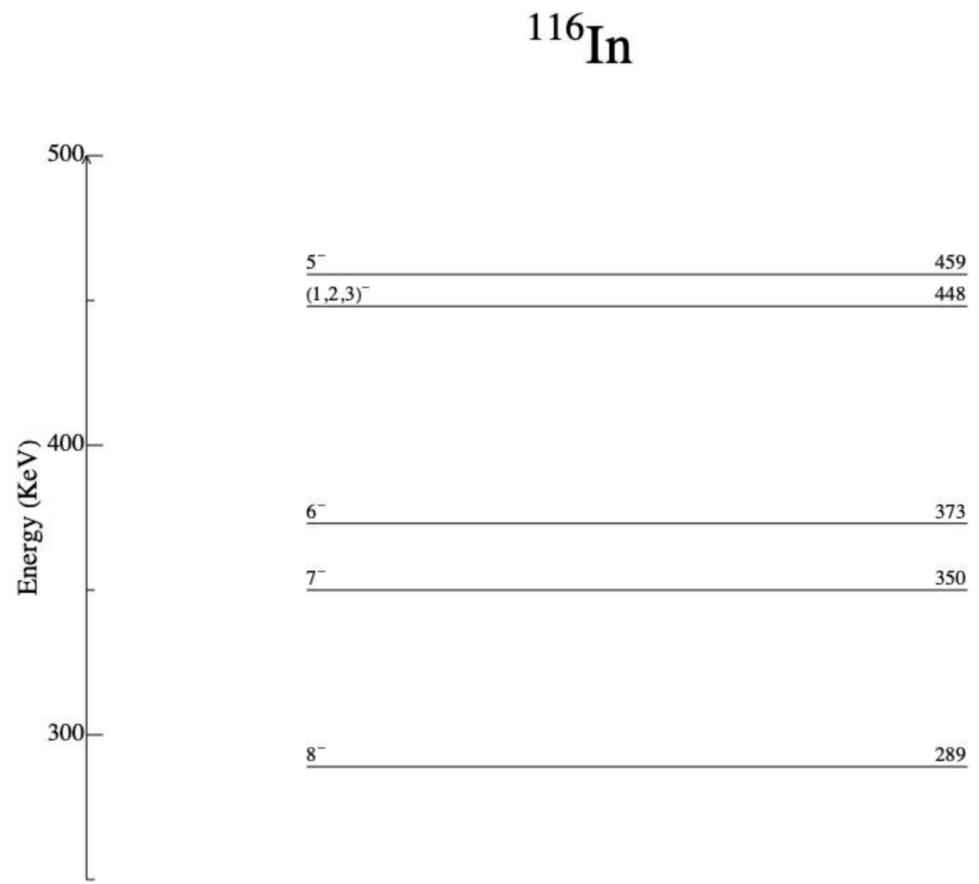
The model space of transition is given by the model space of the odd-odd nucleus. The model is given by five active neutron orbitals, $2d_{5/2}$, $1g_{7/2}$, $3s_{1/2}$, $1h_{11/2}$, and $2d_{3/2}$, and by four active proton orbitals, $2p_{1/2}$, $2p_{3/2}$, $1f_{5/2}$, and $1g_{9/2}$.

