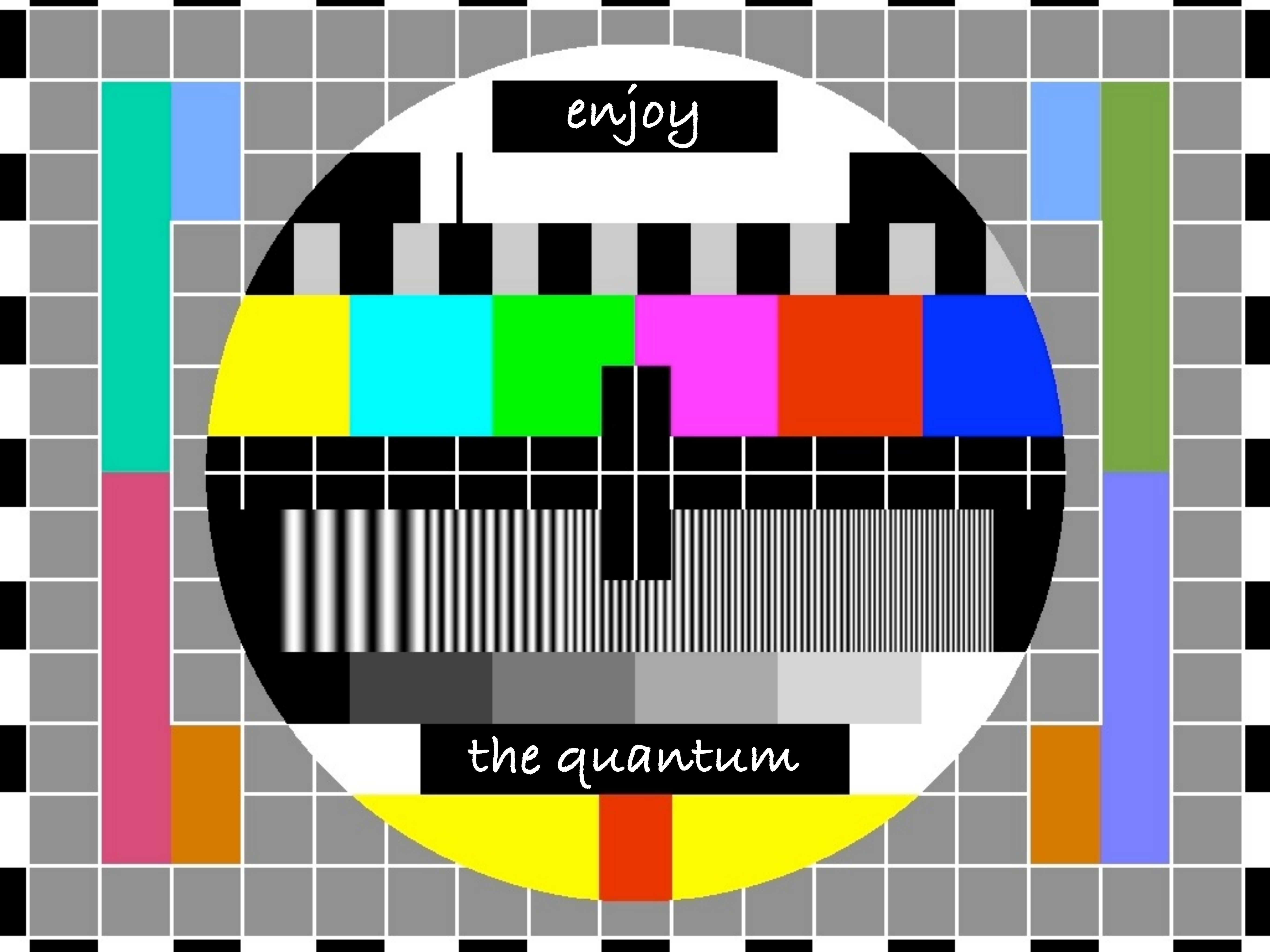
enjoy
the quantum

An invitation to quantum estimation theory

(quantum metrology for fundamental physics)

Matteo G. A. Paris

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INFN & Dipartimento di Fisica

Università degli Studi di Milano, Italy

Quantum technology for fundamental physics

Erice, 1 - 7 September, 2023

place for the dolphin
(if I were Lorenzo)

Measurement and estimation

place for the dolphin (if I were Lorenzo)

(tum.de)

Measurement and estimation

Do we measure physical quantities?

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(if I were Lorenzo)

Measurement and estimation

Or perhaps we are mostly estimating them?

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(if I were Lorenzo)

Measurement and estimation

- ~~direct measurements~~
- indirect measurements

→ influence on a different quantity

$$\boldsymbol{\chi} = (x_1, x_2, \ldots)$$

- choice of the measurement $\quad p(x|\lambda)$

- choice of the estimator $\quad \boldsymbol{\chi} \mapsto \hat{\lambda} = f(\boldsymbol{\chi})$

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(if I were Lorenzo)

Measurement and estimation

- <u>global</u> estimation theory
  (when you have no a priori information)
  look for a measurement which is optimal in average
  (over the possible values of the parameter)

- <u>local</u> estimation theory
  (when you have some a priori information)
  look for a measurement which is optimal for a
  specific value of the parameter (—> ultimate bounds)

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(if I were Lorenzo)**

Measurement and estimation

- ~~direct measurements~~
- indirect measurements

↳ influence on a different quantity

$$\boldsymbol{\chi} = (x_1, x_2, \ldots)$$

- choice of the measurement $\quad p(x|\lambda)$

- choice of the estimator $\quad \boldsymbol{\chi} \mapsto \hat{\lambda} = f(\boldsymbol{\chi})$

