Higher rank equivariant Donaldson-Witten Theory

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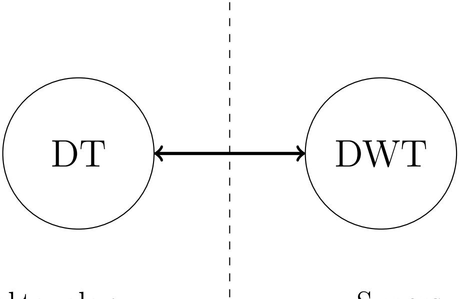
String Theory as a bridge between Gauge Theories
and Quantum Gravity
University of Turin, 23-24 February 2023

Donaldson and Donaldson-Witten theory



Differential topology

Supersymmetry

Donaldson theory

- Framework: 4-manifold X, vector bundle V, connection $\mathcal{A}$
- Anti-self-dual connections

$$*F_{\mathrm{A}} = -F_{\mathrm{A}}$$

$$\mathcal{M}_{\mathrm{ASD}} = \{[\mathcal{A}] = \mathcal{A}/\mathcal{G} | * F_{\mathrm{A}} = -F_{\mathrm{A}}\}$$

- Donaldson map

$$\gamma_* \in H_*(X) \to \mu(\gamma_*) \in H^{4-*}(\mathcal{M}_{\mathrm{ASD}})$$

- Donaldson invariants

$$\mathcal{D}_X^{c_1,k}(\gamma_0, \gamma_1^{(1)}, \ldots, \gamma_1^{(p)}, \gamma_2^{(1)}, \ldots, \gamma_2^{(q)}, \gamma_3^{(1)}, \ldots, \gamma_3^{(r)}) =$$
$$\int_{\mathcal{M}_{\mathrm{ASD}}(c_1,k)} \mu(\gamma_0)^\ell \wedge \prod_{i=1}^{p} \mu(\gamma_1^{(i)}) \wedge \prod_{j=1}^{q} \mu(\gamma_2^{(j)}) \wedge \prod_{m=1}^{r} \mu(\gamma_3^{(m)})$$

$$\dim \mathcal{M}_{\mathrm{ASD}} = 4\ell + 3p + 2q + r$$

Donaldson theory

- Bases of homology groups

$$\{\gamma_1^i\}_{i=1,\dots,b_1}\,, \quad \{\gamma_2^i\}_{i=1,\dots,b_2}\,, \quad \{\gamma_3^i\}_{i=1,\dots,b_3}$$

- Generic cycles

$$\gamma_1 = \sum_i u_i \gamma_1^i\,, \quad \gamma_2 = \sum_i v_i \gamma_2^i\,, \quad \gamma_3 = \sum_i w_i \gamma_3^i$$

- Generating function of Donaldson polynomials

$$\mathcal{Z}^{c_1}(p, \gamma_1, \gamma_2, \gamma_3) = \sum_{k=0}^{\infty} \mathcal{D}_X^{c_1,k}(e^{p\gamma_0+\gamma_1+\gamma_2+\gamma_3})$$

- Polynomials are homogeneous if we assign the degrees

$$\deg(p) = 4\,, \quad \deg(u_i) = 3\,, \quad \deg(v_i) = 2\,, \quad \deg(w_i) = 1$$

TFT of the Witten type

There is an explicit metric dependence of the theory, but the correlation functions do not depend on the metric.

- Q - scalar supersymmetry of the action and

$$T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}} = -\mathrm{i}\, Q G \mu\nu$$

- $\mathcal{O}_i$ - Q-invariant operators

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle = 0$$

- Topological observables

$$\mathcal{O} \in \frac{\mathrm{Ker\,Q}}{\mathrm{Im\,Q}}$$

QFT version of the Donaldson map

We can associate topological observables to homology classes in spacetime.

- For the descent procedure we need an operator such that $QG_\mu = \partial_\mu$
- In our case

$$P_\mu = T_{0\mu} = -\mathrm{i}QG_{0\mu} \triangleq -\mathrm{i}QG_\mu$$

- Descent operators $\phi^{(0)}(x)$ is scalar and Q-invariant

$$\phi^{(n)}_{\mu_1\dots\mu_n}(x) = G_{\mu_1}\dots G_{\mu_n}\phi^{(0)}(x) \qquad \phi^{(n)} = \frac{1}{n!}\phi^{(n)}_{\mu_1\dots\mu_n}\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_n}$$

- Descent equations

$$\mathrm{d}\phi^{(n)} = Q\phi^{(n+1)}$$

- Topological observables

$$\mu_{\phi^{(0)}}(\gamma_n) = \int_{\gamma_n}\phi^{(n)}$$

Donaldson-Witten theory

TFT of the Witten type can be obtained by twisting of $\mathcal{N} = 2$ super Yang-Mills theory.

- Field content: gauge field A_μ, two gluinos $\lambda_{v\alpha}$, compex scalar ϕ, auxiliary fields D_{uv}. Two supercharges $\mathcal{Q}_{\alpha u}$
- Total symmetry group

$$\rlap{$\overbrace{}^{SO(4)}$}SU(2)_- \times \underbrace{SU(2)_+ \times SU(2)_R}_{SU(2)'_+} \times U(1)$$

- $SU(2)_R$ indices u, $v \to \dot\alpha$, $\dot\beta$
- Twisted supercharges

$$\overline{\mathcal{Q}} = \varepsilon^{\dot\alpha\dot\beta}\overline{\mathcal{Q}}_{\dot\alpha\dot\beta}, \quad G_\mu = \frac{1}{2}(\sigma_\mu)^{\dot\alpha\beta}\mathcal{Q}_{\dot\alpha\beta},$$

$$\overline{\mathcal{Q}}^+_{\mu\nu} = \frac{1}{2}\varepsilon^{\alpha\beta}(\sigma_\mu)^{\dot\alpha\alpha}(\sigma_\nu)^{\dot\beta\beta}(\overline{\mathcal{Q}}_{\dot\alpha\dot\beta} + \overline{\mathcal{Q}}_{\dot\beta\dot\alpha})$$

Donaldson-Witten theory

- Twisted group

$$SU(2)_- \times SU(2)_+ \times SU(2)_R \times U(1)_R \to SU(2)_- \times SU(2)'_+ \times U(1)_R$$