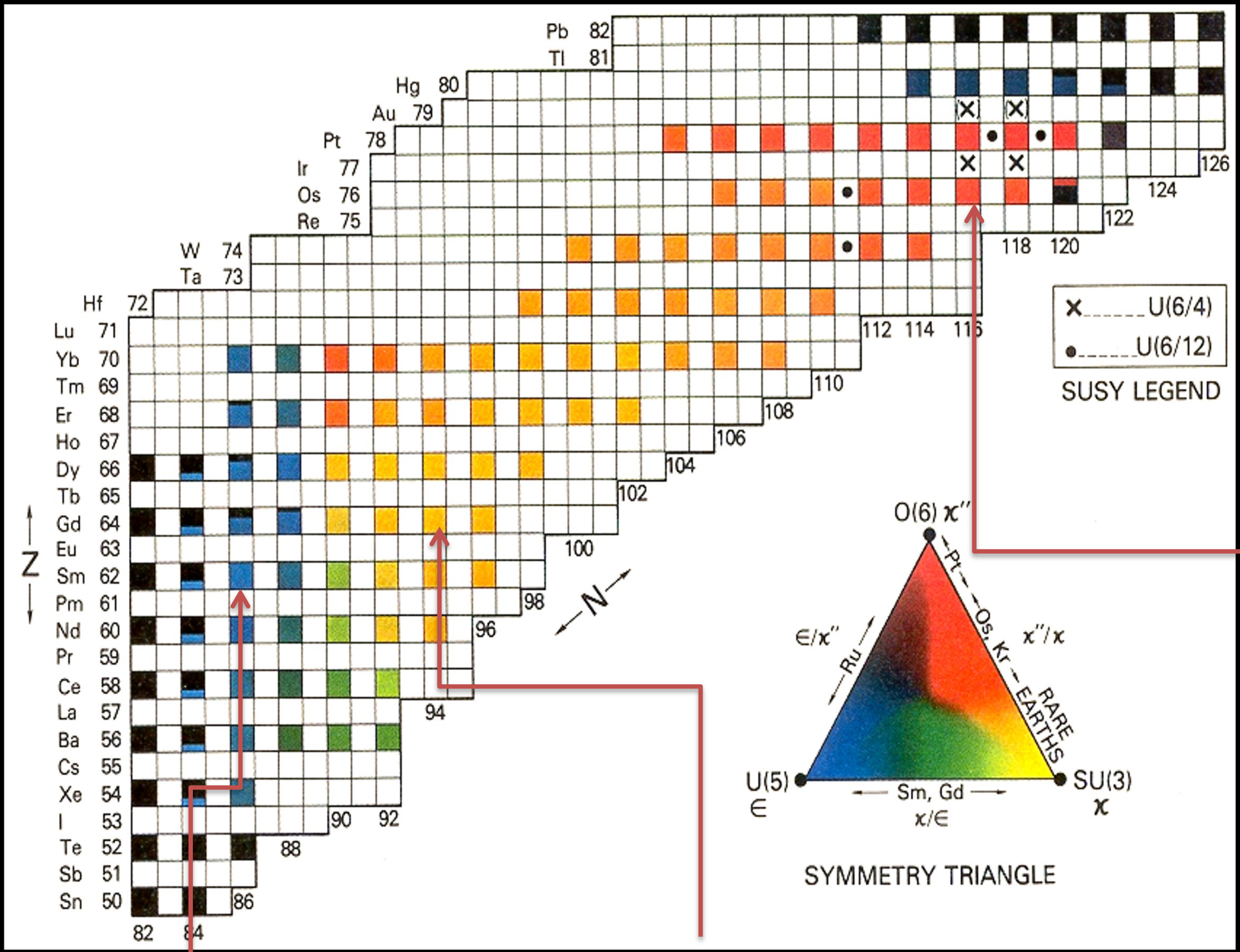
TABLE VI. Spectroscopic amplitudes (SAs) and one-body transition densities (OBTDs) for transitions from the 0_1^+ ground state of ^{116}Sn and ^{116}Cd to the J_i^P states of ^{116}In.

^{116}In			^{116}Sn(0_1^+)		^{116}Cd(0_1^+)	
J_i^P	j_ν	j_π	SA	OBTD	SA	OBTD
1_1^+	$1g_{7/2}$	$1g_{9/2}$	0.0650	0.0375	−0.0788	−0.0455
5_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0300	−0.0090	0.0055	0.0016
	$1g_{7/2}$	$1g_{9/2}$	−0.0270	−0.0081	0.0187	0.0056
	$3s_{1/2}$	$1g_{9/2}$	−0.0677	−0.0204	0.2289	0.0690
	$2d_{3/2}$	$1g_{9/2}$	−0.0338	−0.0102	0.1576	0.0475
	$1h_{11/2}$	$2p_{1/2}$	0.0134	0.0040	−0.1451	−0.0438
	$1h_{11/2}$	$2p_{3/2}$	−0.0045	−0.0014	0.0666	0.0201
	$1h_{11/2}$	$1f_{5/2}$	0.0042	0.0013	−0.0554	−0.0167
4_1^+	$2d_{5/2}$	$1g_{9/2}$	0.0083	0.0028	−0.0226	−0.0075
	$1g_{7/2}$	$1g_{9/2}$	0.0195	0.0065	0.0015	0.0005
	$3s_{1/2}$	$1g_{9/2}$	0.0271	0.0090	−0.1058	−0.0353
	$2d_{3/2}$	$1g_{9/2}$	0.0323	0.0108	−0.2125	−0.0708
	$1h_{11/2}$	$2p_{3/2}$	0.0006	0.0002	−0.0050	−0.0017
	$1h_{11/2}$	$1f_{5/2}$	0.0032	0.0011	−0.0523	−0.0174
2_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0036	−0.0016	0.0055	0.0025
	$1g_{7/2}$	$1g_{9/2}$	0.0453	0.0203	−0.0667	−0.0298
4_2^+	$2d_{5/2}$	$1g_{9/2}$	0.0014	0.0005	0.0064	0.0021
	$1g_{7/2}$	$1g_{9/2}$	−0.0066	−0.0022	−0.0053	−0.0018
	$3s_{1/2}$	$1g_{9/2}$	0.0205	0.0068	−0.0884	−0.0295
	$2d_{3/2}$	$1g_{9/2}$	0.0138	0.0046	−0.0651	−0.0217
	$1h_{11/2}$	$2p_{3/2}$	−0.0001	−0.0000	0.0035	0.0012
	$1h_{11/2}$	$1f_{5/2}$	0.0007	0.0002	−0.0003	−0.0001
3_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0196	−0.0074	0.0262	0.0099
	$1g_{7/2}$	$1g_{9/2}$	−0.0359	−0.0136	0.0301	0.0114
	$2d_{3/2}$	$1g_{9/2}$	−0.0437	−0.0165	0.2567	0.0970
	$1h_{11/2}$	$1f_{5/2}$	0.0110	0.0042	−0.1473	−0.0557