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local estimation theory: Cramer - Rao bound

- variance of unbiased estimators

$$\mathrm{Var}_\lambda[\widehat{\lambda}] \geq \frac{1}{M F(\lambda)}$$

- M -> number of measurements
- F -> Fisher Information

$$F(\lambda) = \int dx\, p(x|\lambda) \left[\partial_\lambda \log p(x|\lambda)\right]^2$$

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Optimal estimation scheme (classical)

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(if I were Lorenzo)

$$\boldsymbol{\chi} = (x_1, x_2, \ldots)$$

$$\boldsymbol{\chi} \mapsto \widehat{\lambda} = f(\boldsymbol{\chi})$$

$$\mathrm{Var}_\lambda[\widehat{\lambda}] \geq \frac{1}{MF(\lambda)}$$

- Optimal measurement -> maximum Fisher
  (no recipes on how to find it)

- Optimal estimator -> saturation of CR inequality
  (e.g. Bayesian or MaxLik asymptotically)

Quantum estimation

- What about time and temperature in quantum mechanics?

- The "resources" involved in quantum-enhanced technology are entanglement, non-locality, entropy, interferometric phase-shift, etc.. In general they are not observable quantities in strict sense (do not correspond to a self-adjoint operator)

  - No correspondence principle
  - No uncertainty relations

here a whale (if I were Lorenzo)

Quantum estimation

- What about time and temperature in quantum mechanics?

- The "resources" involved in quantum-enahnced technology are entanglement, non-locality, entropy, interferometric phase-shift, etc.. In general they are not observable quantities in strict sense (do not correspond to a self-adjoint operator)

Quantum estimation theory

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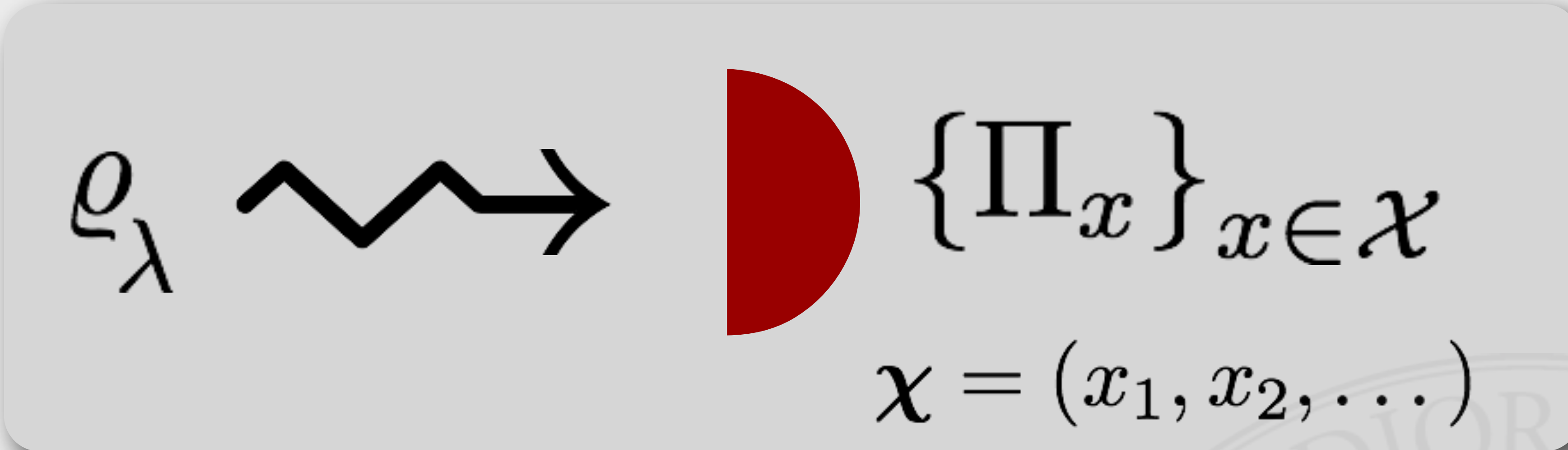
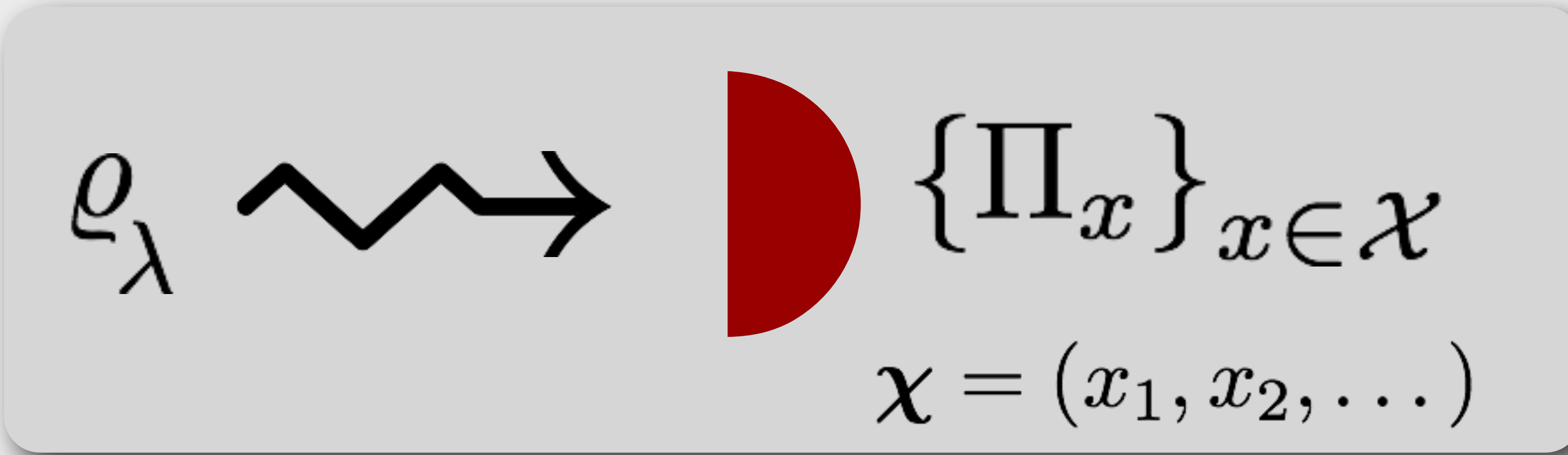
Quantum estimation



