Tracy-Widom distributions in supersymmetric gauge theories

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Outline

Ultimate goal is to solve four-dimensional super Yang–Mills theories

$$\text{SYM theory} = [\text{gauge fields}] + [\text{fermions}] + [\text{scalars}]$$

for arbitrary 't Hooft coupling $\lambda = g^2_{\rm YM} N_c$, to any order in $1/N_c^2$

Integrabilty allows us to realize this program for a special class of observables

- ✔ Weak coupling expansion is easy, what we expect at strong coupling?
- ✔ Tracy-Widom distribution in random matrices
- ✔ Free energy in $\mathcal{N}=2$ SYM on sphere
- ✔ Correlation function of (infinitely) heavy half-BPS operators in $\mathcal{N}=4$ SYM
- ✔ Strong coupling expansion from Szegő–Akhiezer-Kac formula

What we expect at strong coupling

Simple example: circular Wilson loop in planar $\mathcal{N} = 4$ SYM

$$W = \frac{1}{N_c} \langle \mathrm{tr}\, P\, \mathrm{e}^{ig_{\mathrm{YM}} \oint ds\, \dot{x} \cdot A(x(s)) + i\Phi(x(s))|\dot{x}(s)|} \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N_c$

Expansion at weak and strong coupling

$$W \overset{\lambda \ll 1}{=} 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \frac{\lambda^4}{737280} + \dots$$

$$W \overset{\lambda \gg 1}{=} \exp\left(\sqrt{\lambda} - \frac{1}{2} \log\left(\frac{\pi}{2} \lambda^{3/2} \right) - \frac{3}{8\sqrt{\lambda}} - \frac{3}{16\lambda} - \frac{21}{128\lambda^{3/2}} + \dots \right)$$

Semiclassical asymptotics in AdS/CFT

$$\log W = -\sqrt{\lambda} A_0 - A_1 \log(\sqrt{\lambda}) - B - \frac{A_2}{\sqrt{\lambda}} - \frac{A_3}{\lambda} - \frac{A_4}{\lambda^{3/2}} + \dots$$

- ✔ A_0 minimal area in AdS_5
- ✔ A_i and B come from fluctuations (very hard to compute in AdS/CFT)

Tracy-Widom distribution

Describes statistics of the spacing of the eigenvalues of $N \times N$ hermitian matrices for $N \to \infty$

Gaussian Unitary Ensemble

$$Z_{\rm GUE} = \int d^{N\times N} a\, e^{-\frac{1}{2}\,{\rm tr}\, a^2} = \int_{-\infty}^{\infty} d\lambda_1 \ldots d\lambda_N \prod_{i\neq j} (\lambda_i - \lambda_j)^2 e^{-\frac{1}{2}\sum_i \lambda_i^2}$$

Laguerre ensemble (Wishart matrix theory)

$$Z_{\rm Laguerre} = \int_0^{\infty} d\lambda_1 \ldots d\lambda_N \prod_{i\neq j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} \lambda_i^{\ell}\, e^{-\lambda_i}$$

where $\ell > -1$ and eigenvalues are located on semi-axis $[0, \infty)$.

The probability density for eigenvalues

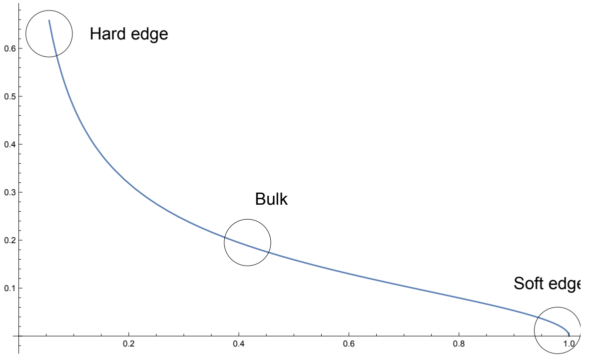
$$R_n(x_1, \ldots, x_n) = \left\langle \prod_{i=1}^{n} \delta(\lambda_i - x_i) \right\rangle = \det K_N(x_i, x_j)\Big|_{i,j=1,\ldots,n}$$

$$K_N(x, y) = \sum_{k=0}^{N-1} \phi_k(x)\phi_k(y)$$

where $\phi_k(x)$ are orthonormal functions $x^k e^{-x^2/2} + \ldots$ (GUE) and $x^k x^{\ell/2} e^{-x/2} + \ldots$ (Laguerre)