- Twisted spin content of the fields

$$\begin{aligned}
A_\mu \,(1/2,1/2,0)^0 \quad &\to \quad A_\mu \,(1/2,1/2)^0 \\
\lambda_{v\alpha} \,(1/2,0,1/2)^{-1} \quad &\to \quad \psi_\mu \,(1/2,1/2)^1 \\
\bar{\lambda}_{v\dot{\alpha}} \,(1/2,0,1/2)^1 \quad &\to \quad \eta \,(0,0)^{-1},\ \chi^+_{\mu\nu} \,(1,0)^{-1} \\
\phi \,(0,0,0)^{-2} \quad &\to \quad \phi \,(0,0)^{-2} \\
\phi^\dagger \,(0,0,0)^2 \quad &\to \quad \phi^\dagger \,(0,0)^2 \\
D_{uv} \,(0,0,1)^0 \quad &\to \quad D^+_{\mu\nu} \,(1,0)^1
\end{aligned}$$

Donaldson-Witten theory on a manifold

- Twisted action

$$S = \{\overline{\mathcal{Q}}, V\} - \frac{1}{2}\int F \wedge F$$

- Descent procedure

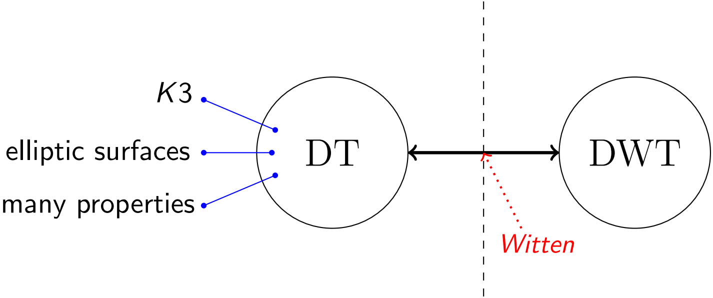
$$\phi^{(0)} = \mathrm{Tr}(\phi^2), \quad G_\mu, \quad \{\overline{\mathcal{Q}}, G_\mu\} = \partial_\mu$$

Semi-classical approximation is exact!

$$\langle \mu_1 \cdots \mu_n \rangle = \int [\mathcal{D}] \mu_1 \cdots \mu_n \, e^{-\frac{1}{g^2} S}$$

$$\frac{\partial}{\partial g} \langle \mu_1 \cdots \mu_n \rangle = \frac{2}{g^3} \langle \mu_1 \cdots \mu_n S \rangle = 0$$

We take $g \to 0$ and the integral reduces to the integral over the classical solutions.



$K3$

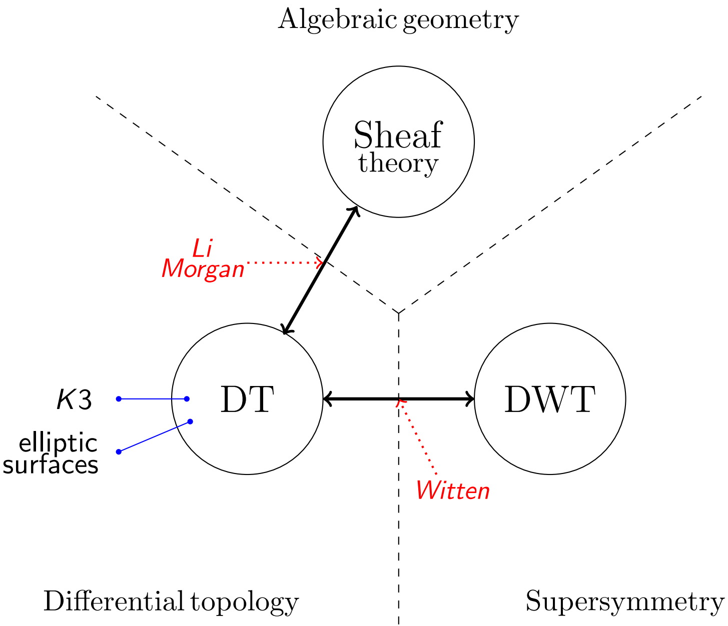
elliptic surfaces

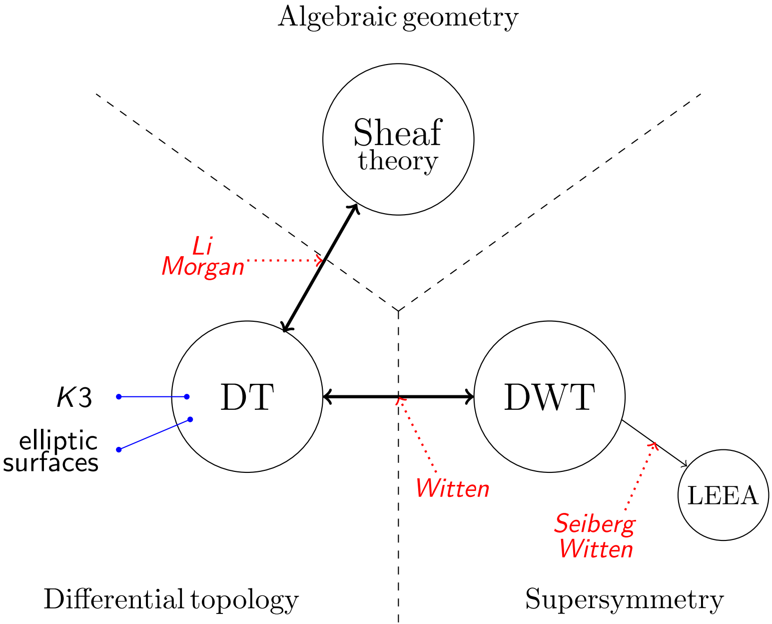
many properties

DT

DWT

Witten


Algebraic geometry
Sheaf theory
Li
Morgan
K3
elliptic surfaces
DT
DWT
Witten
Differential topology
Supersymmetry