- Optimal measurements
- Ultimate bounds to precision

here a whale
(if I were Lorenzo)

Let's go quantum (local) (1)



- probability density $p(x|\lambda) = \mathrm{Tr}\left[\varrho_\lambda\, \Pi_x\right]$

- symm. log. derivative (SLD) $$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda}$$

  selfadjoint, zero mean $\mathrm{Tr}\left[\varrho_\lambda\, L_\lambda\right] = 0$

- Fisher Information $$F(\lambda) = \int dx\, \frac{\mathrm{Re}\left(\mathrm{Tr}\left[\varrho_\lambda \Pi_x L_\lambda\right]\right)^2}{\mathrm{Tr}[\varrho_\lambda \Pi_x]}$$

why not right derivative $\partial_\lambda \varrho_\lambda = \varrho_\lambda R_\lambda$?

here a whale (if I were Lorenzo)

Let's go quantum (local) (2)

$$F(\lambda) \le \int dx \left| \frac{\mathrm{Tr}\left[\varrho_\lambda \Pi_x L_\lambda\right]}{\sqrt{\mathrm{Tr}[\varrho_\lambda \Pi_x]}} \right|^2$$

$$= \int dx \left| \mathrm{Tr}\left[\frac{\sqrt{\varrho_\lambda}\sqrt{\Pi_x}}{\sqrt{\mathrm{Tr}\left[\varrho_\lambda \Pi_x\right]}} \sqrt{\Pi_x} L_\lambda \sqrt{\varrho_\lambda} \right] \right|^2$$

$$\le \int dx \, \mathrm{Tr}\left[\Pi_x L_\lambda \varrho_\lambda L_\lambda\right]$$

$$= \mathrm{Tr}[L_\lambda \varrho_\lambda L_\lambda] = \mathrm{Tr}[\varrho_\lambda L_\lambda^2]$$

Helstrom 1976
Braunstein & Caves 1994

time to switch to Miyazaki pics (if I were Lorenzo)

- Fisher vs Quantum Fisher

$$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda}$$

$$F(\lambda) \le H(\lambda) \equiv \mathrm{Tr}[\varrho_\lambda L_\lambda^2] = \mathrm{Tr}[\partial_\lambda \varrho_\lambda L_\lambda]$$

- ultimate bound on precision $\mathrm{Var}(\lambda) \ge \frac{1}{M H(\lambda)}$

Optimal estimation scheme (quantum, local)

ϱ_λ  $\{\Pi_x\}_{x\in\mathcal{X}}$

$$\boldsymbol{\chi} = (x_1, x_2, \dots)$$

The optimal measurement (i.e. Fisher = quantum Fisher) is projective and equal to the spectral measure of the SLD.

$$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda}$$

Optimal estimator -> saturation of CR inequality (classical postprocessing, e.g. Bayesian or MaxLix)

$$\boldsymbol{\chi} \mapsto \hat{\lambda} = f(\boldsymbol{\chi})$$

here a dynosaur (if I were Lorenzo)

General formulas (basis indepedent)

$$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda}$$

Lyapunov equation

- Symmetric logarithmic derivative

$$L_\lambda = 2\int_0^\infty dt\, \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}$$

- Quantum Fisher Information

$$H(\lambda) = 2\int_0^\infty dt\, \mathrm{Tr}\left[\partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\right]$$

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(if I were Lorenzo)

General formulas

- **Family of quantum states**

$$\varrho_\lambda = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$$

- **Symmetric logarithmic derivative**

$$L_\lambda = \sum_p \frac{\partial_\lambda \varrho_p}{\varrho_p} |\psi_p\rangle\langle\psi_p| + 2\sum_{n\neq m} \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \langle\psi_m|\partial_\lambda\psi_n\rangle\, |\psi_m\rangle\langle\psi_n|$$

- **Quantum Fisher Information**

$$H(\lambda) = \sum_p \frac{(\partial_\lambda \varrho_p)^2}{\varrho_p} + 2\sum_{n\neq m} \frac{(\varrho_n - \varrho_m)^2}{\varrho_n + \varrho_m} |\langle\psi_m|\partial_\lambda\psi_n\rangle|^2$$

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(if I were Lorenzo)

