NNLO Corrections to Bhabha Scattering

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In collaboration with: A. Ferroglia and A. A. Penin

Plan of the Talk

- Introduction
- NNLO QED corrections:
  - Photonic Contributions
  - Electron-Loop Contributions
  - Heavy-Fermion-Loop Contributions
- Numerics
- Summary

Why Bhabha Scattering?

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- Bhabha Scattering is a fundamental process for e^+e^- collider physics because it is chosen for the precise evaluation of the Luminosity:

$$L = \frac{N}{\sigma_{th}}$$

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- Bhabha scattering is a process with a large cross section and it is QED dominated $\Rightarrow$
  - it allows precise experimental measurements (large statistics);
  - it allows precise theoretical calculation of the cross section $\Longrightarrow$ radiative corrections under control at the level of NNLO.

Small and Large Angle Bhabha Scattering

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- Small-Angle

  SABS is important for high-energy accelerators, as for instance LEP or the future ILC.

  For LEP, luminometers were located between 1.4° and 2.9°

  For ILC, they will be located between 0.7° and 2.3°

  The small angle region makes in such a way that the weak contribution can be neglected (the Born with a Z^0 exchanged is already at the level of 0.1%)

Small and Large Angle Bhabha Scattering

In the small-angle limit, the CS is determined only by the Dirac form factor

$$\frac{d\sigma_2^{(\mathrm{ph})}}{d\sigma_0} \stackrel{\theta\to 0}{=} 6\left(F_1^{(1l)}(t)\right)^2 + 4F_1^{(2l)}(t)$$

$$= \frac{1}{(1-\xi+\xi^2)^2}\left\{\ln^2\left(\frac{s}{m^2}\right)\left[\frac{9}{2}+2\ln^2\left(\frac{4\omega^2}{s}\right)\right.\right.$$

$$\left.+6\ln\left(\frac{4\omega^2}{s}\right)\right] + \ln\left(\frac{s}{m^2}\right)\left[6\zeta(3)-3\zeta(2)-\frac{93}{8}+9\ln(\xi)\right.$$

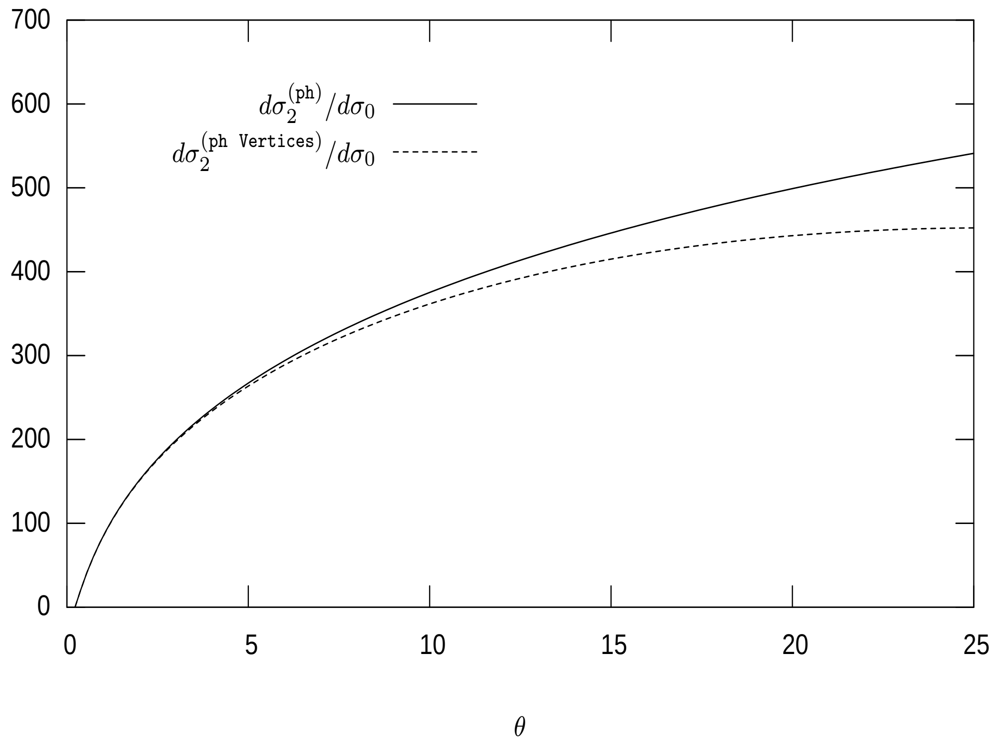
$$\left.-4\ln^2\left(\frac{4\omega^2}{s}\right)[1-\ln(\xi)] - 2\ln\left(\frac{4\omega^2}{s}\right)[7-6\ln(\xi)]\right]$$

$$-9\zeta(3)+\frac{51}{4}\zeta(2)-12\zeta(2)\ln(2)-\frac{32}{5}\zeta^2(2)$$

$$+\frac{27}{2}+6\zeta(3)\ln(\xi)-3\zeta(2)\ln(\xi)-\frac{93}{8}\ln(\xi)+\frac{9}{2}\ln(\xi)^2$$

$$+\ln^2\left(\frac{4\omega^2}{s}\right)\left[2-4\ln(\xi)+2\ln^2(\xi)\right]$$

$$\left.+\ln\left(\frac{4\omega^2}{s}\right)\left[8-14\ln(\xi)+6\ln^2(\xi)\right]+\mathcal{O}(\xi)\right\}$$



$E = 0.5$ GeV and $\omega = E$

Units of $\alpha^2/\pi^2 \sim 5.4\cdot 10^{-6}$

V. S. Fadin, E. A. Kuraev, L. Trentadue, L. N. Lipatov and N. P. Merenkov, Phys. Atom. Nucl. 56 (1993) 1537 [Yad. Fiz. 56N11 (1993) 145]

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- Large-Angle

  LABS is important for low-energy accelerators (meson factories), as for instance DAΦNE.

  The KLOE experiment has luminometers located between 55° and 125°

  The small energy makes in such a way that the weak contributions also in this case are negligible. At 10 GeV they are at the level of 0.1%

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  - Virtual corr. to the cross section with $m = 0$ (Bern-Dixon-Ghinkulov ’00)
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  - $N_F = 1$ with $m_e \neq 0$ (B.-Ferroglia-Mastrolia-Remiddi-van der Bij ’04-’05)
  - Constant term of photonic corrections not suppressed by the ratio m^2/s (Penin ’05)
  - HF contr. in the small-m_f limit (Actis-Czakon-Gluza-Riemann ’07, Becher-Melnikov ’07)
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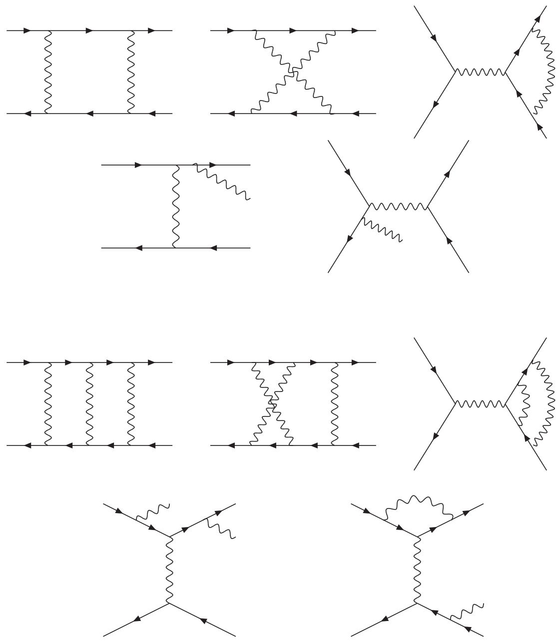