Algebraic geometry

Sheaf theory

Li
Morgan

$K3$

elliptic
surfaces

DT

DWT

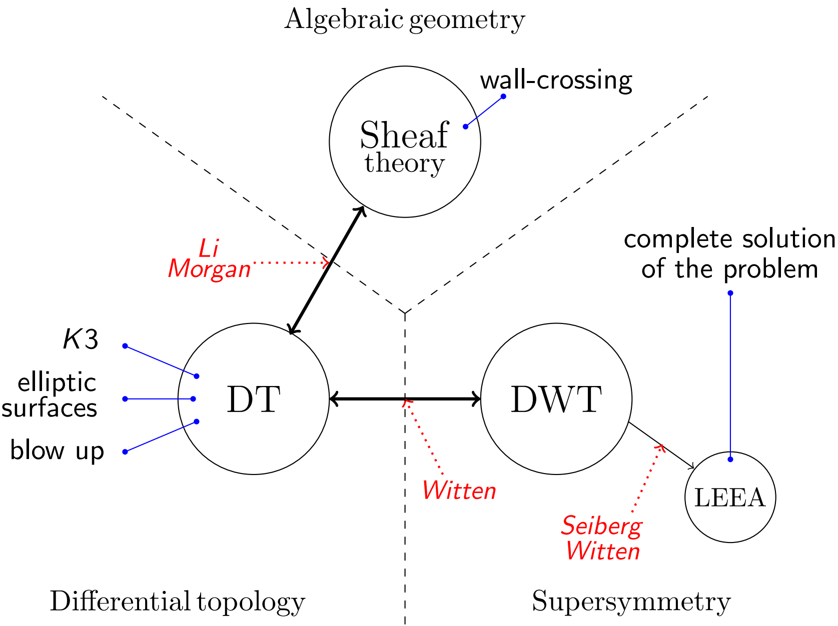
LEEA

Witten

Seiberg
Witten

Differential topology

Supersymmetry



Algebraic geometry

Sheaf theory

wall-crossing

Li
Morgan

complete solution of the problem

*K*3

elliptic surfaces

blow up

DT

DWT

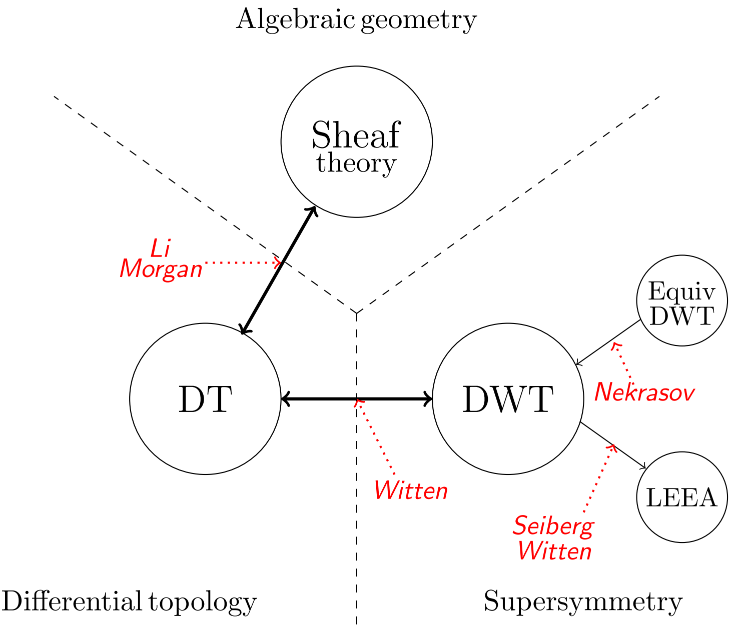
Witten

LEEA

Seiberg
Witten

Differential topology

Supersymmetry



Algebraic geometry

Sheaf theory

Li Morgan

Equiv DWT

DT

DWT

Nekrasov

Witten

LEEA

Seiberg Witten

Differential topology

Supersymmetry

Equivariant $\mathcal{N} = 2$ twisted Yang-Mills theory on $\mathbb{R}^4$

We start with the $\mathcal{N} = 1$ theory in six-dimensional space $\mathbb{C} \times \mathbb{R}^4$.

- Identification

$$(z,\, \bar{z},\, x) \to (z + n + m\tau,\, \bar{z} + n + m\bar{\tau},\, g_1^n g_2^m x)$$

$g_1,\ g_2$ are two commuting rotations on $\mathbb{R}^4$ and the metric is flat

$$\mathrm{d}s^2 = A\,\mathrm{d}z\mathrm{d}\bar{z} + \mathrm{d}x^2$$

- Standard torus identification

$$(z,\, \bar{z},\, \tilde{x}) \to (z + n + m\tau,\, \bar{z} + n + m\bar{\tau}, \tilde{x}),$$

but with the metric

$$\mathrm{d}s^2 = A\mathrm{d}z\mathrm{d}\bar{z} + g_{\mu\nu}(\mathrm{d}x^\mu + \xi_\Omega^\mu \mathrm{d}z + \bar{\xi}_\Omega^\mu \mathrm{d}\bar{z})(\mathrm{d}x^\nu + \xi_\Omega^\nu \mathrm{d}z + \bar{\xi}_\Omega^\nu \mathrm{d}\bar{z}),$$

$\xi^{\mu}_{\Omega} = \Omega^{\mu}_{\nu} x^{\nu}$, $\bar{\xi}^{\mu}_{\Omega} = \overline{\Omega}^{\mu}_{\nu} x^{\nu}$ and Ω^{μ}_{ν} are the generators of the rotation $\mathbb{R}^4$.

$$\Omega^{\mu\nu} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}, \quad \overline{\Omega}^{\mu\nu} = \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\epsilon}_2 \\ 0 & 0 & -\bar{\epsilon}_2 & 0 \end{pmatrix}.$$

Nekrasov: in a theory on the Ω-background, the action is no longer exact with respect to $\overline{\mathcal{Q}}$ and it should be replaced by

$$\tilde{\mathcal{Q}} = \overline{\mathcal{Q}} + \Omega^{\mu}_{\nu} x^{\nu} G_{\mu}.$$

$\tilde{\mathcal{Q}}$ turns out to be an equivariant differential!