General formulas

- Family of quantum states

$$\varrho_\lambda = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$$

- Symmetric logarithmic derivative

$$L_\lambda = \sum_p \frac{\partial_\lambda \varrho_p}{\varrho_p} |\psi_p\rangle\langle\psi_p| + 2 \sum_{n\neq m} \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \langle\psi_m|\partial_\lambda\psi_n\rangle\, |\psi_m\rangle\langle\psi_n|$$

- Quantum Fisher Information

$$H(\lambda) = 8 \lim_{\epsilon\to 0} \frac{1 - F(\varrho_\lambda, \varrho_{\lambda+\epsilon})}{\epsilon^2}$$

here a dynosaur (if I were Lorenzo)

estimability of a parameter

- signal-to-noise ratio (single measurement)

$$R_\lambda = \frac{\lambda^2}{\mathrm{Var}(\lambda)} \leq Q_\lambda \equiv \lambda^2 H(\lambda)$$

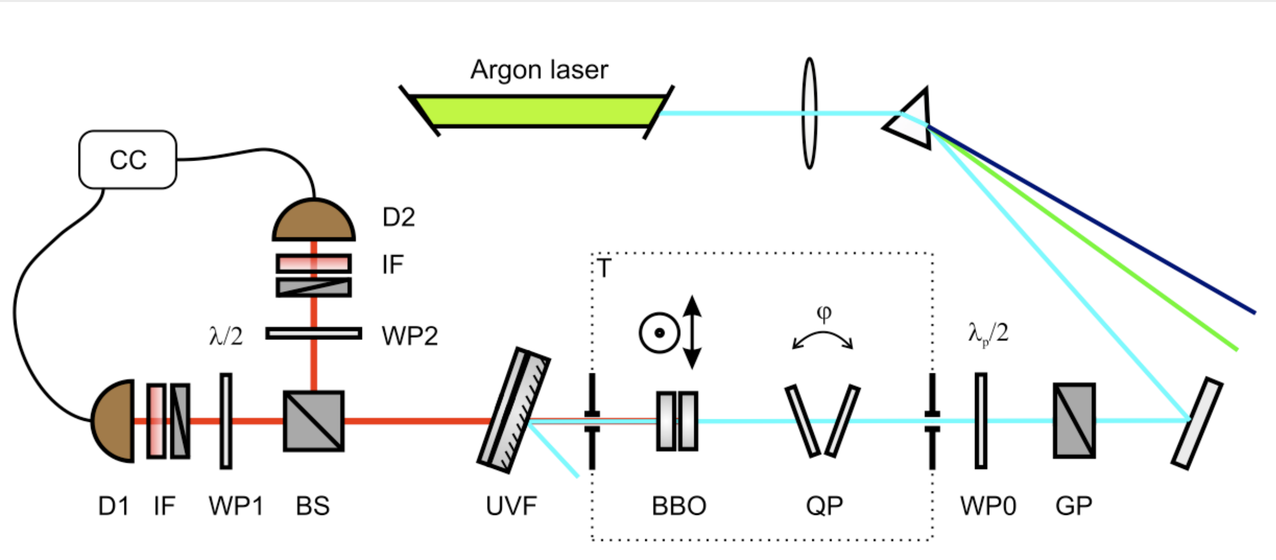
**here a dynosaur
(if I were Lorenzo)**

Applications (madamina il catalogo e' questo)

here Mozart Kugel (everybody loves chocolate, Lorenzo told me)

- Quantum Interferometry
- Estimation of Gaussian states and operations
- Coupling constants (e.g. nonlinear interactions)
- Wave function of finite-dimensional systems
- Estimation of entanglement (and discord)
- Estimation in quantum critical systems
- Assessing quantum probes for complex systems
- Assessing quantum resources in metrology
- Assessing local vs global measurements
- Assessing criticality as a resource in metrology
- Probing quantum phase transitions
- Probing Hamiltonian terms
- New physics at gravity/QM interface
- ...

Estimation of entanglement (@INRIM)