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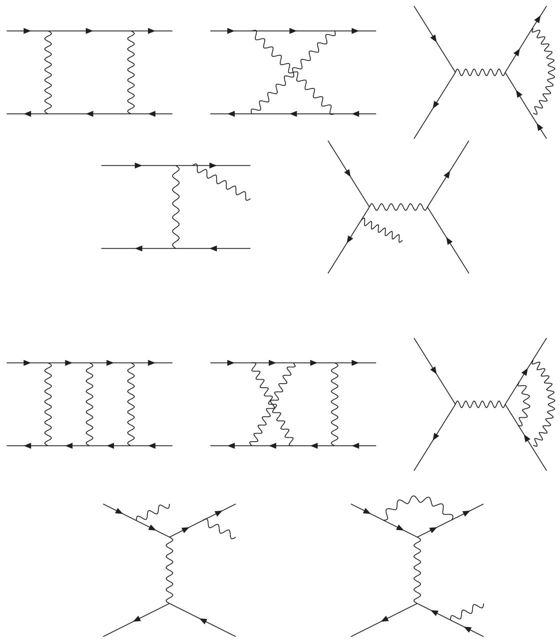

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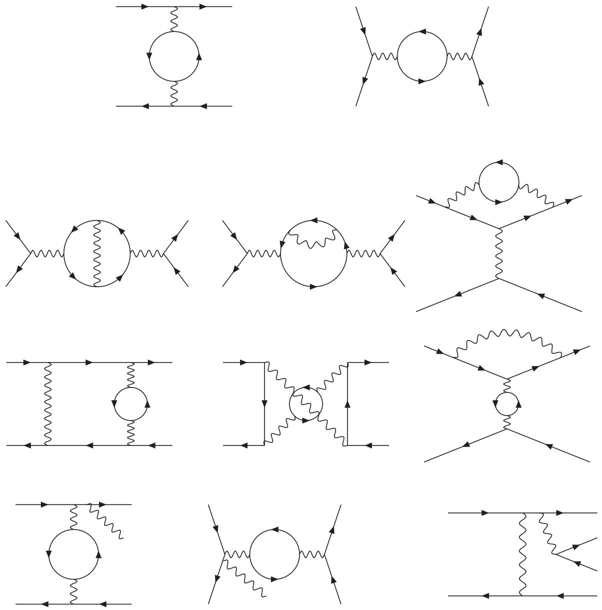

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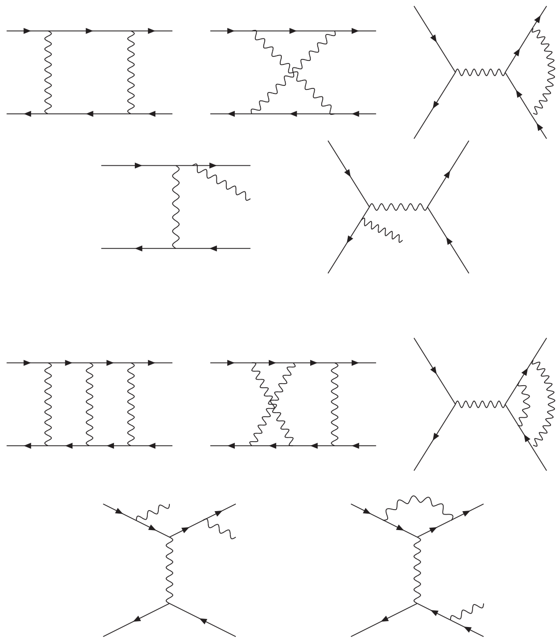

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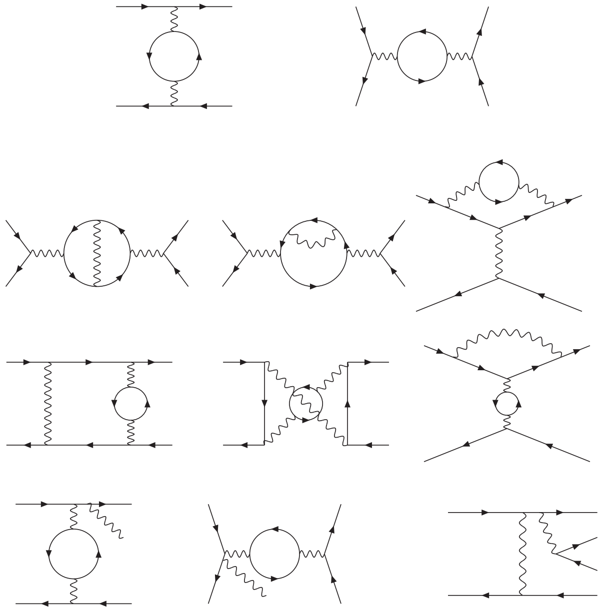

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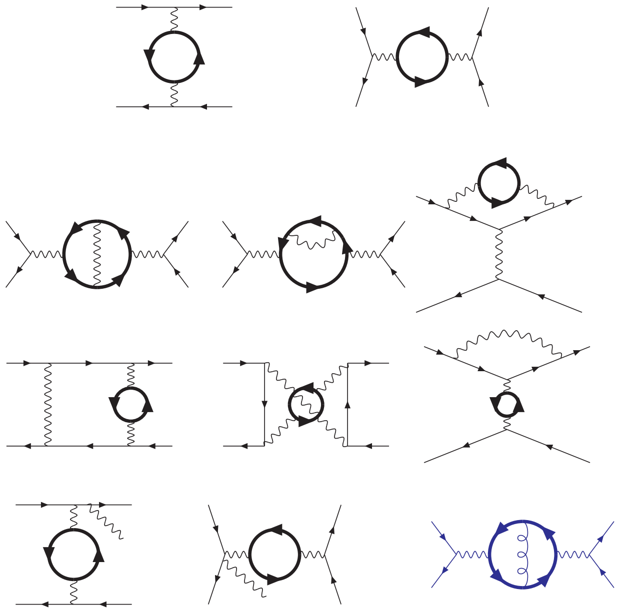

Becher-Melnikov '07
Actis-Czakon-Gluza-Riemann '07
B.-Ferroglia-Penin '07

Mass Hierarchy

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The physical problem is characterized by a well defined mass hierarchy

- Low-Energy Acc.

$$m_e^2 \ll m_\mu^2 < m_c^2 \sim m_\tau^2 \sim m_b^2 \sim s,t,u \ll m_t^2$$

- High-Energy Acc.

$$m_e^2 \ll m_{light-f}^2 \ll m_t^2 \sim s,t,u$$

The electron mass is always small compared to all the scales in the game

In both cases, therefore, the electron contribution provides the biggest fermionic contribution, followed by the muon

This hierarchy allows to calculate radiative corrections neglecting the mass of the electron, or, better, keeping the mass of the electron only in the log-enhanced terms, as a regulator for the collinear divergences