Equivariant localisation

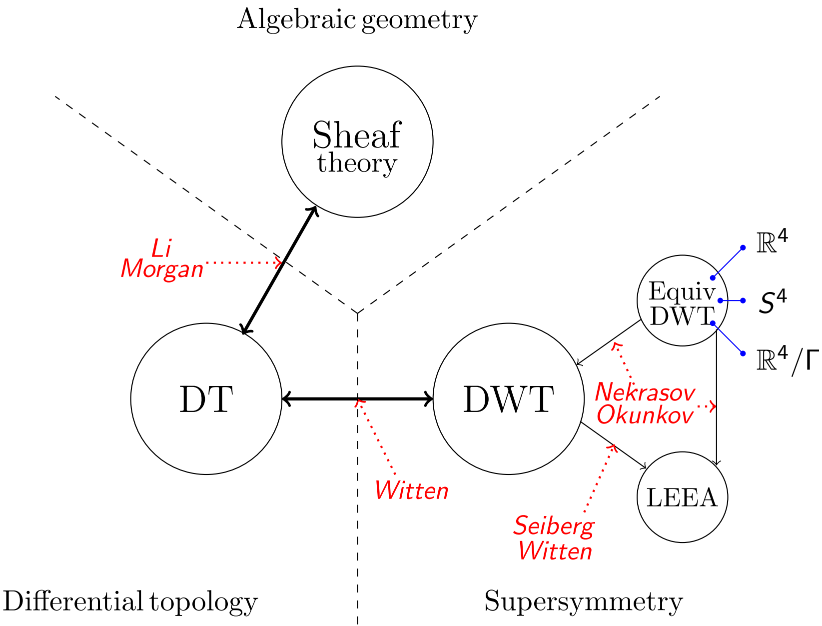
- Now we take observables from the equivariant cohomology $\mathcal{O} \in \frac{\mathrm{Ker}\,\tilde{\mathrm{Q}}}{\mathrm{Im}\,\tilde{\mathrm{Q}}}$
- We can use the Localisation theorem to compute the correlators.

Localisation theorem

Let X be a compact oriented D-dimensional smooth manifold equipped with the action of a group G with only isolated fixed points x_i. Then the integral of a closed equivariant form α is given by the localization formula:

$$\int_X \alpha = (-2\pi)^{\frac{D}{2}} \sum_i \frac{\alpha_0(x_i)}{\sqrt{\det \mathcal{L}_{x_i}}},$$

where $\alpha_0(x_i)$ is the zero-form part of α at the fixed point x_i and $\mathcal{L}_{x_i}$ is the natural action of G on the tangent space $T_{x_i}X$



Algebraic geometry

Sheaf theory

Li
Morgan

DT

DWT

Witten

Equiv DWT

$\mathbb{R}^4$

S^4

$\mathbb{R}^4/\Gamma$

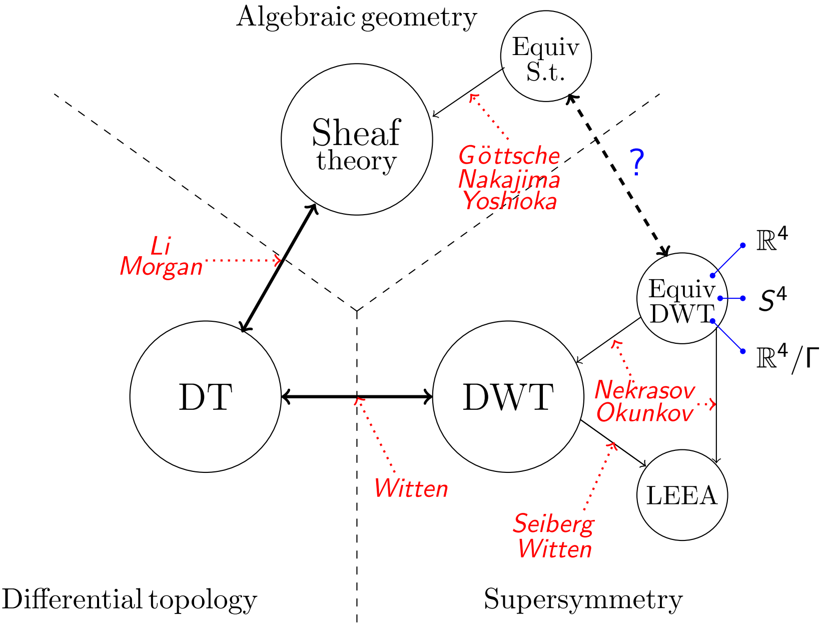
Nekrasov
Okunkov

LEEA

Seiberg
Witten

Differential topology

Supersymmetry



Algebraic geometry

Equiv
S.t.

Sheaf
theory

Göttsche
Nakajima
Yoshioka

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Li
Morgan

$\mathbb{R}^4$

Equiv
DWT

S^4

$\mathbb{R}^4/\Gamma$

DT

DWT

Nekrasov
Okunkov

Witten

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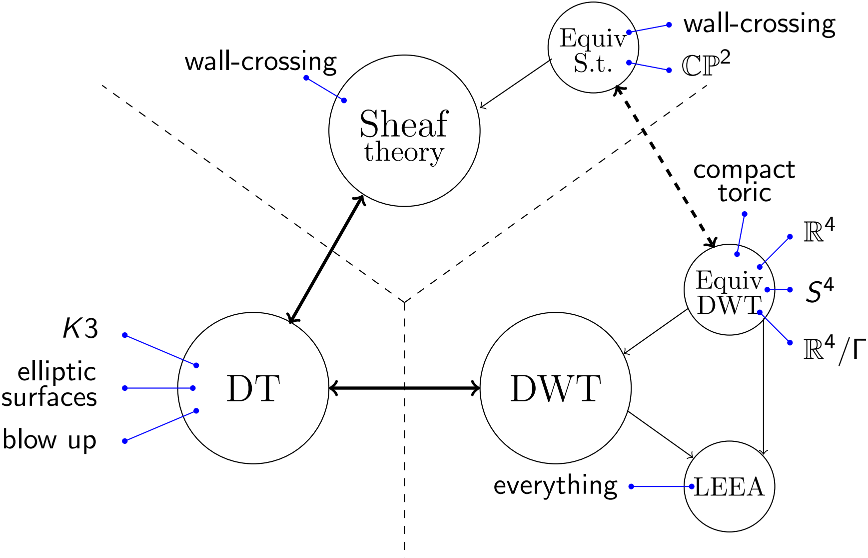
Seiberg
Witten