Tracy-Widom distribution II

The distribution of the eigenvalues in the Laguerre ensemble in the limit $N \to \infty$



$$R_1(4Nx) \sim \frac{1}{2\pi}\sqrt{\frac{1-x}{x}}$$

Scaling behaviour of $K_N(x,y)$ around $x = 0$ (hard edge), $x = 1$ (soft edge) and $0 < x < 1$ (bulk)

bulk : $$\frac{\sin \pi(x-y)}{\pi(x-y)}$$

soft edge : $$\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(x)\mathrm{Ai}'(y)}{x-y}$$

hard edge : $$\frac{J_\ell(\sqrt{x})\sqrt{y}J_\ell'(\sqrt{y}) - \sqrt{x}J_\ell'(\sqrt{x})J_\ell(\sqrt{y})}{2(x-y)}$$

The probability that there are no eigenvalues on the interval $[0, s]$

$$E(0; s) = \det(1-K)_{[0,s]} = 1 + \sum_{n\geq 1} \frac{(-1)^n}{n!} \int_0^s dx_1 \dots dx_n \det \|K(x_i, x_j)\|_{1\leq i,j\leq n}$$

Fredholm determinant of the integral operator: *Sinc* (bulk), *Airy* (soft edge) and *Bessel* (hard edge)

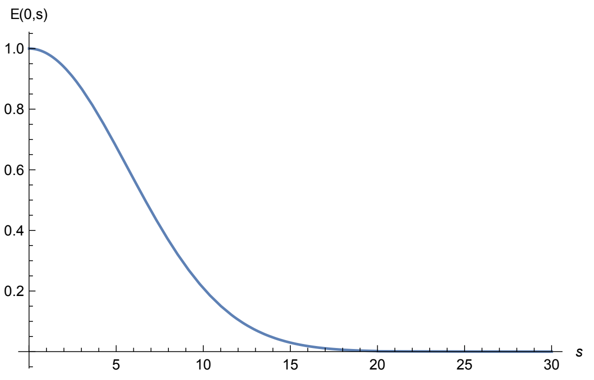
Bessel kernel

Tracy-Widom distribution close to the hard edge

$$E(0,s) = \det(1 - K_{\text{Bessel}})_{[0,s]} = \exp\left(-\frac{1}{4}\int_0^s dx\, \log(s/x)\, Q^2(x)\right)$$

$Q(s)$ satisfies Painlevé V differential equation

Dependence of the probability $E(0,s)$ on the interval length s



Asymptotics of $E(0,s)$ at small and large s

$$E(0,s) \overset{s \ll 1}{=} 1 - \frac{(s/4)^{\ell+1}}{\Gamma^2(\ell+2)} + \dots$$

$$E(0,s) \overset{s \gg 1}{=} \exp\left(-s/4 - \frac{\ell^2}{4}\log s + \frac{\ell}{8} s^{-1/2} + \dots\right)$$

Remarkably similar to weak/strong coupling expansion in gauge theory for $s \sim \sqrt{\lambda}$

Bessel kernel at finite temperature

$$K_\ell(x,y) = \sum_{n\geq 1} \phi_n(x)\phi_n(y)\chi\left(\frac{y}{2g}\right), \qquad \phi_n(x) = \sqrt{2n+\ell-1}\,\frac{J_{2n+\ell-1}(\sqrt{x})}{\sqrt{x}}$$