The nuclear states of the ^{116}In system are in our model restricted to be $J_P = 0^+$, 1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, and 8^+, within the first 14 excited states. The calculated excitation

Negative Parity spectrum of the ^{116}In





Vibrational

Rotational

Gamma Unstable

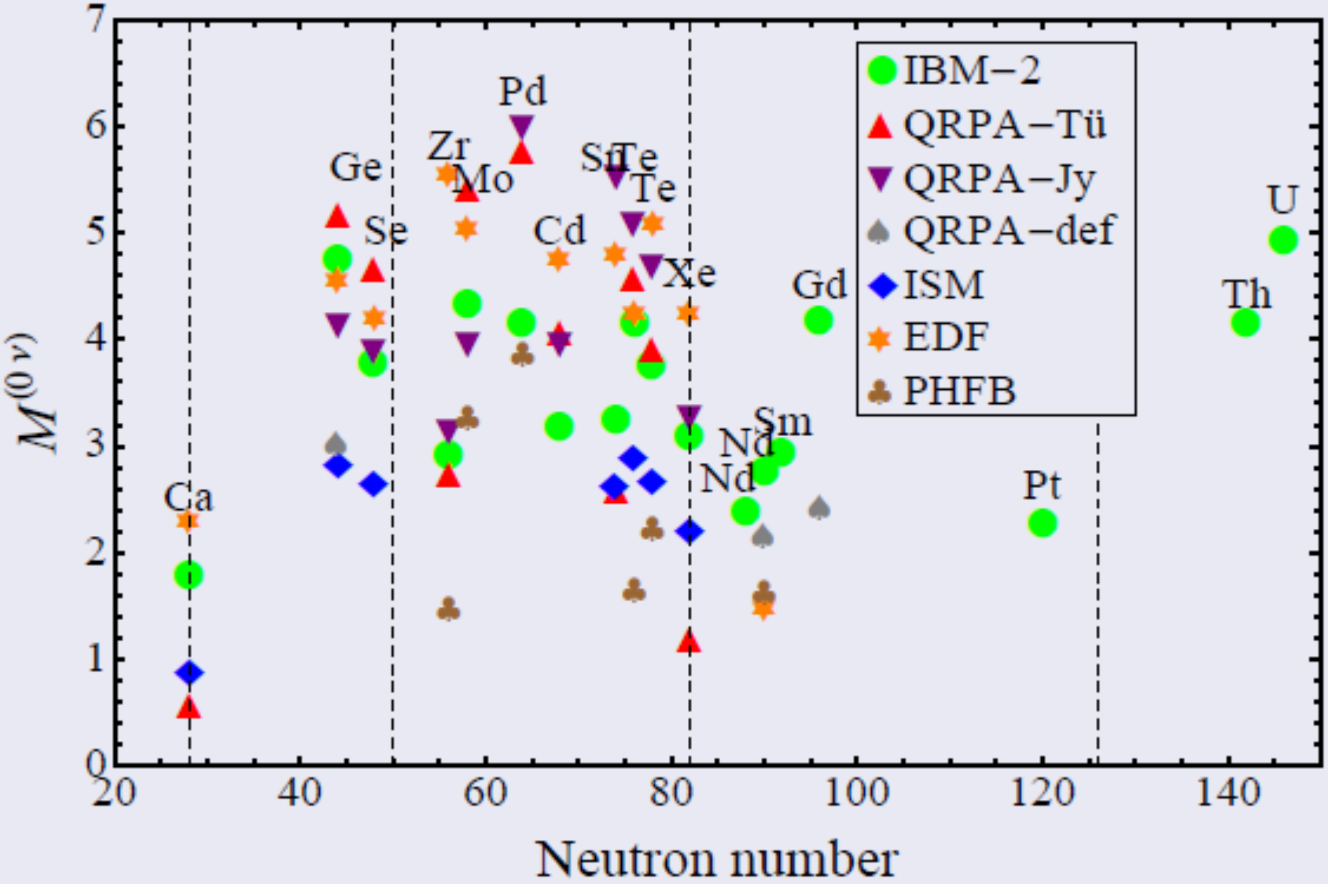
0νββ decay

- Tomoda – Rep Prog Phys 54, 53 (1991)
- Simkovic, Pantis, Vergados & Faessler - Phys. Rev. C 60, 055502 (1999)
- Barea & Iachello – Phys Rev C 79, 044301 (2009)

$$\left[T_{1/2}^{0\nu\beta\beta}(0^+ \to 0^+)\right]^{-1} = G_{0\nu} \left|M^{(0\nu)}\right|^2 \left(\frac{\langle m_\nu \rangle}{m_e}\right)^2$$