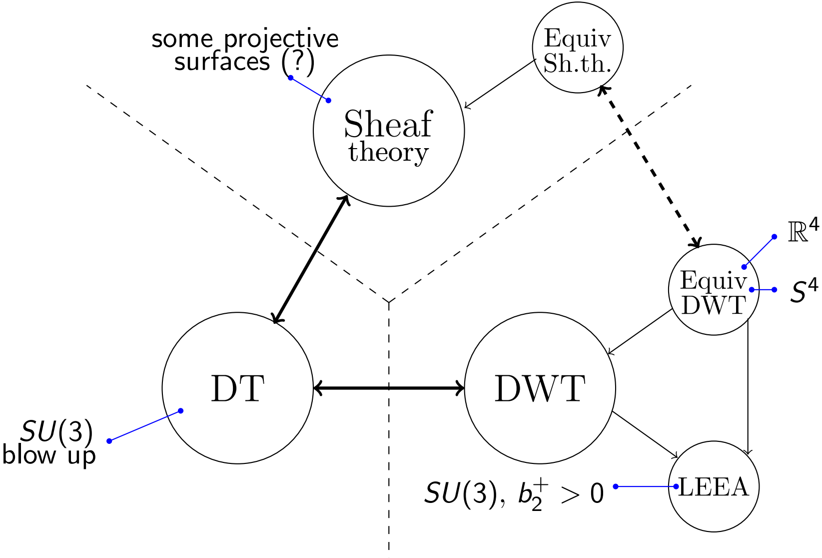
Differential topology

Supersymmetry

Main results in SU(2)

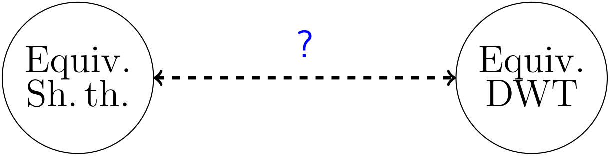


Main results in SU(N)



Outline

1. Recurrence relation for the instanton partition function on $\mathbb{R}^4$
2. Donaldson invariants of compact toric manifolds



Equiv.
Sh. th.

?

Equiv.
DWT

Nekrasov partition function on $\mathbb{R}^4$

- $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{C}^N$ are the vacuum expectation values of the Higgs field. $SU(N)$ theory: $\sum_{u=1}^N a_u = 0$

$$\mathcal{Z}(\mathbf{a}) = Z_{\mathrm{pert}}(\mathbf{a}) Z_{\mathrm{inst}}(\mathbf{a}) \qquad Z_{\mathrm{inst}}(\mathbf{a}) = \sum_{k=0}^{\infty} q^k Z_k(\mathbf{a})$$

- Terms are parametrized by N Young diagrams $\vec{Y} = (Y_1, \dots, Y_N)$ with the total number of boxes $|\vec{Y}| = k$.

$$Z_k(\mathbf{a}) = \sum_{\substack{\vec{Y} \\ |\vec{Y}|=k}} \left(\prod_{(i,j)\in Y_u} (a_{vu} + \epsilon_1(i - \tilde{l}_{Y_v,j}) - \epsilon_2(j - 1 - l_{Y_u,i})) \right.$$
$$\left. \cdot \prod_{(i,j)\in Y_v} (a_{vu} - \epsilon_1(i - 1 - \tilde{l}_{Y_u,j}) + \epsilon_2(j - l_{Y_v,i})) \right)^{-1}$$

$l_{Y,i}$ is the length of the i-th row of diagram Y,
$\tilde{l}_{Y,i}$ is the length of the i-th column of diagram Y.

Zamolodchikov recurrence relation

- There is a recurrence relation in $SU(2)$ theory

$$Z_{\text{inst}}(a) = 1 + \sum_{m,\, n=1}^{\infty} \frac{q^{mn} Z_{\text{inst}}(\epsilon_{m,-n})}{(-a+\epsilon_{m,n})(a+\epsilon_{m,n})} \frac{2\epsilon_{m,n}}{\prod_{\substack{i=-m+1\\(i,j)\neq(0,0)}}^{m} \prod_{j=-n+1}^{n} \epsilon_{i,j}}$$

- $\epsilon_{m,n} = m\epsilon_1 + n\epsilon_2,\ a = a_1 - a_2 \triangleq a_{12}$
- Poles are simple and located at $a = \epsilon_{m,n}$ with $m \cdot n > 0$.

$$\mathrm{Res}_{a=\epsilon_{m,n}} Z_{\text{inst}}(a) = -q^{mn} \frac{Z_{\text{inst}}(\epsilon_{m,-n})}{\prod_{\substack{i=-m+1\\(i,j)\neq(0,0)}}^{m} \prod_{j=-n+1}^{n} \epsilon_{i,j}}$$

- Equivalent form

$$\lim_{\alpha\to 0} \frac{\mathcal{Z}(\alpha+\epsilon_{m,n})}{\mathcal{Z}(\alpha+\epsilon_{m,-n})} = -\mathrm{Sign}(\epsilon_1).$$

Partial Weyl permutation

- $\mathbf{m}$, $\mathbf{n} \in \mathbb{Z}^N$ are points of reference for $\mathbf{a}$

$$\begin{aligned} a_1 &= \alpha_1 + m_1\epsilon_1 + n_1\epsilon_2 \\ &\dots \\ a_u &= \alpha_u + m_u\epsilon_1 + \mathbf{n_u}\epsilon_2 \\ &\dots \\ a_v &= \alpha_v + m_v\epsilon_1 + \mathbf{n_v}\epsilon_2 \\ &\dots \\ a_N &= \alpha_N + m_N\epsilon_1 + n_N\epsilon_2 \end{aligned} \quad \rightarrow \quad \begin{aligned} \hat{a}_1^{(uv)} &= \alpha_1 + m_1\epsilon_1 + n_1\epsilon_2 \\ &\dots \\ \hat{a}_u^{(uv)} &= \alpha_u + m_u\epsilon_1 + \mathbf{n_v}\epsilon_2 \\ &\dots \\ \hat{a}_v^{(uv)} &= \alpha_v + m_v\epsilon_1 + \mathbf{n_u}\epsilon_2 \\ &\dots \\ \hat{a}_N^{(uv)} &= \alpha_N + m_N\epsilon_1 + n_N\epsilon_2 \end{aligned}$$

- α_u is arbitrary (not necessary small)

Residue formula

- Poles are simple and located at $a_{uv} = \epsilon_{mn}$ with $m \cdot n > 0$
- Residue of $Z_{\rm inst}(\mathbf{a})$ w.r.t. a_{uv} is proportional to the value $Z_{\rm inst}(\hat{\mathbf{a}}^{(\mathbf{uv})})$.

$$\lim_{\alpha_{uv}\to 0} \frac{\mathcal{Z}(\mathbf{a})}{\mathcal{Z}(\hat{\mathbf{a}}^{(uv)})} = -\mathrm{Sign}(\epsilon_1)$$

It is **exact** with respect to all variables except α_{uv}.