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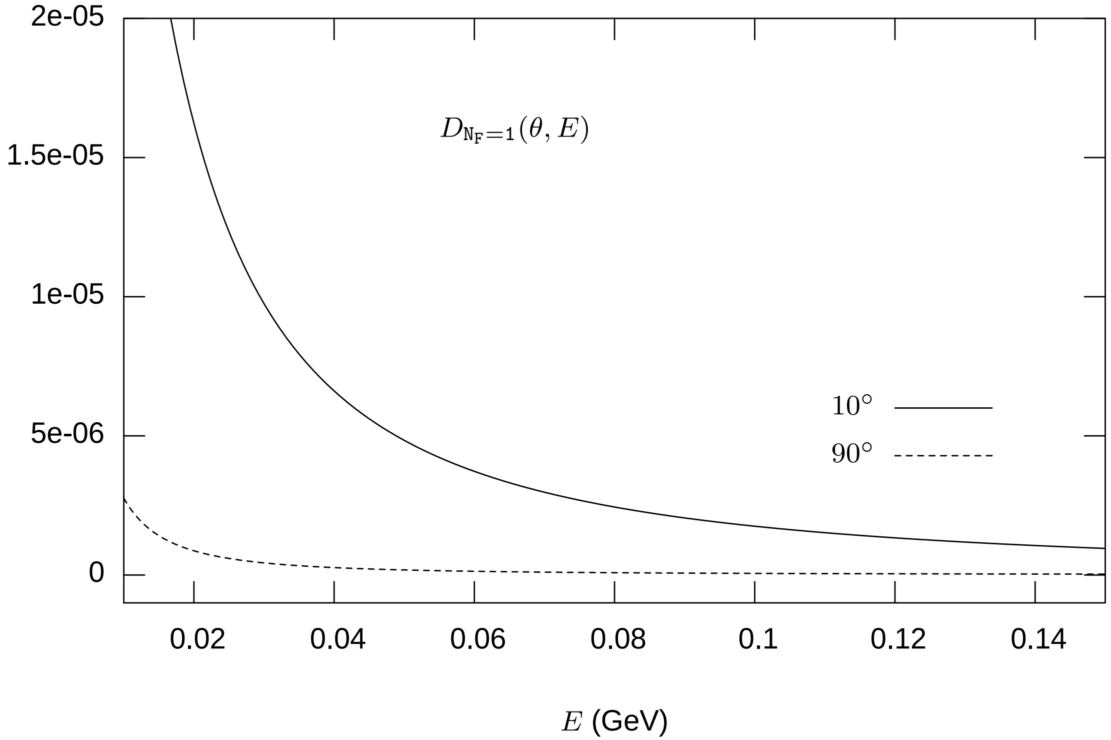

$$D_{\mathrm{N_F}=1}(\theta, E) = \left(\frac{\alpha}{\pi}\right)^2 \left| \left(\frac{d\sigma_2^{(\mathrm{N_F}=1)}}{d\Omega} - \left. \frac{d\sigma_2^{(\mathrm{N_F}=1)}}{d\Omega} \right|_L \right) \right| \left(\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1}{d\Omega} \right)^{-1}$$

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R. B. and A. Ferroglia, Phys. Rev. D 72, 056004 (2005)

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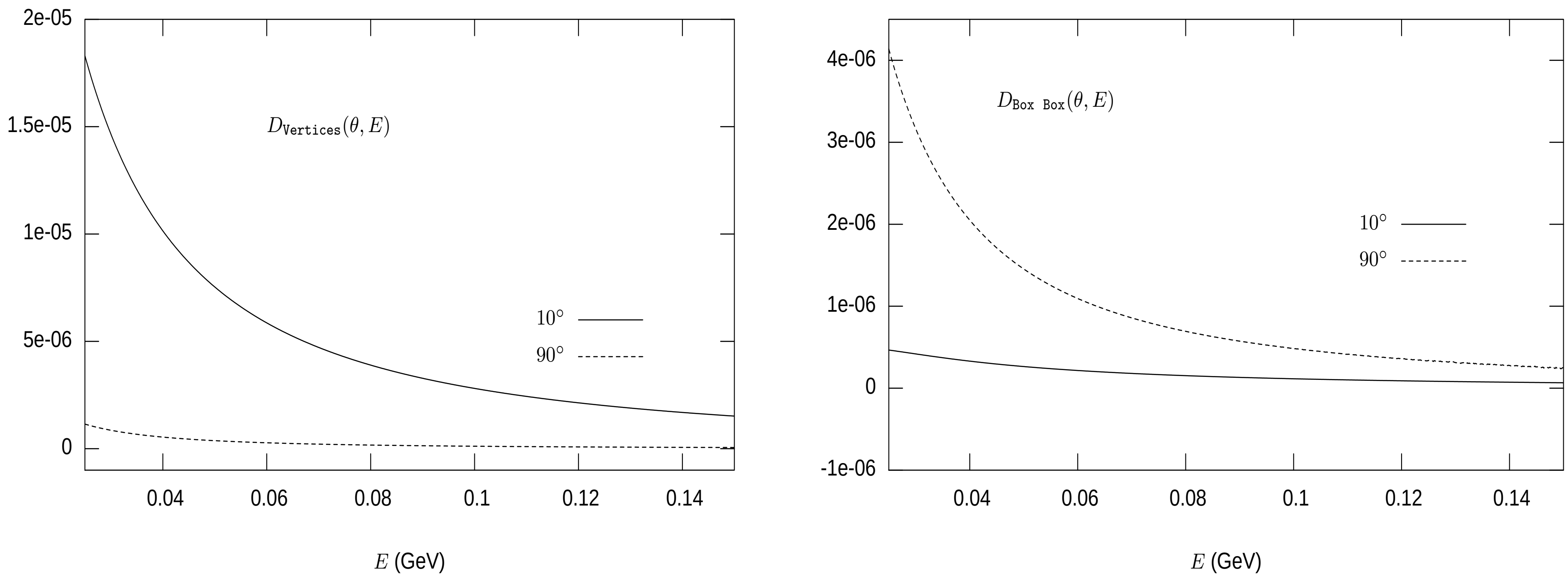

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At the NNLO the Cross Section has the following form:

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)}(\xi)\ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)}(\xi)\ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)}(\xi) + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

where

$$\xi = \frac{1-\cos\theta}{2}$$

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NOTE: this approximation is not valid in the almost-forward ($|t| < m^2$) and in the almost-backward ($|u| < m^2$) directions, where terms of order m^2/t and m^2/u become important

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However, alredy at 1° the terms of order m^2/t are totally negligible.