$$|\psi_\phi\rangle = \cos\phi|HH\rangle + \sin\phi|VV\rangle$$

$$D_\phi = \cos^2\phi|HH\rangle\langle HH| + \sin^2\phi|VV\rangle\langle VV|$$

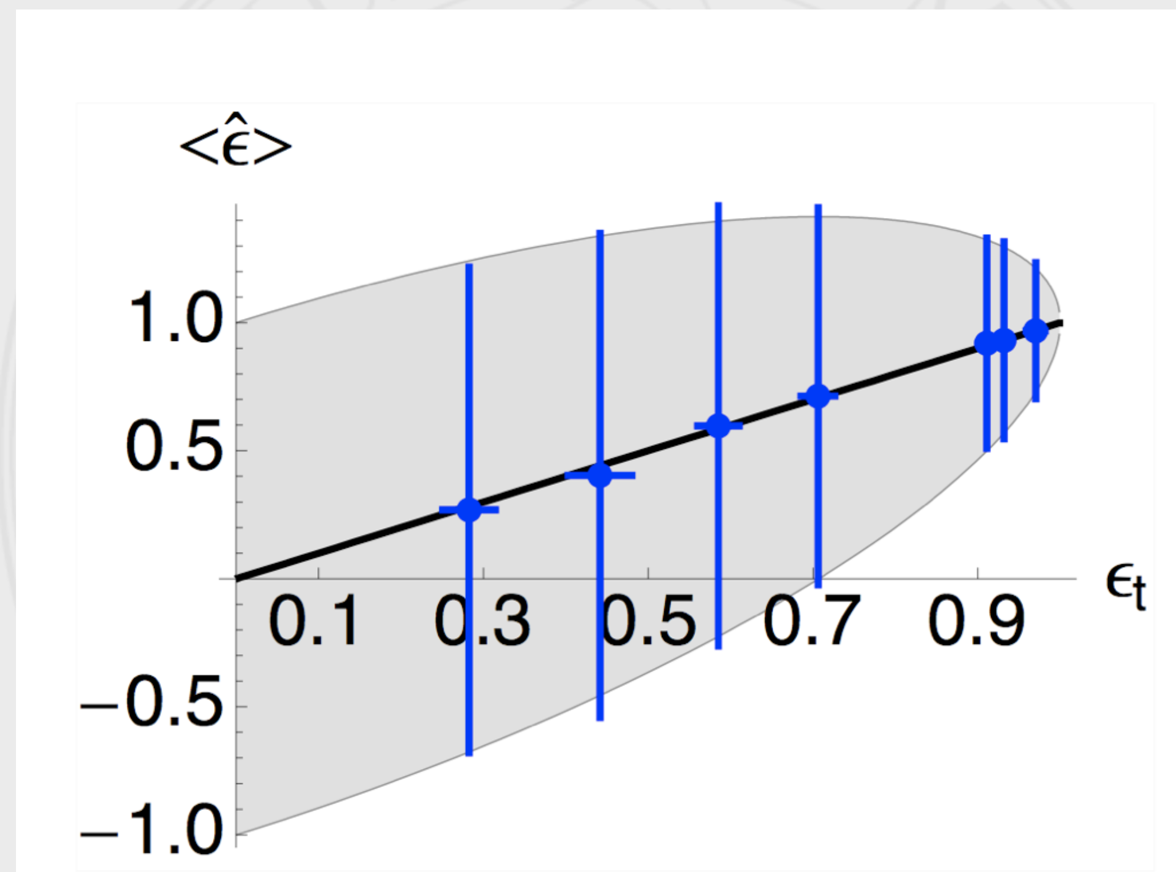
$$\varrho_\epsilon = p|\psi_\phi\rangle\langle\psi_\phi| + (1-p)D_\phi$$

$$\epsilon = p\sin 2\phi$$

optimal estimation by visibility measurements

Fisher information is monotone with entanglement

Estimation of "low" entanglement is inherently inefficient



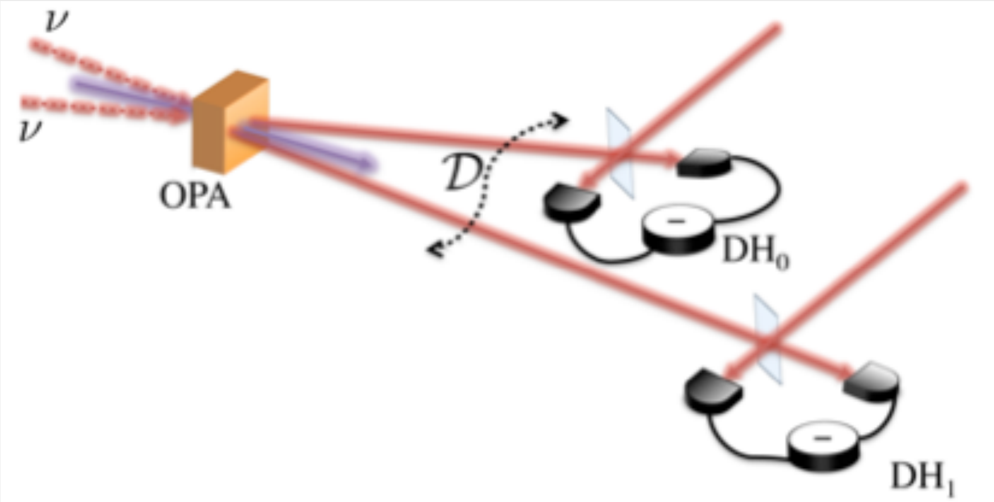
PRL 104, 100501 (2010) PHYSICAL REVIEW LETTERS week ending 12 MARCH 2010