- In terms of only instanton part if has the form

$$\mathrm{Res}_{a_{uv}=\epsilon_{m,n}} Z_{\rm inst}(\mathbf{a}) = q^{mn} \frac{1}{\mathcal{P}_N^{(uv)}(m,n|\mathbf{a})} Z_{\rm inst}(\hat{\mathbf{a}}^{(uv)}),$$

where

$$\mathcal{P}_N^{(uv)}(m,n|\mathbf{a}) = \prod_{i=-m}^{m-1} {\prod_{j=-n}^{n-1}}' \epsilon_{i,j} \cdot \prod_{\substack{w=1\\ w\neq u,v}}^{N} \prod_{i=1}^{m} \prod_{j=1}^{n} [(a_{vw}+\epsilon_{i,j})(-a_{uw}+\epsilon_{i,j})]$$

Refined residue formula

- Difficulty: it is a relation between sums running over all possible N-tuples of the Young diagrams. Too many terms!

$$Z_{\mathrm{inst}}(\mathbf{a}) = \sum_{\mathbf{Y}} Z_{\mathbf{Y}}(\mathbf{a}),$$

- Residue formula can be refined

$$\mathrm{Res}_{a_{uv}=\epsilon_{m,n}} \sum_{\mathbf{Y}\in\mathcal{F}} Z_{\mathbf{Y}}(\mathbf{a}) = q^{mn} \frac{1}{\mathcal{P}_N^{(uv)}(m,n|\mathbf{a})} \sum_{\mathbf{Y}\in\tilde{\mathcal{F}}} Z_{\mathbf{Y}}(\hat{\mathbf{a}}^{(uv)}),$$

$\mathcal{F}$ (or $\tilde{\mathcal{F}}$) is a subset of the set of all the N-tuples of Young diagrams with k cells (or $k-mn$ cells), which we will call a **family** of Young diagrams (or a **dual family**).

How to define a family?

- $Z_{\mathbf{Y}}$ are instanton configurations invariant under $T = U(1)^2 \times U(1)^N$
- $(\epsilon_1, \epsilon_2, a_1, \ldots, a_N)$ are coordinates on T
- $(\epsilon_1, \epsilon_2, \alpha_1, \ldots, \alpha_N)$ are twisted coordinates on T

$$a_u = \alpha_u + m_u \epsilon_1 + n_u \epsilon_2, \qquad u = 1, \ldots, N$$

- Equation $\alpha_{uv} = 0$ defines a subgroup $T^{(uv)} \cong U(1)^2 \times U(1)^{N-1}$

$$\lim_{\alpha_{uv} \to 0} \frac{Z_{\text{pert}}(\mathbf{a}) \sum_{\mathbf{Y} \in \mathcal{F}} Z_{\mathbf{Y}}(\mathbf{a})}{Z_{\text{pert}}(\hat{\mathbf{a}}^{(uv)}) \sum_{\mathbf{Y} \in \tilde{\mathcal{F}}} Z_{\mathbf{Y}}(\hat{\mathbf{a}}^{(uv)})} = -\text{Sign}(\epsilon_1)$$

- Equivariant localisation: behaviour of $Z_{\mathbf{Y}} Z_{\text{pert}}$ is defined by the representation of $T^{(uv)}$ on the tangent space at the fixed point
- It is difficult to analyse...

Sheaf theory point of view

Let us look at the sheaf theory instead!

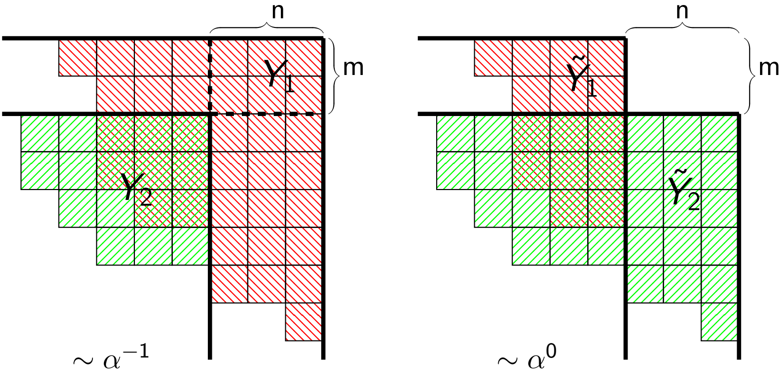
Compactified $\mathcal{M}_{\mathrm{ASD}}$ $\longleftrightarrow$ $\mathcal{M}$ of framed torsion-free sheaves on $\mathbb{CP}^2$

Fixed-point instanton configurations $\longleftrightarrow$ Equivariant splitted torsion-free sheaves with trivial framing

- T, $T^{(uv)}$ act on a space of sections of the fixed-point sheaf on $\mathbb{C}^2$ and their representations are easy to describe.

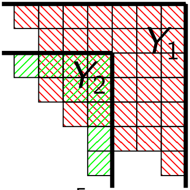
We gather in a **family** all the fixed-point sheaves which transform in the same way under $T^{(uv)}$.
We gather in the **dual family** all the fixed-point sheaves which transform in the same way as the **family** members under $T^{(uv)}$ with the partially permuted twisting parameters n_u, n_v.