The Photonic Contribution

$$\frac{d\sigma_{phot}}{d\sigma_0} = \delta^{(2)}_{phot,2} \ln^2\left(\frac{s}{m^2}\right) + \delta^{(2)}_{phot,1} \ln\left(\frac{s}{m^2}\right) + \delta^{(2)}_{phot,0}$$

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$\delta^{(2)}_{phot,2}$ known since **Arbuzov-Kuraev-Shaikhatdenov '98**

$\delta^{(2)}_{phot,1}$ known since **Glover-Tausk-van der Bij '01**

$\delta^{(2)}_{phot,1}$ known since **Penin '05**

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- Reconstruction from massless CS

For a generic QED/QCD process without closed fermion loops

$$\mathcal{M}^{(m\neq 0)} = \prod_{i\in\{\texttt{all legs}\}} Z_i^{\frac{1}{2}}(m,\epsilon)\mathcal{M}^{(m=0)}$$

where Z is the ratio between the massive and massless Dirac form factor

$$F^{(m\neq 0)}(Q^2) = Z(m,\epsilon)\, F^{(m=0)}(Q^2) + \mathcal{O}(m^2/Q^2)$$

Therefore, starting from the totally massless result of Bern-Dixon-Ghinkulov ’00, one can reconstruct the photonic cross section where the collinear divergences are regulated with the mass of the electron.

A. Mitov and S. Moch, JHEP 0705 (2007) 001.
T. Becher and K. Melnikov, JHEP 0706 (2007) 084.

The Electron-Loop Contribution

$$\frac{d\sigma_{N_F=1}}{d\sigma_0} = \delta^{(2)}_{N_F=1,3}\ln^3\left(\frac{s}{m^2}\right) + \delta^{(2)}_{N_F=1,2}\ln^2\left(\frac{s}{m^2}\right) + \delta^{(2)}_{N_F=1,1}\ln\left(\frac{s}{m^2}\right) + \delta^{(2)}_{N_F=1,0}$$

where:

$$\delta^{(2)}_{N_F=1,3} = \frac{1}{(1-\xi+\xi^2)^2}\Big[-\frac{1}{9}+\frac{2}{9}\xi-\frac{1}{3}\xi^2+\frac{2}{9}\xi^3-\frac{1}{9}\xi^4\Big]$$

$$\delta^{(2)}_{N_F=1,2} = \frac{1}{(1-\xi+\xi^2)^2}\Big[\ln\left(\frac{4w^2}{s}\right)\Big(-\frac{4}{3}+\frac{8}{3}\xi-4\xi^2+\frac{8}{3}\xi^3-\frac{4}{3}\xi^4\Big)-\Big(\frac{17}{18}-\frac{17}{9}\xi+\frac{17}{6}\xi^2-\frac{17}{9}\xi^3+\frac{17}{18}\xi^4\Big)$$
$$-\Big(\frac{1}{3}-\frac{2}{3}\xi+\xi^2-\frac{2}{3}\xi^3+\frac{1}{3}\xi^4\Big)\ln(1-\xi)-\Big(\frac{1}{3}-\frac{1}{3}\xi+\frac{1}{3}\xi^3-\frac{1}{3}\xi^4\Big)\ln(\xi)\Big]$$

$$\delta^{(2)}_{N_F=1,1} = \frac{1}{(1-\xi+\xi^2)^2}\Big[\ln\left(\frac{4w^2}{s}\right)\Big[\Big(\frac{4}{3}-\frac{8}{3}\xi+4\xi^2-\frac{8}{3}\xi^3+\frac{4}{3}\xi^4\Big)\ln(1-\xi)+\Big(\frac{32}{9}-\frac{64}{9}\xi+\frac{32}{3}\xi^2-\frac{64}{9}\xi^3$$
$$+\frac{32}{9}\xi^4\Big)-\Big(\frac{8}{3}-\frac{14}{3}\xi+6\xi^2-\frac{10}{3}\xi^3+\frac{4}{3}\xi^4\Big)\ln(\xi)\Big]\cdots\Big]$$

$$\delta^{(2)}_{N_F=1,0} = \frac{1}{(1-\xi+\xi^2)^2}\Big[\ln\left(\frac{4w^2}{s}\right)\Big[\Big(-\frac{20}{9}+\frac{40}{9}\xi-\frac{20}{3}\xi^2+\frac{40}{9}\xi^3-\frac{20}{9}\xi^4\Big)\ln(1-\xi)-\Big(\frac{20}{9}-\frac{40}{9}\xi+\frac{20}{3}\xi^2$$
$$-\frac{40}{9}\xi^3+\frac{20}{9}\xi^4\Big)\cdots\Big]$$

In agreement with Becher-Melnikov '07 and Actis-Czakon-Gluza-Riemann '07

A. B. Arbuzov, E. A. Kuraev, N. P. Merenkov and L. Trentadue, Phys. Atom. Nucl. 60 (1997) 591 [Yad. Fiz. 60N4 (1997) 673]

R. B., A. Ferroglia, P. Mastrolia, E. Remiddi and J. J. van der Bij, Nucl. Phys. B716 (2005) 280

Soft-Pair Production

$$\frac{d\sigma_{Pair}}{d\sigma_0} = \delta^{(2)}_{Pair,3}\ln^3\left(\frac{s}{m^2}\right) + \delta^{(2)}_{Pair,2}\ln^2\left(\frac{s}{m^2}\right) + \delta^{(2)}_{Pair,1}\ln\left(\frac{s}{m^2}\right) + \delta^{(2)}_{Pair,0}$$

where:

$$\delta^{(2)}_{Pair,3} = \frac{1}{(1-\xi+\xi^2)^2}\left(\frac{1}{9} - \frac{2}{9}\xi + \frac{1}{3}\xi^2 - \frac{2}{9}\xi^3 + \frac{1}{9}\xi^4\right)$$