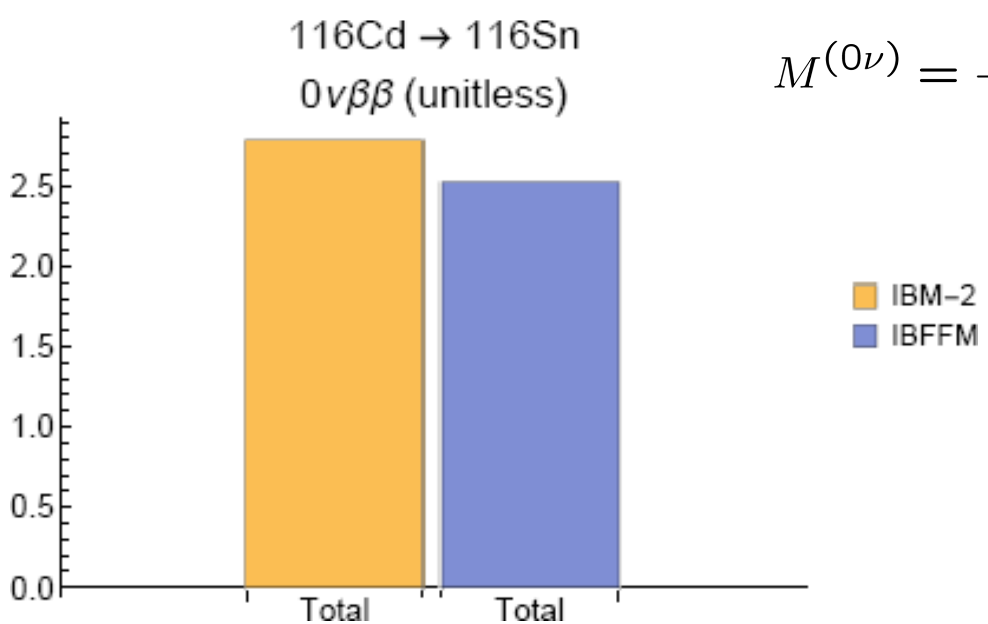
$$M^{(0\nu)} = -\left(\frac{g_V}{g_A}\right)^2 M_F^{(0\nu)} + M_{GT}^{(0\nu)} + M_T^{(0\nu)}$$

$$M^{(0\nu)} = M_{GT}^{(0\nu)} - \left(\frac{g_V}{g_A}\right)^2 M_F^{(0\nu)} + M_T^{(0\nu)}$$



Some results

	AA+VV	AP	PP	MM	Sum(fm^{-1})	$M_F^{(0\nu)}$	$M_{GT}^{(0\nu)}$	$M_T^{(0\nu)}$	$M^{0\nu}$
IBM-2	0.251	−0.0304	0.00696	0.00925	0.237	−0.2246	2.481	0.152	2.77
IBFFM-2	0.219	−0.0186	0.00322	0.00413	0.208	−0.0594	2.320	0.081	2.44



$$M^{(0\nu)} = -\left(\frac{g_V}{g_A}\right)^2 M_F^{(0\nu)} + M_{GT}^{(0\nu)} + M_T^{(0\nu)}$$

IBM-2: closure
(Barea & Iachello)

