Operational Quantum Mereology

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“ Mereology is the philosophical study of the relationships between parts and wholes, examining how individual components combine to form larger entities and the principles governing their composition”

ChatGPT

Some Motivational Blah Blah

- The reductionistic approach: to explain the behavior of complex systems in terms of that of their simpler constituents e.g., particles, their properties and their mutual interactions

- Division into simpler components, it is *not* unique; it depends e.g., on the questions one is trying to address, experimental limitations, the physical regime one is operating at.

- *Examples:* **a)** in scattering experiments sometimes a system is seen as elementary i.e., with no sub-parts, and sometimes not; **b**) Fock spaces factorize in continuously infinite many ways (single particle basis choice); **c)** Decreasing energy different DOFs freeze and associated subsystems decouple. from the Hilbert space.

The Question(s)

How does one select, the subdivision of a system into sub-parts? Can one establish a "natural" connection between, the intrinsic dynamics of the system, the operational capabilities of the observer, and a emergent multi-partite structure?

This Talk: a Quantum Information & Operator Algebras approach...

Generalized Tensor Product Structures

- Let $\mathcal{H}$ a d-dimensional Hilbert space with generalized Tensor Product Structure (**gTPS**),

$$\mathcal{H} \cong \bigoplus_{J=1}^{d_{\mathcal{Z}}} \left(\mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}\right) \tag{1}$$

- This is *quivalent* to consider a $*$-closed algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ and its $\mathcal{A}'$ its *commutant*

$$\mathcal{A} \cong \bigoplus_{J=1}^{d_{\mathcal{Z}}} \mathbb{1}_{n_J} \otimes \mathcal{L}(\mathbb{C}^{d_J}), \qquad \mathcal{A}' \cong \bigoplus_{J=1}^{d_Z} \mathcal{L}(\mathbb{C}^{n_J}) \otimes \mathbb{1}_{d_J}. \tag{2}$$

- $(\mathcal{A}, \mathcal{A}')$ defines a *generalized quantum subsystem.* Standard case $d_{\mathcal{Z}} = 1$ (factors). Quantum Error Correction and Noiseless Subsystems are examples of gTPS!

Scrambling of Algebras: $\mathcal{A}$-OTOC

Given an algebra $\mathcal{A}$ and a unitary channel $\mathcal{U}$, the $\mathcal{A}$-**OTOC** is defined by Haar averaging over the unitary subgroups of $\mathcal{A}$ and $\mathcal{A}'$ ($\|\cdot\|_2$ Hilbert-Schmidt norm).

$$G_{\mathcal{A}}(\mathcal{U}) := \frac{1}{2d}\,\mathbb{E}_{X\in\mathcal{A},Y\in\mathcal{A}'}\left[\|[X,\mathcal{U}(Y)]\|_2^2\right] \tag{3}$$

- It quantifies the degree of noncommutativity i.e., symmetry-breaking, induced b $\mathcal{U}$.
- $G_{\mathcal{A}}(\mathcal{U}) = 0 \Rightarrow$ *no* information, encoded in $\mathcal{A}'$ or $\mathcal{A}$, leaks across the bipartition $(\mathcal{A},\, \mathcal{A}')$.
- $G_{\mathcal{A}}(\mathcal{U}) \neq 0 \Rightarrow$ Information gets dynamically “scrambled” across the bipartition $(\mathcal{A},\, \mathcal{A}')$.

Factors & Operator Entanglement

For $\mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_B$, $\mathcal{A} \cong \mathbb{1}_A \otimes \mathcal{L}(\mathcal{H}_B)$, and $\mathcal{U} = \mathrm{Ad}U_t$ we get

$$G(t) = 1 - \frac{1}{d^2} \mathrm{Tr}\left[S_{AA'} U_t^{\otimes 2} S_{AA'} U_t^{\dagger\,\otimes 2}\right]. \tag{4}$$

Bipartite OTOC = **Operator entanglement** entropy of U_t

- Also, equal to average entropy production due to information scrambling.
- The long-time average of $G(t)$ is related to entanglement properties of the eigenstates of the Hamiltonian and tells apart integrable from chaotic phases.
- Applications: Fun& Entangling Power, Q-Chaos, Q-Complexity,...

Maximal Abelian algebra & Coherence Generating Power

- For $\mathcal{A}_B = \mathbf{C}\{P_\mu = |\mu\rangle\langle\mu|\}_{\mu=1}^d$, where $B := \{|\mu\rangle\}_{\mu=1}^d$ is an orthonormal basis we have

$$G_{\mathcal{A}_B}(\mathcal{U}) = \frac{1}{d}\sum_{\mu=1}^{d} \|\mathbb{Q}_B\,\mathcal{U}(P_\mu)\|_2^2 \tag{5}$$

  where $\mathbb{Q}_B := \mathbb{1} - \mathbb{P}_{\mathcal{A}_B}$ is the projector on B off-diagonal matrices.