Experimental Estimation of Entanglement at the Quantum Limit

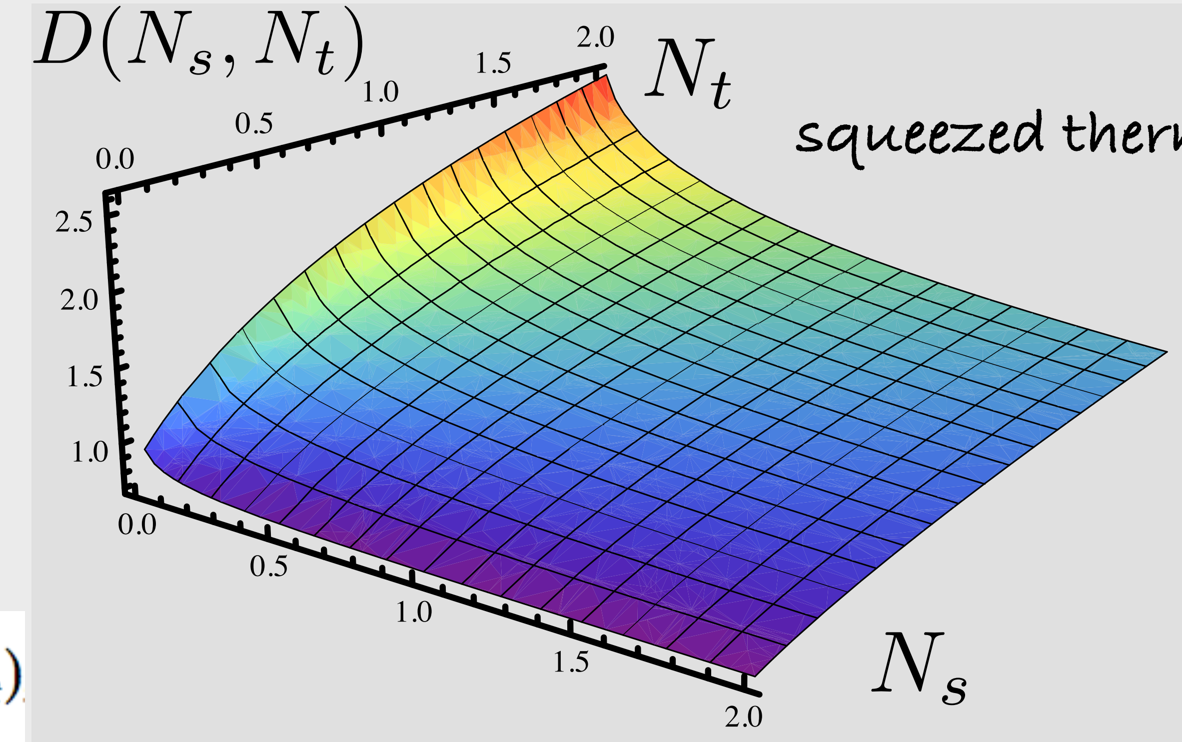
Giorgio Brida,[1] Ivo Pietro Degiovanni,[1] Angela Florio,[1,2] Marco Genovese,[1] Paolo Giorda,[3] Alice Meda,[1] Matteo G. A. Paris,[4,5] and Alexander Shurupov[6,1,7]

here a dolphin
(if I were Lorenzo)

Estimation of quantum discord (@Charles Fabry)



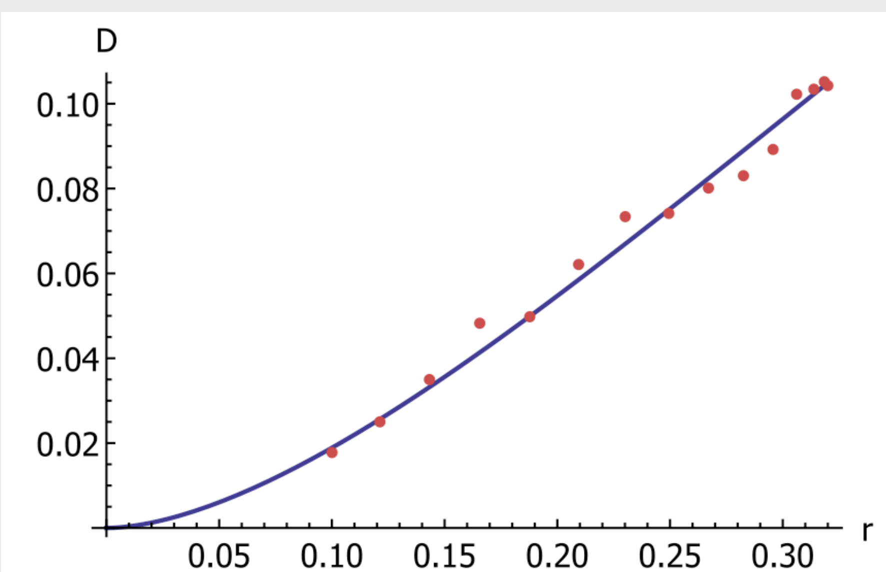
$$\varrho(N_s, N_t) = S_2(s)\nu(N_t) \otimes \nu(N_t) S_2(s)^\dagger$$