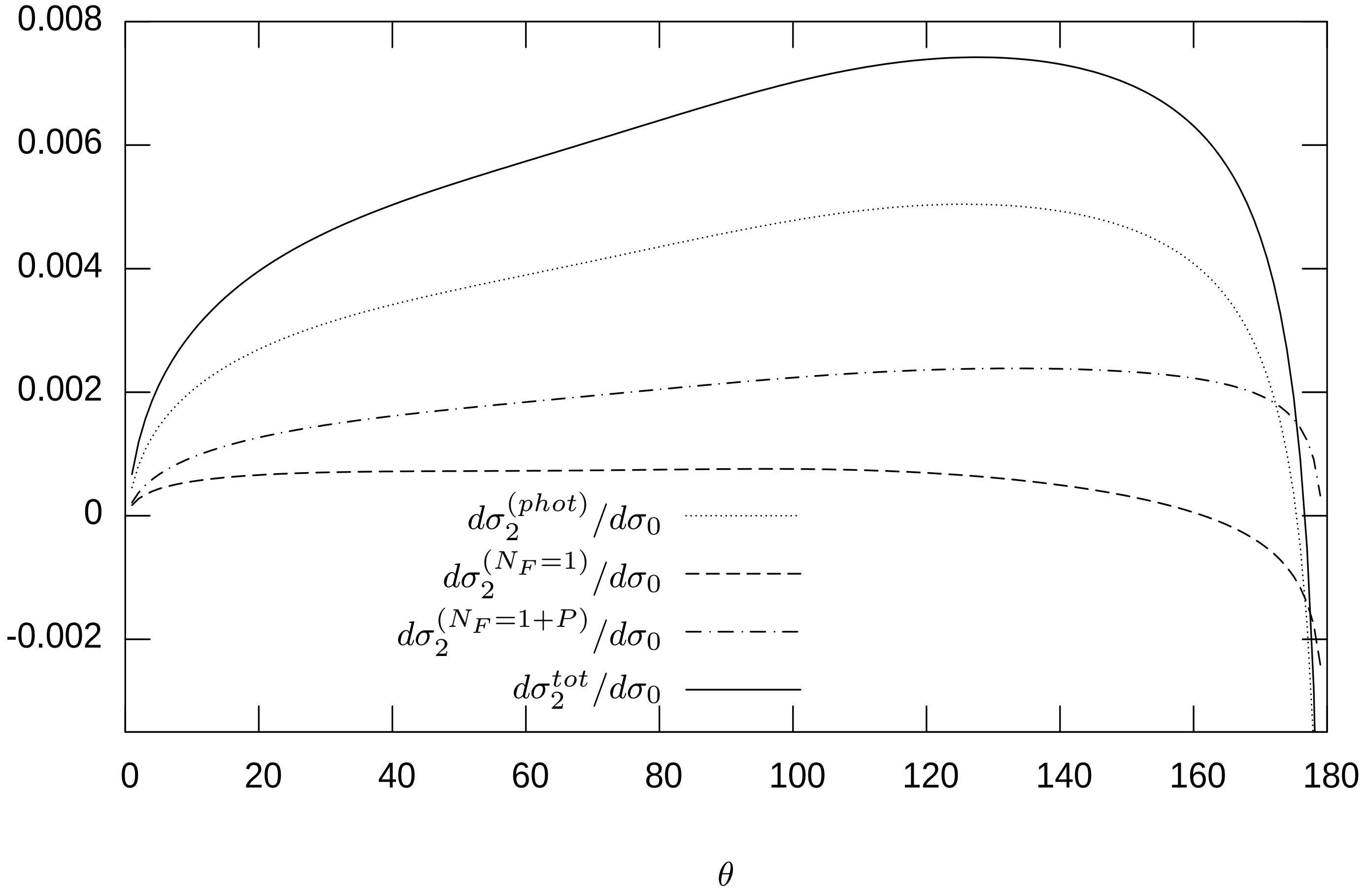
$$\delta^{(2)}_{Pair,2} = \frac{1}{(1-\xi+\xi^2)^2}\Big[\ln\left(\frac{4w^2}{s}\right)\left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right) - \left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right)\ln(1-\xi)$$
$$-\left(\frac{5}{9} - \frac{10}{9}\xi + \frac{5}{3}\xi^2 - \frac{10}{9}\xi^3 + \frac{5}{9}\xi^4\right) + \left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right)\ln(\xi)\Big]$$

$$\delta^{(2)}_{Pair,1} = \frac{1}{(1-\xi+\xi^2)^2}\Big[\ln^2\left(\frac{4w^2}{s}\right)\left(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\right) + \ln\left(\frac{4w^2}{s}\right)\Big[\Big(-\frac{2}{3} + \frac{4}{3}\xi - 2\xi^2 + \frac{4}{3}\xi^3$$
$$-\frac{2}{3}\xi^4\Big)\ln(1-\xi) - \left(\frac{10}{9} - \frac{20}{9}\xi + \frac{10}{3}\xi^2 - \frac{20}{9}\xi^3 + \frac{10}{9}\xi^4\right) + \left(\frac{2}{3} - \frac{4}{3}\xi + \cdots\right]\Big]$$

$$\delta^{(2)}_{Pair,0} = \frac{1}{(1-\xi+\xi^2)^2}\Big[\ln^2\left(\frac{4w^2}{s}\right)\Big[\Big(-\frac{1}{3} + \frac{2}{3}\xi - \xi^2 + \frac{2}{3}\xi^3 - \frac{1}{3}\xi^4\Big)\ln(1-\xi) + \Big(\frac{1}{3} - \frac{2}{3}\xi + \xi^2$$
$$-\frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\Big)\ln(\xi)\Big]\cdots\Big]$$

A. B. Arbuzov, E. A. Kuraev, N. P. Merenkov and L. Trentadue, Phys. Atom. Nucl. 60 (1997) 591 [Yad. Fiz. 60N4 (1997) 673]; Nucl. Phys. B474 (1996) 271.

Photonic and Electron-Loop Corrections



$E = 0.5$ GeV and $\Omega = \omega = E$

Heavy-Fermion Contribution

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$$m_e^2 \ll m_f^2 \ll s, t, u$$

T. Becher and K. Melnikov, JHEP 0706 (2007) 084.
S. Actis, M. Czakon, J. Gluza and T. Riemann, Nucl. Phys. B 786 (2007) 26.

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- but it is also possible to keep the full dependence on the heavy-fermion mass

$$\boxed{m_e^2 \ll m_f^2 \sim s, t, u}$$

R. B., A. Ferroglia, A. A. Penin, Phys. Rev. Lett. 100 (2008) 131601; JHEP 0802 (2008) 080.

Heavy-Fermion Contribution: small-m_f

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- Reconstruction from massless CS

  If we include closed fermion loops, the formula changes a bit

$$\mathcal{M}^{(m\neq 0)} = Z^2(m,\epsilon)\mathcal{M}^{(m=0)}S(s,t,u,m_f,\epsilon)$$

  where Z is the ratio between the massive and massless Dirac form factor and S is the "soft" function, calculated in SCET.

  Again, from the totally massless result of Bern-Dixon-Ghinkulov '00, one can reconstruct the N_F part of the CS, in the limit $m_e \ll m_f \ll s,t,u$

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  T. Becher and K. Melnikov, JHEP 0706 (2007) 084.

- **Diagrammatic Calculation**

  - reduction to the MIs with the Laporta algorithm
  - calculation of the MIs directly in the $m_e/s \to 0$ limit with Mellin-Barnes

  S. Actis, M. Czakon, J. Gluza and T. Riemann, Nucl. Phys. B 786 (2007) 26.

Heavy-Fermion Contribution: small-m_f

The constraint $\boxed{m_e^2 \ll m_f^2 \ll s, t, u}$ is well verified for instance for leptons at high-energy accelerators (ILC) and for the muon at meson factories.

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- Contributions coming from a tau loop at KLOE energies ($m_\tau \sim \sqrt{s}$)
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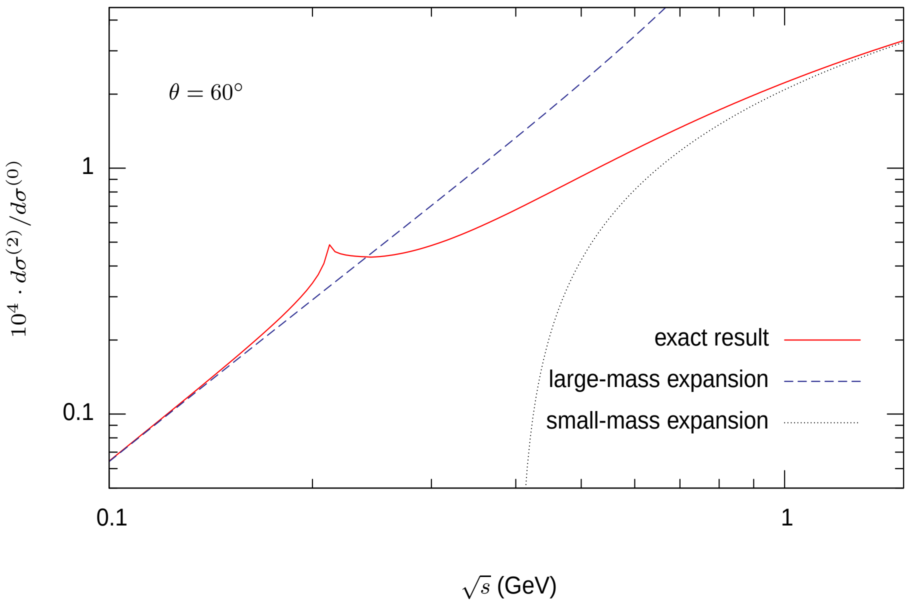

Two-loop corrections to the Bhabha scattering differential cross section at $\theta = 60^\circ$ due to a closed loop of muon. The solid line represents the exact result. The dashed and dotted lines represent the results of the large-mass expansion and small-mass expansion, respectively.