Family with one member

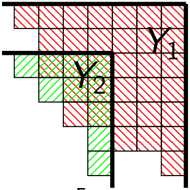


$$\mathrm{Res}_{a=\epsilon_{m,n}} \sum_{\mathbf{Y}\in\mathcal{F}} Z_{\mathbf{Y}}(a) = q^{mn} \frac{1}{\mathcal{P}_2(m,n)} \sum_{\mathbf{Y}\in\tilde{\mathcal{F}}} Z_{\mathbf{Y}}(\epsilon_{m,-n})$$

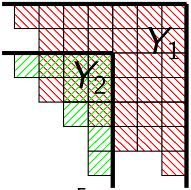
Family with many members



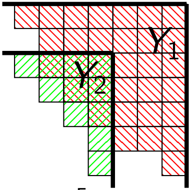
$\sim \alpha^{-5}$



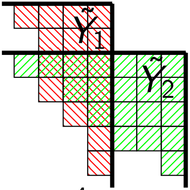
$\sim \alpha^{-5}$



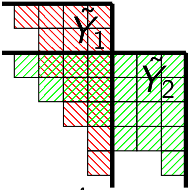
$\sim \alpha^{-5}$



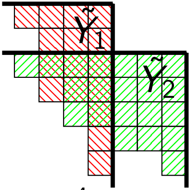
$\sim \alpha^{-5}$



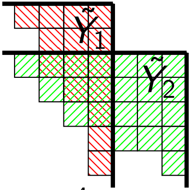
$\sim \alpha^{-4}$



$\sim \alpha^{-4}$



$\sim \alpha^{-4}$



$\sim \alpha^{-4}$

Recurrence relation

- We chose $N-1$ independent variables to be a_{uN}, $u = 1 \dots N-1$ and assume that $a_{\hat{u}N}$, $\hat{u} = 2 \dots N-1$ are away from the poles as well as their differences $a_{\hat{u}N} - a_{\hat{v}N}$.
- $Z_{\text{inst}}(\mathbf{a})$ has poles only with respect to a_{1N} at the points $a_{1N} = \epsilon_{m,n}$ and $a_{1N} = \epsilon_{m,n} + a_{\hat{u}N}$

$$Z_{\text{inst}}(\mathbf{a}) = 1 + \sum_{w=2}^{N} \sum_{m,\, n=1}^{\infty} \frac{q^{mn} Z_{\text{inst}}(\hat{\mathbf{a}}^{(1w)})}{(a_{1N} - a_{wN} + \epsilon_{m,n})(a_{1N} - a_{wN} - \epsilon_{m,n})} \cdot \frac{2\epsilon_{m,n}}{\mathcal{P}_N^{(1w)}(m, n|\mathbf{a})}$$

Compact toric manifold

Compact toric manifold - algebraic torus $(\mathbb{C}^*)^2$ appropriately compactified by gluing with $\mathbb{CP}^1$ surfaces (divisors).

- Toric manifold comes with $U(1)^2$ action.
- There is a standard cover by χ coordinate patches with a fixed point of $U(1)^2$ at the origin.
- On the coordinate patches $U(1)^2$ acts with local weights ϵ_1^ℓ, ϵ_2^ℓ.

Nekrasov conjecture

$$\mathcal{Z}^{(X)}(\Omega, q) = \sum_{\mathbf{k}\in\mathcal{R}} \mathrm{Res}_{\alpha=0} \prod_{\ell=1}^{\chi} \mathcal{Z}^{(\mathbb{C}^2)}(\alpha + k^\ell \epsilon_1^\ell + k^{\ell+1}\epsilon_2^\ell, q e^{\Omega_\ell}),$$

$\mathcal{R}$ is restricted by the stability conditions.

DI of compact toric manifolds in $SU(2)$

[G. Bonelli, F. Fucito, J. F. Morales, M. Ronzani, E. Sysoeva, A. Tanzini. (2021). Gauge theories on compact toric manifolds.]

- Integral localizes on the fixed points of the torus action

$$\mathcal{Z}^{(X)}_{c_1}(\Omega, q) = \sum_{k^\ell \in \Gamma_{c_1}} \mathrm{sign}(\kappa) \mathrm{Res}_{\alpha=0} \underbrace{\prod_{\ell=1}^{\chi} \mathcal{Z}^{(\mathbb{C}^2)}(\alpha + k^\ell \epsilon_1^\ell + k^{\ell+1} \epsilon_2^\ell, q e^{\Omega_\ell}),}_{\mathcal{Z}(\alpha, \mathbf{k}, q e^{\Omega_\ell})}$$

- Γ_{c_1} is defined by c_1

$$\Gamma_{c_1} = \{(k^1, \dots, k^\chi) \in \mathbb{Z}^\chi | \sum_{\ell=1}^{\chi} k^\ell w_\ell = c_1 \,(\mathrm{mod} 2)\}$$

- Stability conditions are hidden in $\mathrm{sign}(\kappa)$

$$\kappa = \int_X \omega \wedge \left(\sum_{\ell=1}^{\chi} k^\ell w_\ell \right) = \sum_{\ell=1}^{\chi} \beta_\ell \, k^\ell \qquad \beta_\ell = \int_X \omega \wedge w_\ell$$

Non-equivariant limit is finite

- κ and Γ_{c_1} depend only on $\chi - 2$ gauge fluxes $r_s = k^\ell \int w_\ell \wedge w_s$
- Thus we can write

$$\begin{aligned}
\mathcal{Z}^{(X)}_{c_1}(\Omega, q) &= \sum_{\substack{r_s \in 2\mathbb{Z} + c_1 \wedge w_s \\ s=3,\ldots,\chi}} \mathrm{sign}(\kappa) \sum_{k^1, k^2 \in \mathbb{Z}} \mathrm{Res}_{\alpha=0} \mathcal{Z}(\alpha, \mathbf{k}, q e^{\Omega_\ell}) \\
&= \sum_{r_s} \mathrm{sign}(\kappa) \sum_{k^1, k^2 \in \mathbb{Z}} \mathrm{Res}_{\alpha=\epsilon_{k^1,k^2}} \mathcal{Z}(\alpha, \tilde{\mathbf{k}}, q e^{\Omega_\ell}) \\
&= \sum_{r_s} \mathrm{sign}(\kappa) \mathrm{Res}_{\alpha=\infty} \mathcal{Z}(\alpha, \tilde{\mathbf{k}}, q e^{\Omega_\ell})
\end{aligned}$$

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \mathcal{Z}^{(X)}_{c_1}(\Omega, q) = \sum_{r^s} \mathrm{sign}(\kappa) \mathrm{Res}_{\alpha=\infty} \lim_{\epsilon_1, \epsilon_2 \to 0} \mathcal{Z}(\alpha, \tilde{\mathbf{k}}, q e^{\Omega_\ell})$$

- $\mathcal{Z}(\alpha, \tilde{\mathbf{k}}, q e^{\Omega_\ell})$ has finite non-equivariant limit