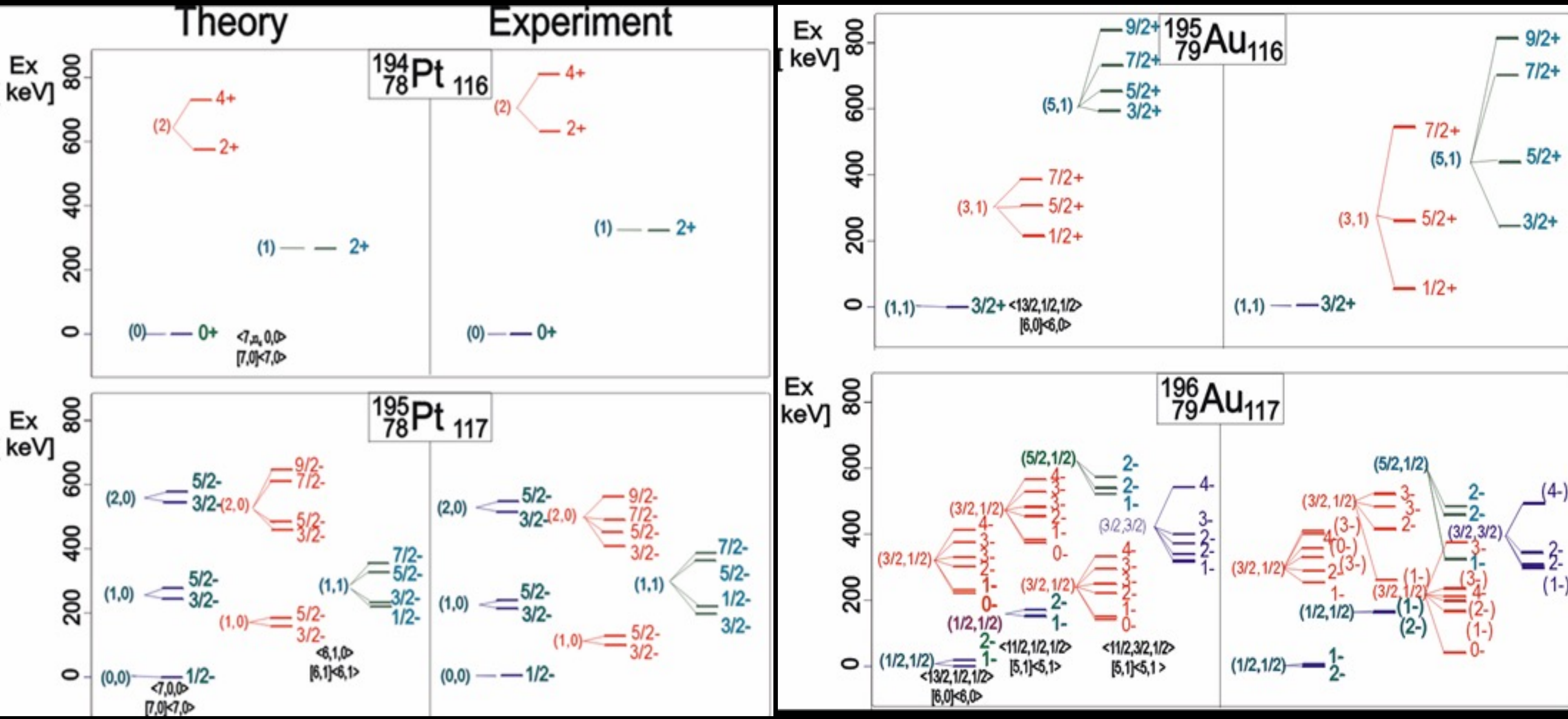
IBFFM-2 closure: sum over intermediate IBFFM states but still with the closure energy used in IBM-2, i.e. 12.5MeV