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A solution with the full dependence on m_f is desirable

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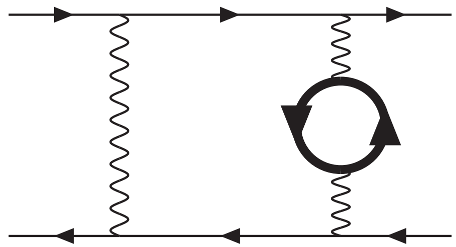
- Each two-loop heavy-fermion box diagram in a physical gauge is collinear-safe! (Frenkel-Taylor ’76)



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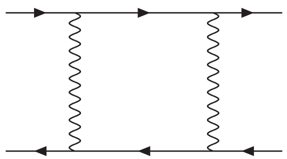

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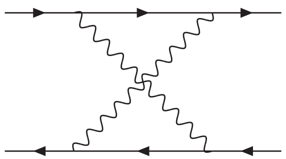

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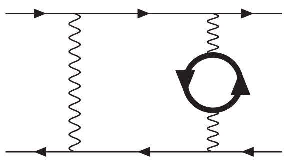

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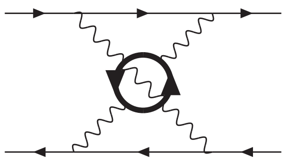

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  Moreover, we can evaluate the boxes in Feynman gauge.

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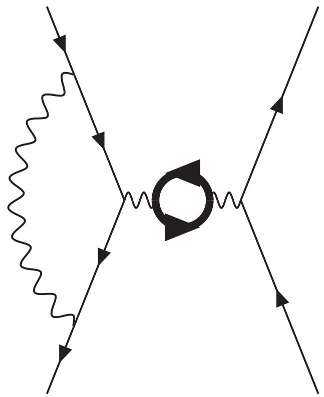
- The collinear divergence comes from the other sets of graphs. In particular it is possible to show that it comes from the reducible ones!



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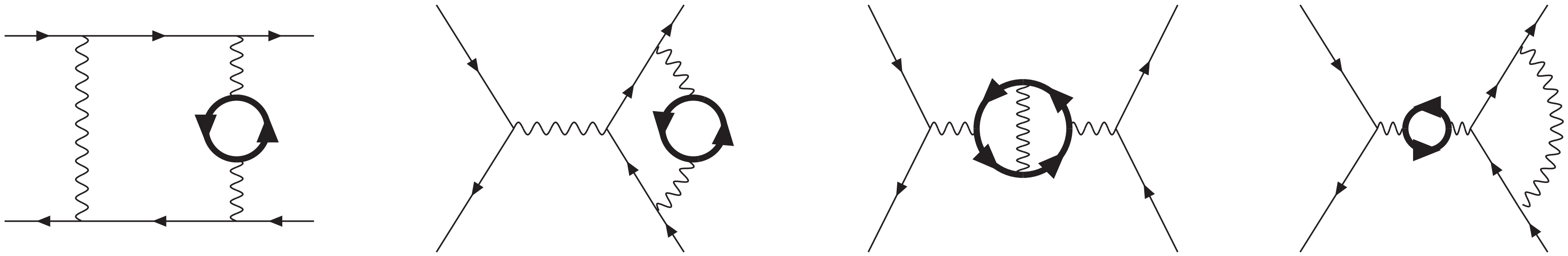
The collinear structure of the cross section is

$$\frac{d\sigma_{N_F>1}}{d\sigma_0} = \delta^{(2)}_{N_F>1,1}(s,t,m_f^2)\,\ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_{N_F>1,0}(s,t,m_f^2)$$

Boxes and two-loop vertices contribute to $\delta^{(2)}_{N_F>1,0}(s,t,m_f^2)$ while the reducible diagrams contribute to $\delta^{(2)}_{N_F>1,1}(s,t,m_f^2)$

Heavy-Fermion Contribution: exact m_f

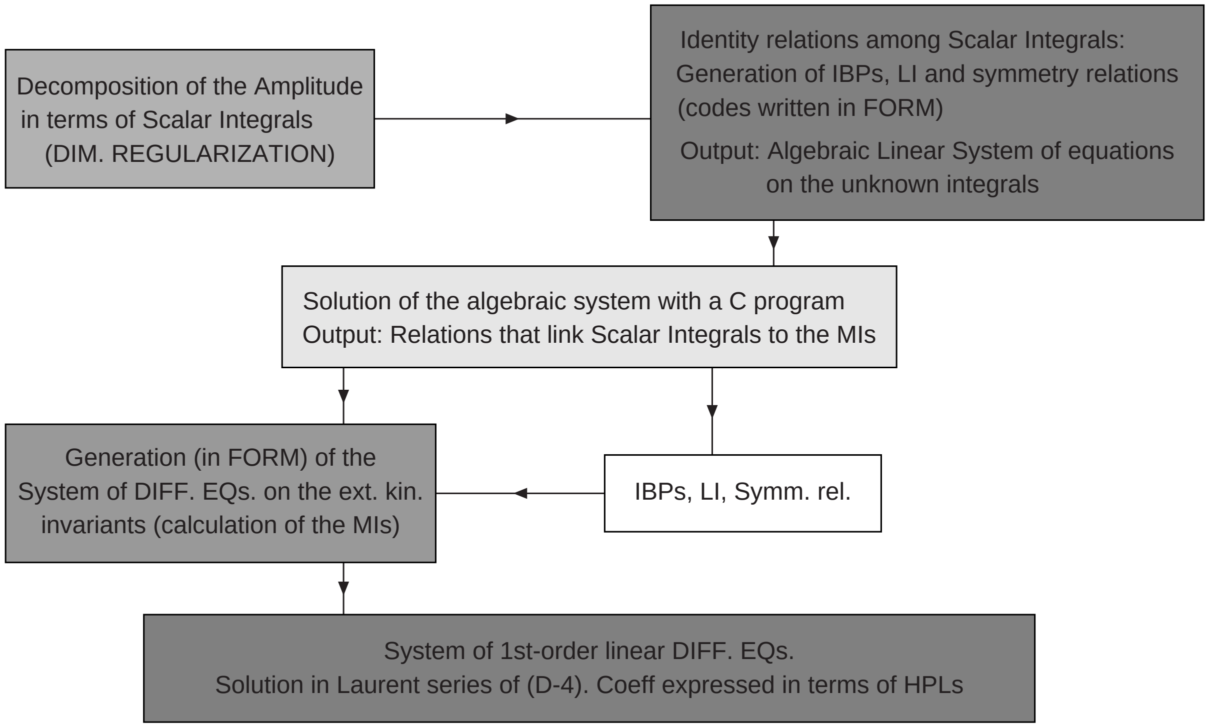


Laporta Algorithm

- Reduction to the MIs

Differential Equations

- Analytic evaluation of the MIs



(B-Ferroglia-Penin '07-'08)