Can be represented by a semi-infinite matrix

$$\int_0^\infty dy\, K_\ell(x,y)\,\phi_n(x) = K_{nm}\phi_m(x)$$

$$K_{nm} = 2(-1)^{n+m}\sqrt{(2n+\ell-1)(2m+\ell-1)}\int_0^\infty \frac{dx}{x} J_{2n+\ell-1}(x)J_{2m+\ell-1}(x)\chi\left(\frac{x}{2g}\right)$$

$\chi(x)$ is the *symbol* of the Bessel operator

$$\det(1-\mathbf{K}_\chi) = \det\left(\delta_{nm} - K_{nm}\right)\Big|_{n,m\geq 1}$$

- ✔ For $\chi(x) = \theta(1-x)$ coincides with the Tracy-Widom distribution $E(0,s)$ for $s = (2g)^2$
- ✔ Finite-temperature generalization: $\chi(x) = 1/(1+e^{\frac{x-\mu}{T}})$
- ✔ In supersymmetric gauge theories we encounter symbol of the form

$$\chi_{\rm loc}(x) = -\frac{1}{\sinh^2(x/2)}, \qquad \chi_{\rm oct}(x) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh\sqrt{x^2+\xi^2}}$$

y and ξ are kinematical variables

Free energy in $\mathcal{N}=2$ super Yang-Mills theory

- $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ coupled to matter multiplets in the symmetric $(N_S=1)$ and anti-symmetric $(N_A=1)$ representations

  The beta function vanishes $\beta_0 = 2N - N_S(N+2) - N_A(N-2) = 0$,

- The partition function on sphere S^4 is given by a matrix integral [Pestun]

$$Z_{S^4} = e^{-F} = \int da\, e^{-\frac{8\pi^2 N}{\lambda}\,\mathrm{tr}\,a^2} |Z_{1-\mathrm{loop}}(a) Z_{\mathrm{inst}}(a)|^2$$

  Non-perturbative instanton contribution $Z_{\mathrm{inst}}(a)$ is exponentially small at large N

- Perturbative corrections $Z_{1-\mathrm{loop}}(a) = \exp(-S_{\mathrm{int}}(a))$ only come from one loop

$$S_{\mathrm{int}}(a) = \sum_{i,j} \Big[\log H(\lambda_i + \lambda_j) - \log H(\lambda_i - \lambda_j) \Big] \qquad (\lambda_i \text{ are eigenvalues of } a)$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \zeta_{2n+1} \sum_{p=0}^{n} \binom{2n+2}{2p+1}\, \mathrm{tr}\,a^{2p+1}\ \mathrm{tr}\,a^{2(n-p)+1}$$

$$H(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}} = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \zeta_{2n+1}\, x^{2n+2} \right)$$

  Matrix model with double-trace interaction

Large N expansion

$$e^{-F} = \left(\frac{8\pi^2}{\lambda}\right)^{-(N^2-1)/2} \int da\, \exp\left(-N\,\mathrm{tr}\, a^2 + \frac{1}{2}\sum_{k,n} C_{kn} O_{2k+1} O_{2n+1}\right)$$

The interaction term is a sum over double traces $O_k = \mathrm{tr}\, a^k$ with the couplings

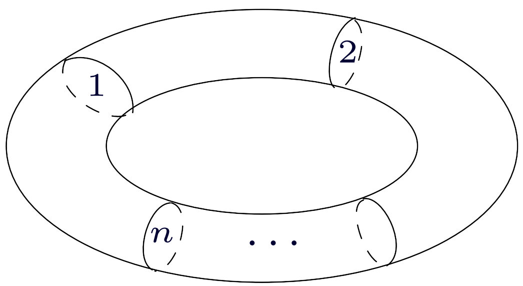
$$C_{kn} = 4\frac{(-1)^{k+n+1}}{k+n+1}\zeta_{2(k+n)+1}\binom{2(k+n+1)}{2k+1}\left(\frac{\lambda}{8\pi^2}\right)^{k+n+1}$$