- This is the *coherence generating power* measure: average coherence generated by U acting on the simplex I_B of B-diagonal density matrices:

$$G_{\mathcal{A}_B}(\mathcal{U}) \propto \mathbb{E}_{\rho\in I_B}\left[\|\mathbb{Q}_B\mathcal{U}(\rho)\|_2^2\right]. \tag{6}$$

- Applications: Q-Chaos, Many Body Localization and Hilbert Space Fragmentation.

Subsystem Emergence & Scrambling Time

Mereological Principle: *out of a family of operationally available gTPSs, dynamics selects those whose with the most resilient "informational identity". Quasi-Stable entities out there...*

- One approach: given a Hamiltonian H and an algebra $\mathcal{A}$, the "scrambling time" $\tau_s(H, \mathcal{A})$ is such that, if $t \ll \tau_s$, then $\mathcal{U}_t = \mathrm{Ad}\, e^{iHt}$, generates negligible scrambling.

$$\{\textit{Emergent}\,\mathcal{A}\} := \mathrm{argmax}_{\mathcal{A}\in\mathbb{A}}\, \tau_s(H, \mathcal{A}). \tag{7}$$

- Short-time expansion: $G_{\mathcal{A}}(\mathcal{U}_t) = 2(t/\tau_s)^2 + O(t^3)$, gives the "Gaussian scrambling rate"

$$\tau_s^{-1}(H, \mathcal{A}) = D(\tilde{H}, \mathcal{A} + \mathcal{A}'). \tag{8}$$

Hilbert-Schmidt distance between $\tilde{H} := d^{-1/2} H$, and $\mathcal{A} + \mathcal{A}' := \{a + b \,|\, a \in \mathcal{A},\, b \in \mathcal{A}'\}$.

Spatial Bipartitions

- $\mathcal{H} = (\mathbf{C}^2)^{\otimes N}$. If $S \subset \{1, \ldots, N\}$ (and $\bar{S}$ is its complement) then $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{\bar{S}}$ and the local observables Algebras $\mathcal{A}_S := L(\mathcal{H}_S) \otimes \mathbf{1}_{\bar{S}}$ and $\mathcal{A}'_S = \mathbf{1}_S \otimes L(\mathcal{H}_{\bar{S}})$.

$$\tau_s^{-1}(H, \mathcal{A}_S) = \|\tilde{H} - \frac{\mathbf{1}_S}{d_S} \otimes \mathrm{Tr}_S(\tilde{H}) - \mathrm{Tr}_{\bar{S}}(\tilde{H}) \otimes \frac{\mathbf{1}_{\bar{S}}}{d_{\bar{S}}}\|_2, \tag{9}$$

  This is just the norm of the $S - \bar{S}$ interaction part of the Hamiltonian!

- Intuitive “subsystems emergence from minimal scrambling:
  Given a family $\mathbb{S}$ of subsets of $\{1, \ldots, N\}$ the Hamiltonian H will select those which are minimally coupled with the rest of the system.
  The whole breaks down into fragments along the “fault lines”...

Maximal Abelian Algebras

- If $\mathcal{A}_B = \mathrm{span}\{|j\rangle\langle j|\} = \mathcal{A}'_B$, where the $|j\rangle$'s form a basis B,

$$\tau_s^{-1}(H, \mathcal{A}_B) = \|\mathbb{Q}_B(\tilde{H})\|_2$$

This quantity is as small when H is as B-diagonal as possible.

- If H is non degenerate with an eigenbasis B_H,

$$\tau_s^{-1}(H, \mathcal{A}_B) \leq \eta_H \, D(\mathcal{A}_B, \, \mathcal{A}_{B_H}). \tag{10}$$

Here $\eta_H := \|\tilde{H}\|_2$ and D is a distance between maximal abelian algebras:

The dynamically selected $\mathcal{A}_B$'s are the closest to the abelian algebra generated by H.

Summary

- Generalized quantum subsystem: **gTPS** $(\mathcal{A}, \mathcal{A}')$ pair of commuting algebras
- Quantum scrambling of observable algebras: $\mathcal{A}$-**OTOC**
- Connections to **operator entanglement** and **coherence generating power**
- Dynamically emergent mereology by slow scrambling: **gaussian scrambling time**

Wanna know more? P.Z., E. Dallas and S. Lloyd, **arXiv:2212.14340**