Magaña, Santopinto, Bijker, PRC (2022), accepted, in press

$g_A = 1.269.$

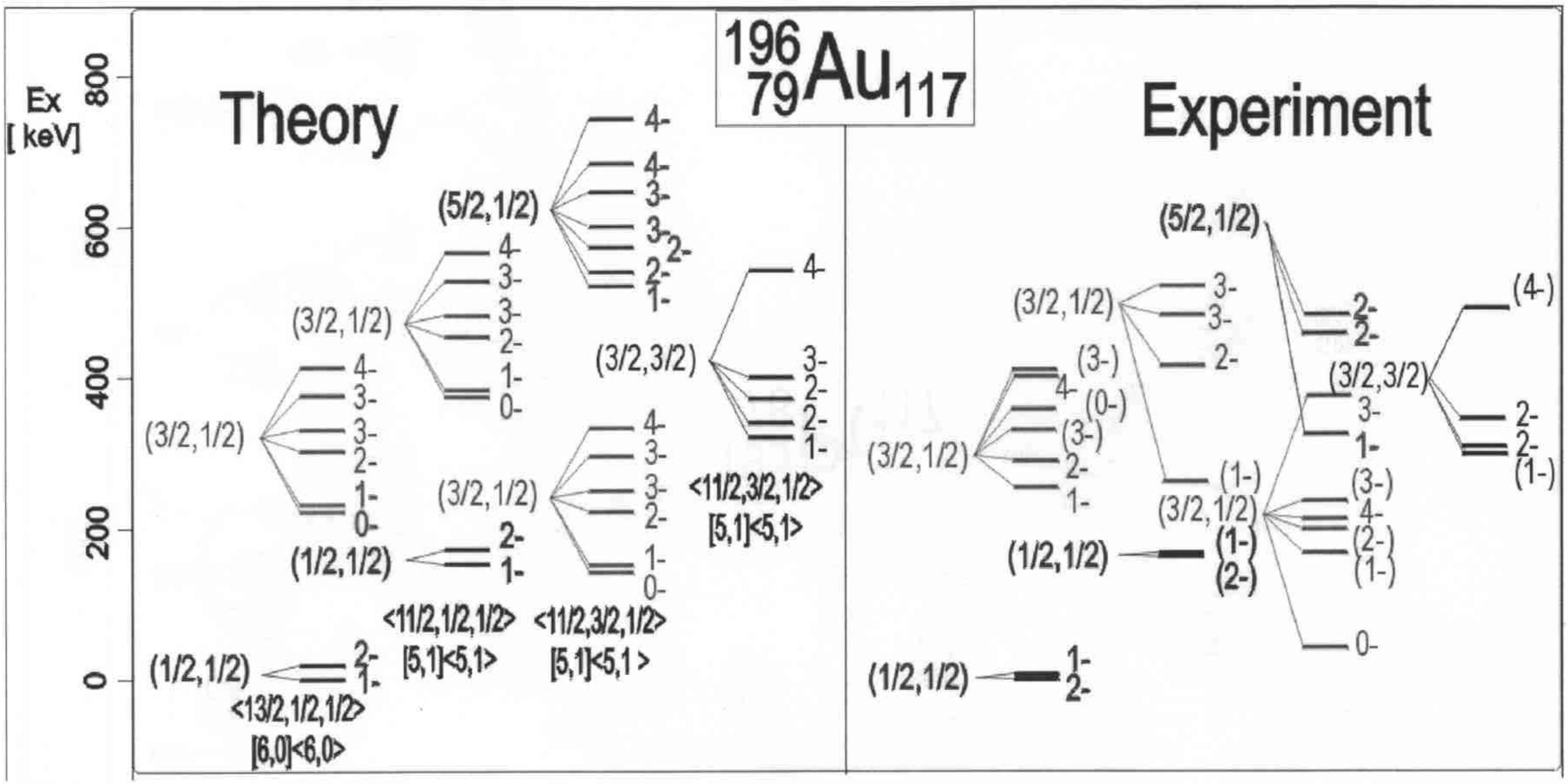
$g_V = 1$

Supersymmetric quartet



*A.Metz, J.Jolie, G. Graw, R. Hertenberger, J. Groger, C. Gunther, N. Warr, and Y. Eisermann, Phys. Rev. Lett, **83**, 1542 **(1999).***

Odd-Odd Nucleus



Metz et al, PRL 83, 1542 (1999)

Two-Nucleon Transfer

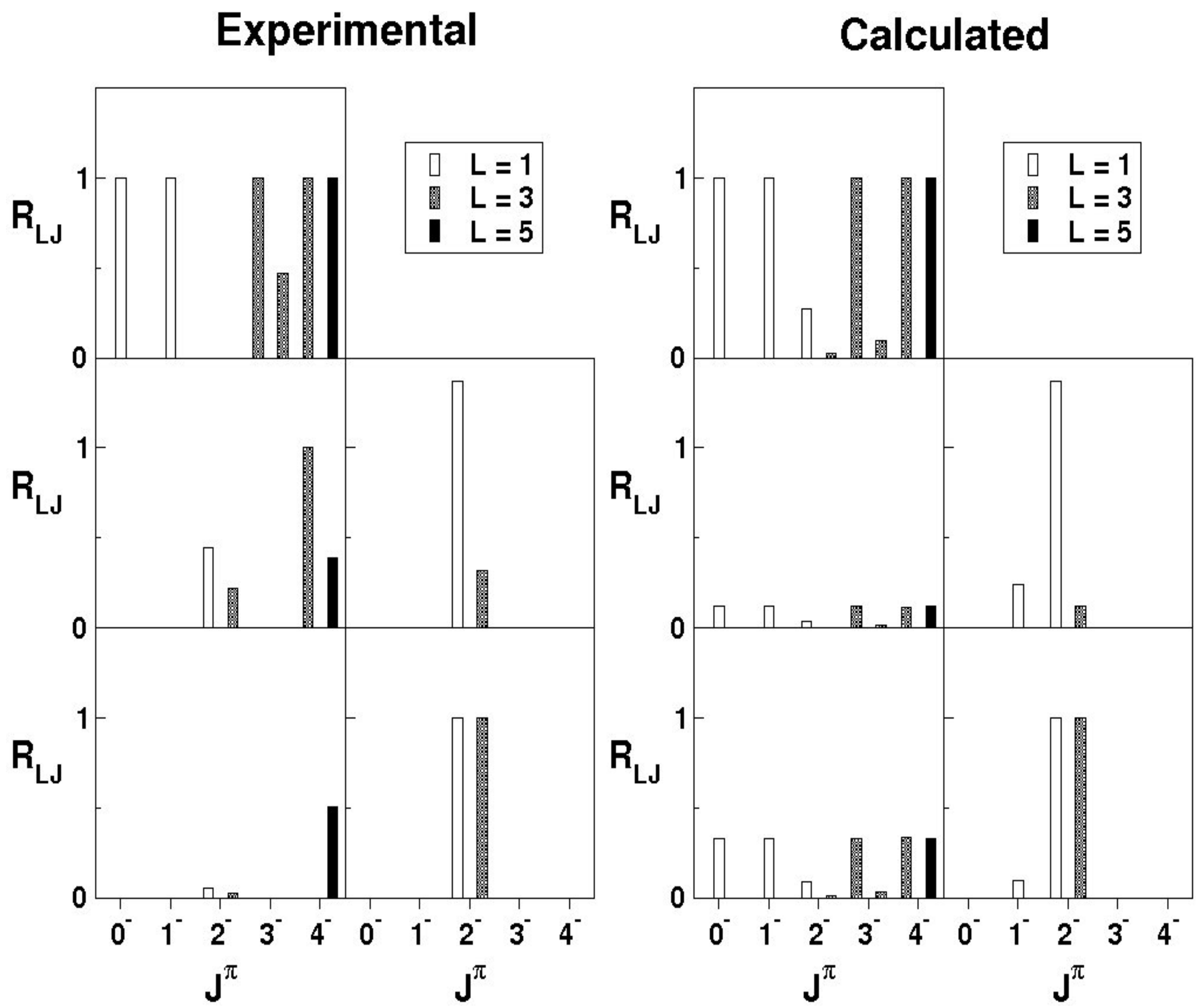
Reaction $^{198}\mathrm{Hg}\,(\vec{d},\alpha)\,^{196}\mathrm{Au}$