Large N expansion

$$F = N^2 F_0(\lambda) + F_1(\lambda) + F_2(\lambda)/N^2 + \ldots$$

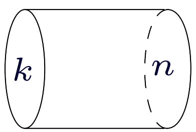
The interaction term does not contribute to F_0

$$F_1 = \sum_{n\geq 1}$$



$$= -\sum_{n\geq 1}\frac{1}{2n}\,\mathrm{tr}[(QC)^n] = \frac{1}{2}\log\det(1-QC)$$

Cylinders $Q_{kn} =$



$= \langle \mathrm{tr}\, a^{2k+1}\, \mathrm{tr}\, a^{2n+1}\rangle_{\mathrm{GUE}}$ are glued together with the weight C_{kn}

Relation to Bessel kernel

Explicit expressions for semi-infinite matrices

$$Q_{kn} = \frac{2\beta_k \beta_n}{k+n+1} + O(1/N^2)\,, \qquad \beta_n = \frac{2^n n \Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+2)}$$

$$C_{kn} = 4\frac{(-1)^{k+n+1}}{k+n+1}\zeta_{2(k+n)+1}\binom{2(k+n+1)}{2k+1}\left(\frac{\lambda}{8\pi^2}\right)^{k+n+1}$$

The matrix (QC) is related to the Bessel kernel by a similarity transformation

[Beccaria,Billò,Galvagno,Hasan,Lerda]

$$\begin{aligned} K_{nm} &= (U^{-1}QCU)_{nm} \\ &= 2\,(-1)^{n+m}\sqrt{2n+1}\sqrt{2m+1}\,\int_0^\infty \frac{dt}{t} J_{2n+1}(t)\,J_{2m+1}(t)\chi\left(\frac{x}{2g}\right) \end{aligned}$$

Special form of the symbol

$$\chi(x) = -\frac{1}{\sinh^2(x/2)}\,, \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

The free energy coincides with the Tracy-Widom distribution at the hard edge for $\ell = 2$

$$F_1 = \frac{1}{2}\log\det(1-QC) = \frac{1}{2}\,\mathrm{tr}\log(1-\mathbf{K}_\chi)$$

Correlation functions in $\mathcal{N} = 4$ SYM

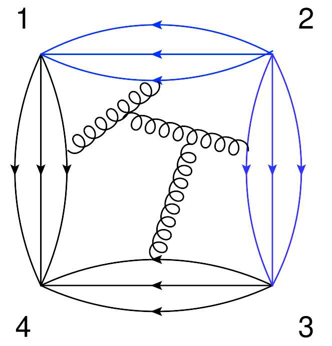
- ✔ Half-BPS operators

$$O_1 = \mathrm{tr}(Z^{K/2}\bar{X}^{K/2}) + \text{permutations}\,, \qquad O_2 = \mathrm{tr}(X^K)\,, \qquad O_3 = \mathrm{tr}(\bar{Z}^K)$$

Exact scaling dimension (R-charge) $\Delta = K$

Two– and three–point functions are protected

- ✔ "Simplest" four-point function



$$\langle O_1(x_1)O_2(x_2)O_1(x_3)O_3(x_4)\rangle = \frac{\mathcal{G}_K(z,\bar{z})}{(x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2)^{K/2}}$$

Depends on two cross ratios and 't Hooft coupling $g^2 = g^2_{\mathrm{YM}} N_c/(4\pi)^2$

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}\,, \qquad \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

- ✔ Examine $\mathcal{G}_K(z,\bar{z})$ in the limit $K \to \infty$ (infinitely heavy operators) with g^2 kept fixed

Weak coupling expansion

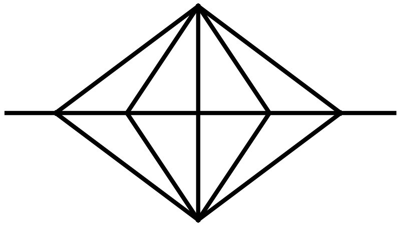
$$\lim_{K\to\infty} \mathcal{G}_K = [\mathbb{O}(z,\bar{z})]^2$$