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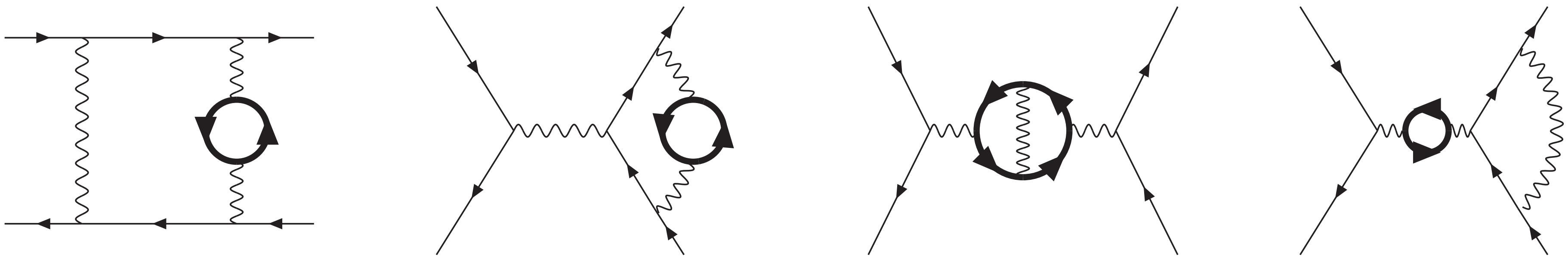

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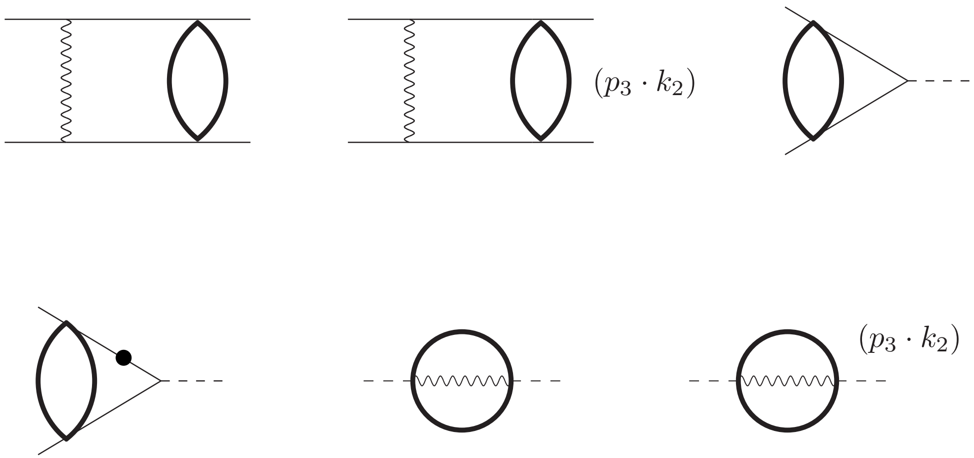

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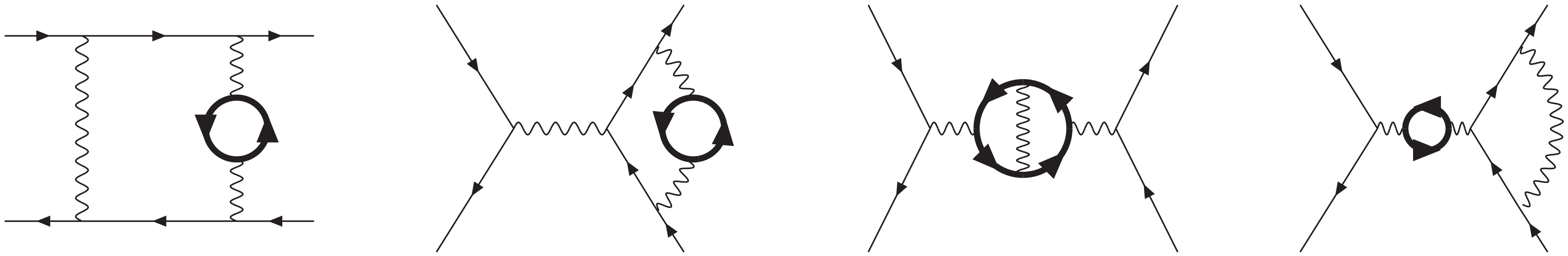

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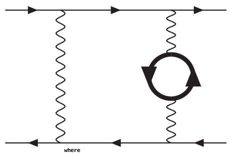


$$= \sum_{i=-3}^{0} M_{1,i}(m_f^2, x; y)\,(D-4)^i + \mathcal{O}(D-4)$$

$$M_{1,-3} = -\frac{1}{2m_f^2 x}$$

$$M_{1,-2} = \frac{1}{4m_f^2 x}\left[2 - G(0;x) - \frac{y+4}{\sqrt{y(y+4)}}G(-\mu;y)\right]$$

$$M_{1,-1} = \frac{1}{8m_f^2 x}\Big[-4 + \zeta(2) + 2G(0;x) - G(0,0;x) + 2G(-\mu,-\mu;y)$$

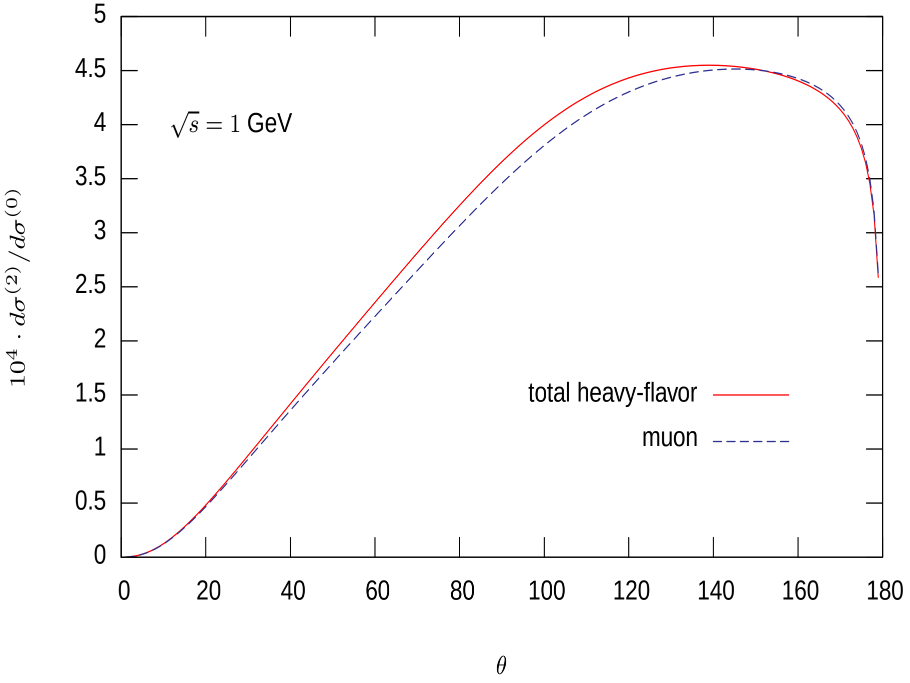
$$+ \frac{y+4}{\sqrt{y(y+4)}}\left[2G(-\mu;y) - 3G(-4,-\mu;y) - G(0;x)G(-\mu;y)\right]\Big]$$