Spectroscopic factors

$$G_{LJ} = \left| \sum_{j_\nu j_\pi} g^{LJ}_{j_\nu j_\pi} \left\langle {}^{196}\mathrm{Au} \right\| (a^\dagger_{j_\nu} a^\dagger_{j_\pi})^{(\lambda)} \left\| {}^{198}\mathrm{Hg} \right\rangle \right|^2$$

Relative strength

$$R_{LJ} = \frac{G_{LJ}}{G_{LJ}(\mathrm{ref})} = \begin{cases} \frac{N+4}{15N} & = \; 0.12 \\ \frac{2(N+4)(N+6)}{15N(N+3)} & = \; 0.33 \end{cases}$$



Barea, Bijker, Frank, PRL 94, 152501 (2005)

Double Charge Exchange Experiment

- Canditates isotopes: ^{48}Ca, ^{82}Se, ^{100}Mo, ^{124}Sn, ^{128}Te, ^{130}Te, ^{136}Xe, ^{148}Nd, ^{150}Nd, ^{154}Sm, ^{160}Gd,^{198}Pt .
  - The NUMEN Project: Nuclear Matrix Elements for Neutrinoless double beta decays, EPJA (2018)54: 72, F. Cappuzzello, .., E. Santopinto et al. May 2018

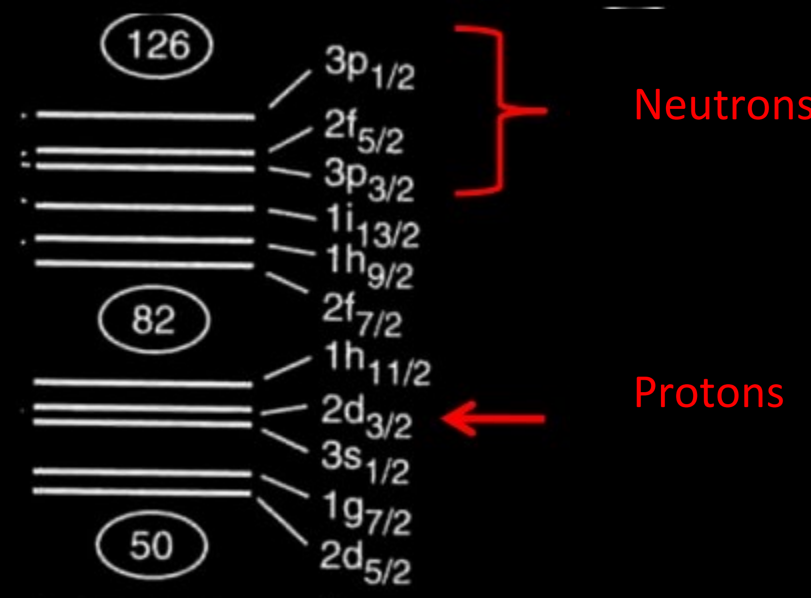
IBM	$b_i^\dagger b_j$	N	$U(6)$
IBFM	$b_i^\dagger b_j \;,\; a_k^\dagger a_l$	$N,\, M$	$U(6) \quad U(m)$
SUSY	$b_i^\dagger b_j \;,\; a_k^\dagger a_l \;,\; b_i^\dagger a_k \;,\; a_k^\dagger b_i$	$\mathcal{N} = N + M$	$U(6/m)$

even-even	s, d	$U(6) \supset SO(6)$
odd proton	$j_\pi = 2d_{3/2}$	$U(6/4)$
odd neutron	$j_\nu = 3p_{1/2}, 3p_{3/2}, 2f_{5/2}$	$U(6/12)$
odd-odd	$j_\pi = 2d_{3/2}$ $j_\nu = 3p_{1/2}, 3p_{3/2}, 2f_{5/2}$	$U(6/4)_\pi \quad U(6/12)_\nu$