$\mathbb{O}(z,\bar{z})$ is a multilinear combination of ladder integrals [Coronado]

$$\mathbb{O}(z,\bar{z}) = 1 + g^2 f_1 - 2g^4 f_2 + 6g^6 f_3 + g^8(-20f_4 - \tfrac{1}{2}f_2^2 + f_1 f_3) + \dots$$
$$= 1 + \sum_{\ell\geq 1}(g^2)^\ell \times \sum_{i_1+\dots+i_n=\ell} d_{i_1\dots i_n} f_{i_1}\dots f_{i_n}$$

The expansion coefficients $d_{i_1\dots i_n}$ can be found to all loops from different OPE limits of $\mathcal{G}_K$

Ladder integrals

$f_L =$ 

$$= \frac{1}{z-\bar{z}} \sum_{m=0}^{L} \frac{(-1)^m(2L-m)!}{L!(L-m)!m!} \ln^m(z\bar{z}) \Big[\underbrace{\mathrm{Li}_{2L-m}(z)}_{\text{polylog}} - \mathrm{Li}_{2L-m}(\bar{z})\Big]$$

The weak coupling expansion can be resummed to all orders in the coupling [Kostov,Petkova,Serban]

Relation to Bessel kernel

$$\mathbb{O}(z,\bar{z}) = \exp\Big[-\frac{1}{2}\sum_{n\geq 1}\mathrm{tr}(CH)^n\Big] = \sqrt{\det(1-CH)}$$

Semi-infinite matrices

$$H_{nm} = \frac{g}{2i}\int_{|\xi|}^{\infty} dt\,\frac{\left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{m-n} - \left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{n-m}}{\cosh y + \cosh t}\underbrace{J_m(2g\sqrt{t^2-\xi^2})}_{\text{Bessel function}}J_n(2g\sqrt{t^2-\xi^2})$$

$$C_{nm} = 2(\cosh y + 1)(\delta_{n+1,m} - \delta_{n,m+1})\,,$$

Kinematical variables : $z = -\,\mathrm{e}^{-y-\xi}$, $\bar{z} = -\,\mathrm{e}^{+y-\xi}$

Similarity transformation [Belitsky,GK],[Kostov,Petkova]

$$K_{nm} = (\Omega^{-1}CH\Omega)_{nm}$$
$$= 2\,(-1)^{n+m}\sqrt{2n+1}\sqrt{2m+1}\int_0^{\infty}\frac{dt}{t}J_{2n+1}(t)\,J_{2m+1}(t)\chi_{\mathrm{oct}}\left(\frac{x}{2g}\right)$$

$\mathbb{O}(z,\bar{z})$ coincides with the Tracy-Widom distribution for $\ell = 0$ and the symbol

$$\chi_{\mathrm{oct}}\left(\frac{x}{2g}\right) = \frac{\cosh y + \cosh\xi}{\cosh y + \cosh(\sqrt{x/(2g)^2+\xi^2})}$$

depends on g, y, ξ

Tracy-Widom distribution in super Yang-Mills theories

Different observables in SYM theories are given by the Tracy-Widom distribution $\det(1-\mathbf{K}_\chi)$

Choice of the observable fixes the form of the symbol:

- ✔ Free energy of $\mathcal{N}=2$ SYM

$$\chi(x) = -\frac{1}{\sinh^2(x/2)}$$

- ✔ Four-point correlator in $\mathcal{N}=4$ SYM

$$\chi(x) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh(\sqrt{x+\xi^2})}$$

- ✔ Circular Wilson loop

$$\chi(x) = -\frac{4}{x^2}$$

The coupling constant defines the interval length in the TW distribution $s \sim g^2$

- ✔ Weak coupling expansion is easy

$$\log\det(1-\mathbf{K}_\chi) = -\operatorname{tr}\mathbf{K}_\chi - \frac{1}{2}\operatorname{tr}(\mathbf{K}_\chi^2) + \dots = c_1 g^2 + c_2 g^4 + \dots$$