Orbits and stability conditions

To make correspondence with the Sheaf theory side explicit we introduce a formal operator

$$\mathcal{P}_\ell(k^{\ell'}) = (-1)^{\delta_{\ell\ell'}} k^{\ell'}$$

and go to the sum over orbits

$$\mathcal{Z}^{(X)}(\Omega, q) = \sum_{\mathbf{k}\in\Gamma} \mathcal{Z}_{\mathrm{orb}}(\mathbf{k}, qe^{\Omega_\ell}), \qquad \Gamma = \{\mathbf{k} |\, k^\ell \geq 0\ \forall \ell\},$$

where

$$\mathcal{Z}_{\mathrm{orb}}(\mathbf{k}, qe^{\Omega_\ell}) = \frac{1}{2^{\zeta_{\mathbf{k}}}} \mathrm{Res}_{\mathfrak{a}=0} \left[\prod_{\ell=1}^{\chi} (1 + \mathcal{P}_\ell) \mathrm{sign}(\kappa) \mathcal{Z}(\alpha, \mathbf{k}, qe^{\Omega_\ell}) \right]$$

$\zeta_{\mathbf{k}}$ is the number of zero entries in $\mathbf{k}$

Orbits and stability conditions

- We divide orbits in groups

$$\begin{array}{rll}
\text{Stable} & \forall\, k^i & 2\beta_i\, k^i < \sum_{\ell=1}^{\chi} \beta_\ell\, k^\ell \\
\text{Semi stable} & \exists\, k^i & 2\beta_i\, k^i = \sum_{\ell=1}^{\chi} \beta_\ell\, k^\ell \\
\text{Unstable} & \exists\, k^i & 2\beta_i\, k^i > \sum_{\ell=1}^{\chi} \beta_\ell\, k^\ell
\end{array}$$

- **k** corresponds to parameters in Klyachko's conditions of sheaf stability
- Contribution of unstable orbits vanishes
- Thus we have **finite** number of terms contributing to every order of q
- DI changes when some of sheaves change their stability type

DI on compact toric manifolds in $SU(N)$

- Our conjecture is that

$$\boxed{\mathcal{Z}^{(X)}_{c_1}(\Omega, q) = \lim_{\delta\to 0} \sum_{\mathbf{k}\in\Gamma_{c_1}} \mathrm{JK}_{\mathcal{C}}\left[e^{i\delta\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}}\mathcal{Z}(\boldsymbol{\alpha}, \mathbf{k}, qe^{\Omega_\ell})\right]}$$

- Γ_{c_1} is defined by c_1

$$\Gamma_{c_1} = \{(\mathbf{k}^1, \ldots, \mathbf{k}^\chi) \in \mathbb{Z}^{\chi(N-1)} | k^\ell_N = 0\, \forall\ell;\, \sum_{\ell=1}^{\chi} k^\ell\omega_\ell = c_1 \,(\mathrm{mod} N)\}$$

- Stability conditions are hidden in $e^{i\delta\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}}$

$$\boldsymbol{\kappa}\cdot\boldsymbol{\alpha} = \sum_{u=1}^{N} \kappa_u\alpha_u \qquad \kappa_u = \int_X \omega\wedge\left(\sum_{\ell=1}^{\chi} k^\ell_u\omega_\ell\right) = \sum_{\ell=1}^{\chi}\beta_\ell\, k^\ell_u$$

- $\mathcal{Z}^{(X)}_{c_1}(\Omega, q)$ has finite non-equivariant limit exactly for the same reason as in $SU(2)$ case

Let us see that conjectured $\mathcal{Z}_{c_1}^{(X)}(\Omega, q)$ has the same walls as appear in Klyachko's stability conditions.

- Let us look at some pole: $\mathcal{Z} \underset{\boldsymbol{\alpha}\to 0}{\sim} \left(\prod_{u<v} \alpha_{uv}\right)^{-1}$
- Set of all α_{uv} is over complete.
- We can chose a basis $\sigma_i = \{\alpha^{(1)}, \ldots, \alpha^{(N-1)}\}$ in different ways.
- We can make partial fraction decomposition of the integrand

$$e^{i\delta\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}}\mathcal{Z} = \begin{bmatrix} non \\ complete \\ denominator \end{bmatrix} + \underbrace{\sum_{\sigma\in\Sigma} \frac{C_\sigma}{\prod_{\alpha^{(i)}\in\sigma} \alpha^{(i)}}}_{\substack{\text{only this contribute} \\ to\ JK-residue}} + \begin{bmatrix} over \\ complete \\ denominator \end{bmatrix}$$

- Contribution is nonzero iff $\boldsymbol{\kappa} \in \ Cone(\sigma)$
- It recovers exactly the walls expected from Klyachko's stability conditions

Equivariant DI in SU(3) theory

- Computing the JK-residue we get Donaldson invariants as

$$\mathcal{Z}_{c_1}^{(X)}(\Omega, q) = \sum_{\mathbf{k}\in\Gamma_{c_1}} \mathrm{sign}(\kappa_2+\kappa_3-2\kappa_1)\,\mathrm{sign}(\kappa_1+\kappa_3-2\kappa_2)$$
$$\mathrm{Res}_{\alpha_{12}=0}\mathrm{Res}_{\alpha_{23}=0}\mathcal{Z}(\boldsymbol{\alpha}, \mathbf{k}, qe^{\Omega_\ell})$$

- We can again introduce stable, semi-stable and non-stable orbits of an operator permuting k_u^ℓ, $u = 1, \ldots, N$ on every patch.
- Contribution of non-stable orbits (corresponding to non-stable sheaves) vanishes (generalised Zamolodchikov relation to the rescue).
- $\mathcal{N} = 4$ partition function gives the Euler characteristic of the moduli space of instantons on the manifold. Our approach to the computation of the partition function leads to the correct Euler characteristic of the moduli space of instantons on $\mathbb{CP}^2$.

Summary

We can state in $SU(N)$ case that our conjectured Donaldson invariants

- Have finite non-equivariant limit
- Change when some sheaves change their stability type

In SU(3)

- Contributions of non-stable orbits in the Donaldson invariants vanish
- Our conjecture recovers Euler characteristic of moduli space of torsion-free sheaves over $\mathbb{CP}^2$

In SU(2)

- Formally recovers the answer obtained by means of the equivariant localisation, which surpassed numerous checks

Thank you for the attention