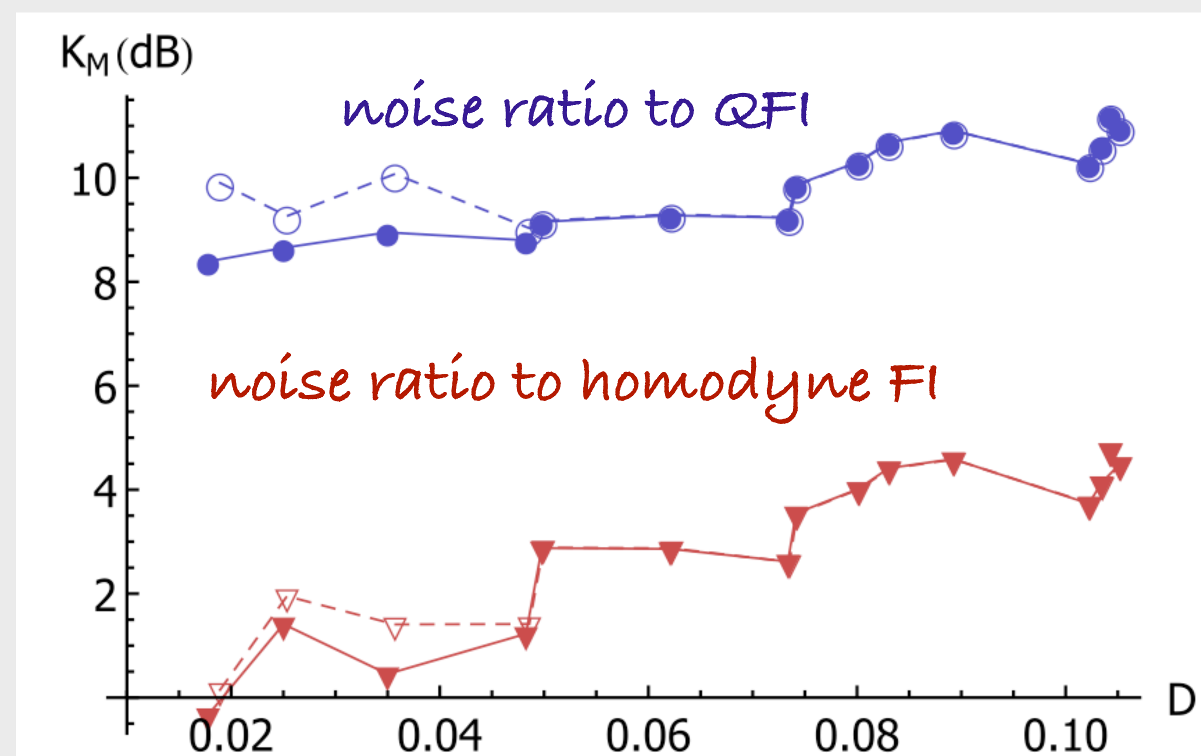


squeezed thermal states

Homodyne detection + inversion or Bayesian analysis

$$p(N_s, N_t | X) = \frac{1}{\mathcal{N}} p(X | N_s, N_t) p_0(N_s) p_0(N_t)$$





PRL **109,** 180402 (2012) PHYSICAL REVIEW LETTERS week ending 2 NOVEMBER 2012

Homodyne Estimation of Gaussian Quantum Discord

Rémi Blandino,[1,*] Marco G. Genoni,[2,†] Jean Etesse,[1] Marco Barbieri,[1] Matteo G. A. Paris,[3,4] Philippe Grangier,[1] and Rosa Tualle-Brouri[5,1]

here a dolphin (if I were Lorenzo)

Is this theory not even wrong?

here nothing
(so people ask why)

Quantum probes for universal gravity corrections

The existence of a minimum length and generalized uncertainty principle (GUP), influence all quantum Hamiltonians

Das et al PRL 101, 221301 (2008)

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}[1 + \beta((\Delta p)^2 + \langle p \rangle^2) + 2\beta(\Delta p_i^2 + \langle p_i \rangle^2)],$$

$$[x_i, p_j] = i\hbar(\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta p_i p_j)$$

$$x_i = x_{0i}, \qquad p_i = p_{0i}(1 + \beta p_0^2)$$

here a dolphin (if I were Lorenzo)

Quantum probes for universal gravity corrections

The existence of a minimum length and generalized uncertainty principle (GUP), influence all quantum Hamiltonians

Das et al PRL 101, 221301 (2008)

$$H = H_0 + H_1 + \mathcal{O}(\beta^2)$$

$$H_0 = \frac{p_0^2}{2m} + V(\vec{r}) \qquad H_1 = \frac{\beta}{m} p_0^4$$

$$\beta = \beta_0/(M_{\rm Pl}c)^2 = \ell_{\rm Pl}^2/2\hbar^2$$

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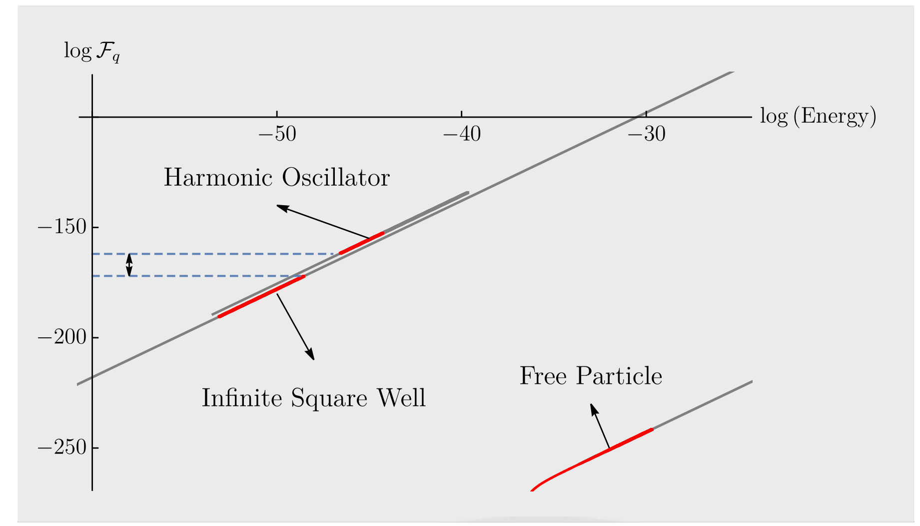
Quantum probes for universal gravity corrections

the largest values of QFI are obtained with a quantum probe subject to a harmonic potential and initially prepared in a superposition of perturbed energy eigenstates



QFI is super-additive with the dimension of the system ($\propto d^3$ for a square well), which therefore represents a metrological resource. The gain in precision is not due to the appearance of entanglement of the state but rather to the increasing number of superposed states generated by the perturbation.

PHYSICAL REVIEW D **102,** 056012 (2020)

Quantum probes for universal gravity corrections

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(if I were Lorenzo)

More than a tool, new physics from QET

PHYSICAL REVIEW LETTERS **124**, 120504 (2020)

Critical Quantum Metrology with a Finite-Component Quantum Phase Transition

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PHYSICAL REVIEW D **94**, 024014 (2016)

Probing deformed quantum commutators

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(Received 21 June 2016; published 6 July 2016)

Gravitational time dilation as a resource in quantum sensing

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Quantum **7**, 946 (2023).