- ✔ Strong coupling expansion is hard

Szegő-Akhiezer-Kac formula

- Asymptotic behaviour for sufficiently smooth symbol $\chi(z)$

$$\det(1-\mathbf{K}_\chi) = \mathrm{e}^{-gA_0+B+O(1/g)}$$

SAK formula (1915-1966)

$$A_0 = -2\widetilde{\psi}(0)\,, \qquad B = \frac{1}{2}\int_0^\infty dk\, k\big(\widetilde{\psi}(k)\big)^2\,,$$

$$\widetilde{\psi}(k) = \int_0^\infty \frac{dz}{\pi}\cos(kz)\log(1-\chi(z))$$

B diverges for $\chi(z) \sim 1 - z^{2\beta}$ or $\widetilde{\psi}(k) \sim -\beta/k$ at large k *Fisher-Hartwig singularity*

- The SAK formula for the Bessel kernel with Fisher-Hartwig singularity has not been derived yet

- Our conjecture [Belitsky,GK]

$$\det(1-\mathbf{K}_\chi) = \mathrm{e}^{-gA_0+A_1\log g+B'+O(1/g)}$$

$$A_1 = \frac{1}{2}\beta^2\,,$$

$$B' = \frac{1}{2}\int_0^\infty dk\left[k\big(\widetilde{\psi}(k)\big)^2 - \beta^2\frac{1-\mathrm{e}^{-k}}{k}\right] + \frac{\beta}{2}\log(2\pi) - \log G(1+\beta)\,,$$

Power suppressed $O(1/g)$ corrections are determined using the *method of differential equations*

Method of differential equations

A powerful method for computing correlators in integrable models [Its,Izergin,Korepin,Slavnov]

Logarithmic derivative of the octagon

$$U(g,y,\xi) = -2g\partial_g \log\det(1-\mathbf{K}_\chi)$$

Satisfies the system of *exact* integro-differential equations [Belitsky,GK]

$$\partial_y U = \int_0^\infty dx\, Q^2(x)\partial_y\chi(x)\,,$$
$$g\partial_g U = -2\int_0^\infty dx\, Q^2(x) x\partial_x\chi(x)\,,$$
$$\partial_\xi U = 8g^2\xi\int_0^\infty dx\, Q^2(x)\partial_x\chi(x) + \frac{\sinh\xi}{\cosh y+\cosh\xi}\int_0^\infty dx\, Q^2(x)\chi(x)$$
$$\chi = \frac{\cosh y+\cosh\xi}{\cosh y+\cosh(\sqrt{x/(2g)^2+\xi^2})}$$

Auxiliary function $Q(x)$ obeys a PDE

$$(g\partial_g + 2x\partial_x)^2\, Q(x) + (x - g\partial_g U + U)\, Q(x) = 0$$

For $\chi(x) = \theta(1-x)$ reduces to Painleve V equation

Tracy-Widom distribution at strong coupling

- Strong coupling expansion:

$$\log\det(1-\mathbf{K}_\chi) = \underbrace{-gA_0 + A_1\,\log g + B}_{\text{SAK formula}} + \frac{A_2}{4g} + \frac{A_3}{12g^2} + \frac{A_4}{24g^3} + \dots$$

- Exact expressions for the expansion coefficients

$$A_0 = 2I_0\,, \qquad A_1 = \frac{1}{2}\,,$$

$$A_2 = -\frac{3I_1}{4}, \qquad A_3 = -\frac{9I_1^2}{16},$$

$$A_4 = -\frac{3I_1^3}{8} + \frac{15I_2}{128}, \qquad A_5 = -\frac{15I_1^4}{64} + \frac{75I_1I_2}{256}, \qquad \dots$$

Dependence on symbol (=choice of observable) enters through a *profile function*

$$I_n(y,\xi) = \int_0^\infty \frac{dz}{\pi}\,\frac{\left(z^{-1}\partial_z\right)^n}{(2n-1)!!}\,z\partial_z\log\left(1-\chi(z)\right)$$