$N = \sum_i b_i^\dagger b_i$ total number of bosons

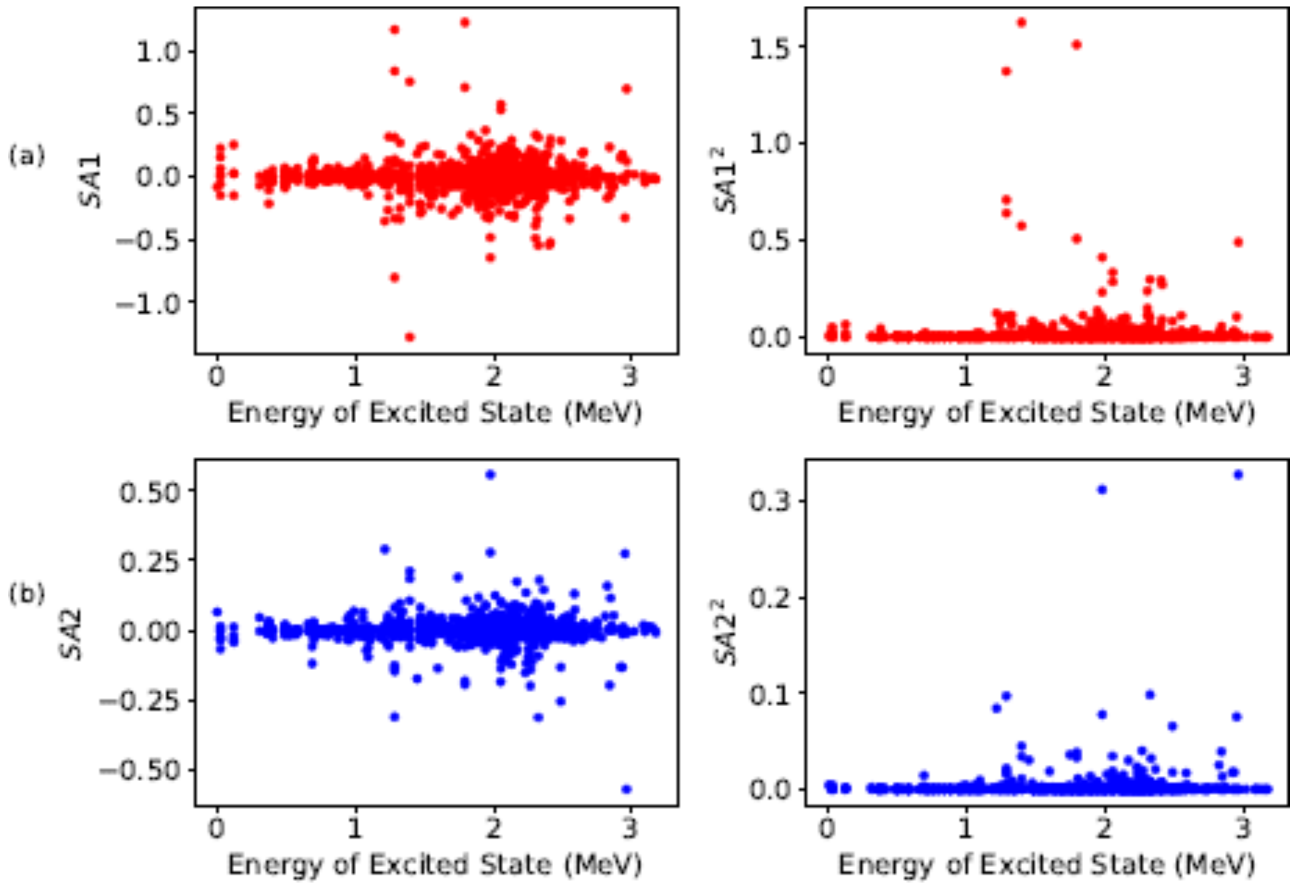
$M = \sum_\mu a_\mu^\dagger a_\mu$ total number of fermions

$\mathcal{N} = N + M$ total number of bosons and fermions



Arima & Iachello, PRL 40, 385 (1978)
Iachello, PRL 44, 772 (1980)
Balantekin, Bars, Bijker & Iachello, PRC 27, 1761 (1983)
Van Isacker, Jolie, Heyde & Frank, PRL 54, 653 (1985)

Spectroscopic Amplitudes



$^{116}Cd \rightarrow {}^{116}In$

$^{116}Sn \rightarrow {}^{116}In$

FIG. 7. (a) Spectroscopic Amplitudes for the transition ^{116}Cd $\rightarrow$ ^{116}In denoted as SA1 and (b) transitions ^{116}Sn $\rightarrow$ ^{116}In denoted as SA2. The squared of the spectroscopic amplitudes are denoted as $SA1^2$ and $SA2^2$ respectively.