International Journal of Theoretical Physics (2019) 58:2914–2935
https://doi.org/10.1007/s10773-019-04174-9

Quantum Sensing of Curvature

Daniele Bonalda[1] · Luigi Seveso[1] · Matteo G. A. Paris[1]

PHYSICAL REVIEW E **104**, 014136 (2021)

Role of topology in determining the precision of a finite thermometer

Alessandro Candeloro,[1,*] Luca Razzoli,[2,†] Paolo Bordone,[2,3,‡] and Matteo G. A. Paris[1,4,§]
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**here a dolphin
(if I were Lorenzo)**

- More than one parameter
- Quantum probes for complex systems
- Beyond the Cramer-Rao bound
- ...

here a dolphin
(if I were Lorenzo)

Summary/conclusions

Quantum estimation theory is a relevant tool to design and assess quantum enhanced measurements (estimation schemes)

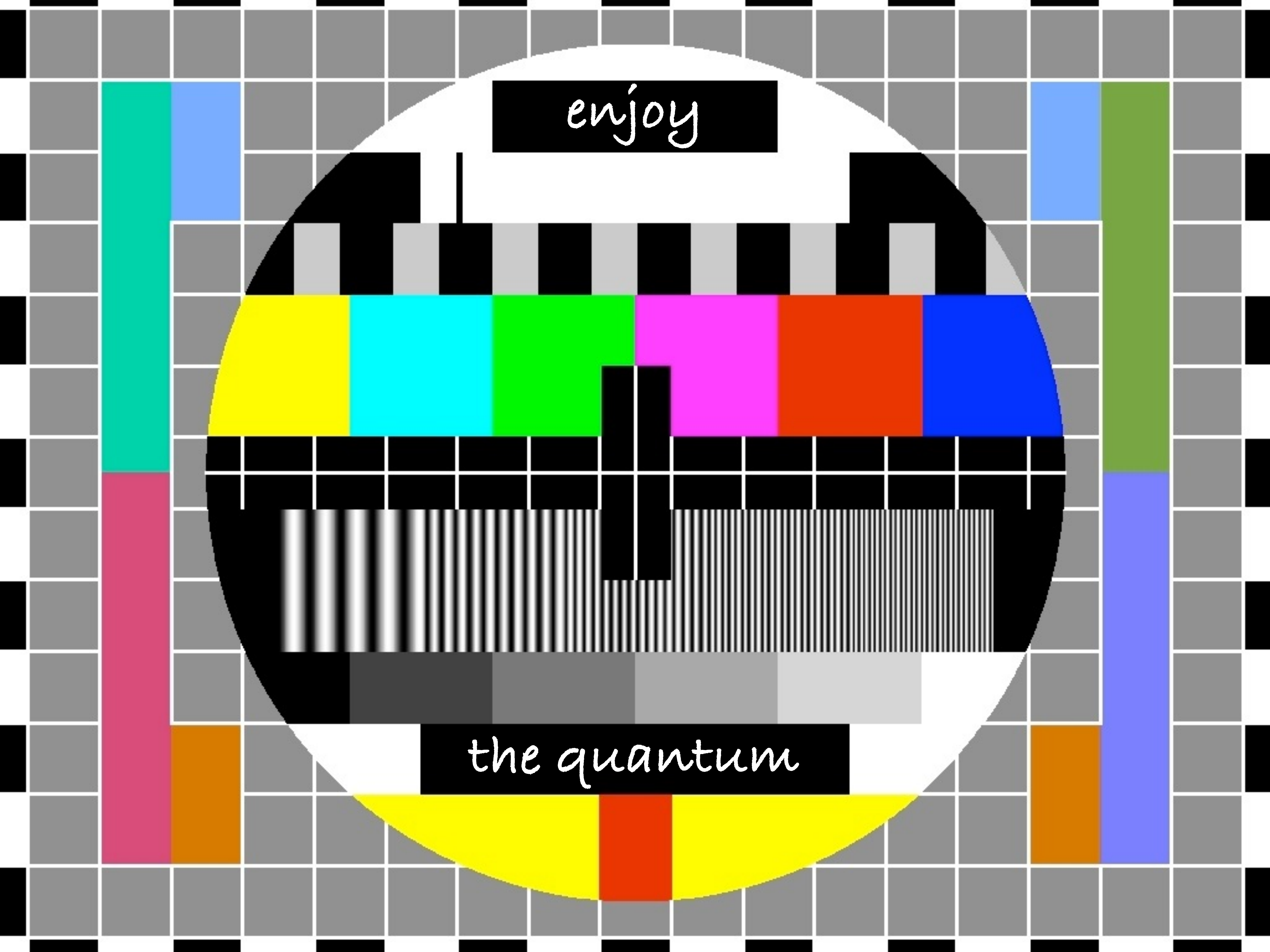
The single parameter QCR provides the ultimate quantum limit to precision. More precisely, it bounds precision of schemes exploiting quantumness of probes.

QET provides tools to assess the intrinsic estimability of a parameter and to determine whether a given theory is worth investigation

Quantum-based measurements may be further improved by exploiting detector dependence on the parameter of interest and thus the quantumness of detectors, i.e. quantum-enhanced measurements may be more precise than previously thought.

Current research is about joint estimation of more than one parameter, e.g. signal and noise to realize self calibrating estimation schemes.

a pic of Lorenzo


enjoy
the quantum