A_1 is universal, generated by the Fisher-Hartwig singularity

B is the Dyson-Widom constant

Application to free energy in $\mathcal{N}=2$ SYM

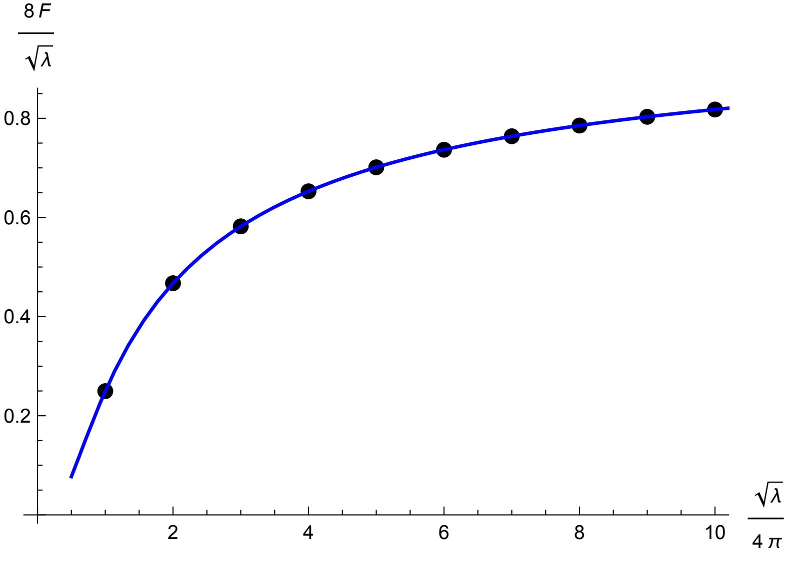
$$F=\frac{1}{2}\log\det(1-\mathbf{K}_\chi)\,,\qquad \chi(x)=-\frac{1}{\sinh^2(x/2)}$$

✔ Weak coupling expansion in $\widehat{\lambda}=\lambda/(8\pi^2)$

$$F=5\zeta_5\widehat{\lambda}^3-\frac{105}{2}\zeta_7\widehat{\lambda}^4+441\zeta_9\widehat{\lambda}^5-(25\zeta_5^2+3465\zeta_{11})\widehat{\lambda}^6+\left(525\zeta_5\zeta_7+\frac{212355\zeta_{13}}{8}\right)\widehat{\lambda}^7+\dots$$

✔ Strong coupling expansion in $1/\sqrt{\lambda}$

$$F=\frac{1}{8}\lambda^{1/2}-\frac{3}{8}\log\lambda-3\log \mathsf{A}+\frac{1}{4}-\frac{11}{12}\log 2+\frac{3}{4}\log(4\pi)$$
$$+\frac{3}{32}\log\left(\lambda'/\lambda\right)-\frac{15\zeta_3}{64\,\lambda'^{3/2}}-\frac{945\zeta_5}{512\,\lambda'^{5/2}}-\frac{765\zeta_3^2}{128\,\lambda'^3}+\dots$$
$$-\frac{i}{4}\lambda^{1/2}\,\mathrm{e}^{-\sqrt{\lambda}}\left(1+O(\lambda^{-1/2})\right),\qquad \lambda'^{1/2}=\lambda^{1/2}-4\log 2\,.$$



Series in $1/\sqrt{\lambda}$ has factorially growing coefficients

Borel singularities are in one-to-one correspondence with nonperturbative corrections

Conclusions and open questions

Various quantities (free energy, correlation functions, Wilson loop) in *different* 4d super Yang-Mills theories are expressed in terms of the *same* (temperature dependent) Tracy-Widom distribution

This relation is powerful enough to predict the dependence on 't Hooft coupling

- ✔ Who ordered this universality?
- ✔ What is the reason why the Bessel kernel appears in all cases?
- ✔ Nonplanar corrections?

Thank you for your attention!