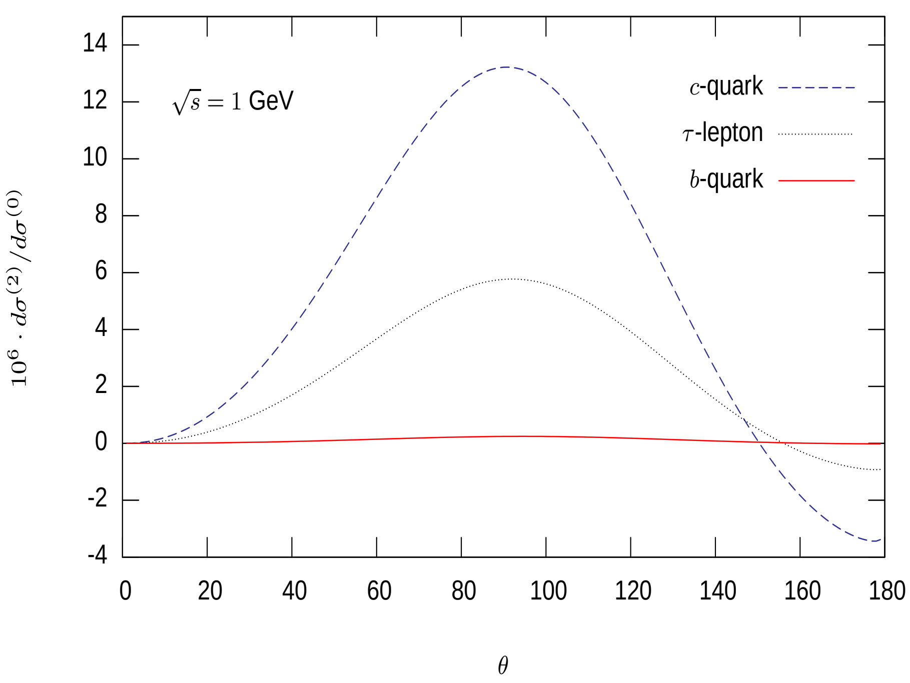
$$M_{1,0} = \cdots$$

Numerical Analysis

Numerical Analysis

$\sqrt{s} = 1$ GeV QED Corrections



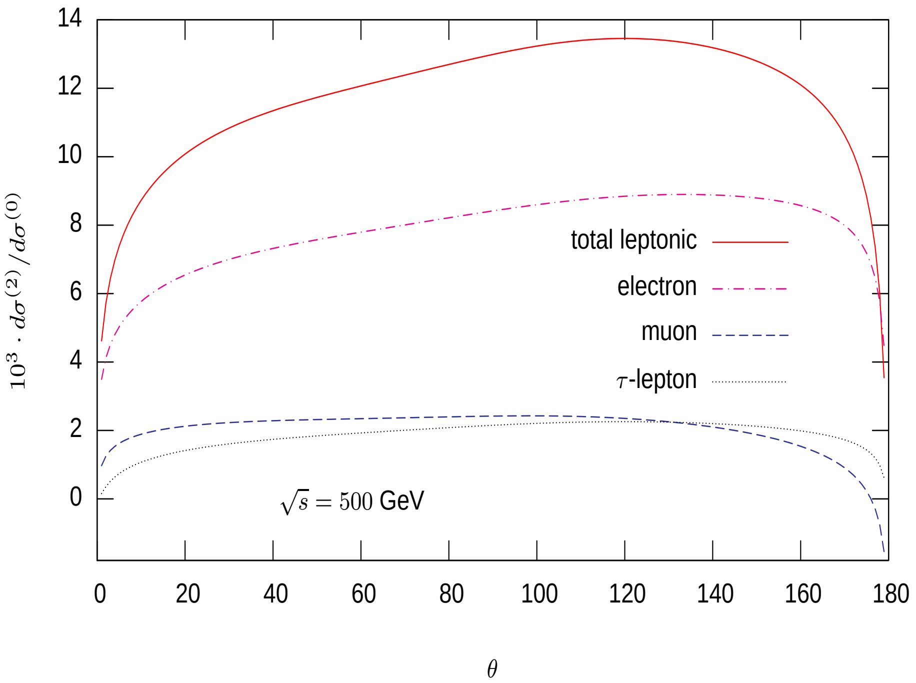


Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 1$ GeV due to a closed loop of muon (dashed line). The solid line represents the sum of the contributions of the muon, τ-lepton, c-quark and b-quark.

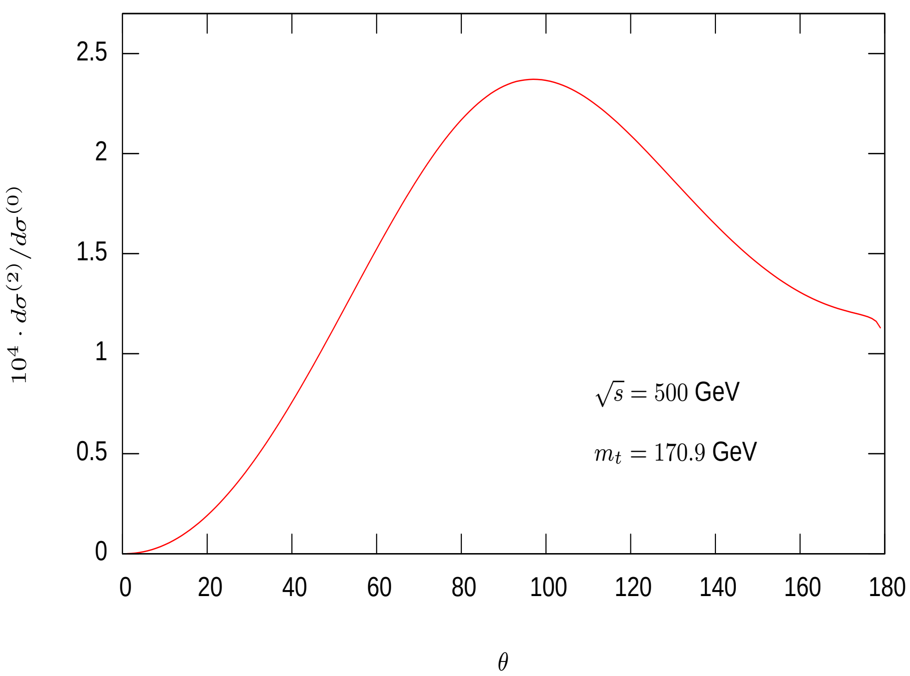
Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 1$ GeV due to a closed loop of τ-lepton (dotted line), c-quark (dashed line) and b-quark (solid line) for $m_c = 1.25$ GeV and $m_b = 4.7$ GeV.

Numerical Analysis

$\sqrt{s} = 500$ GeV   QED Corrections



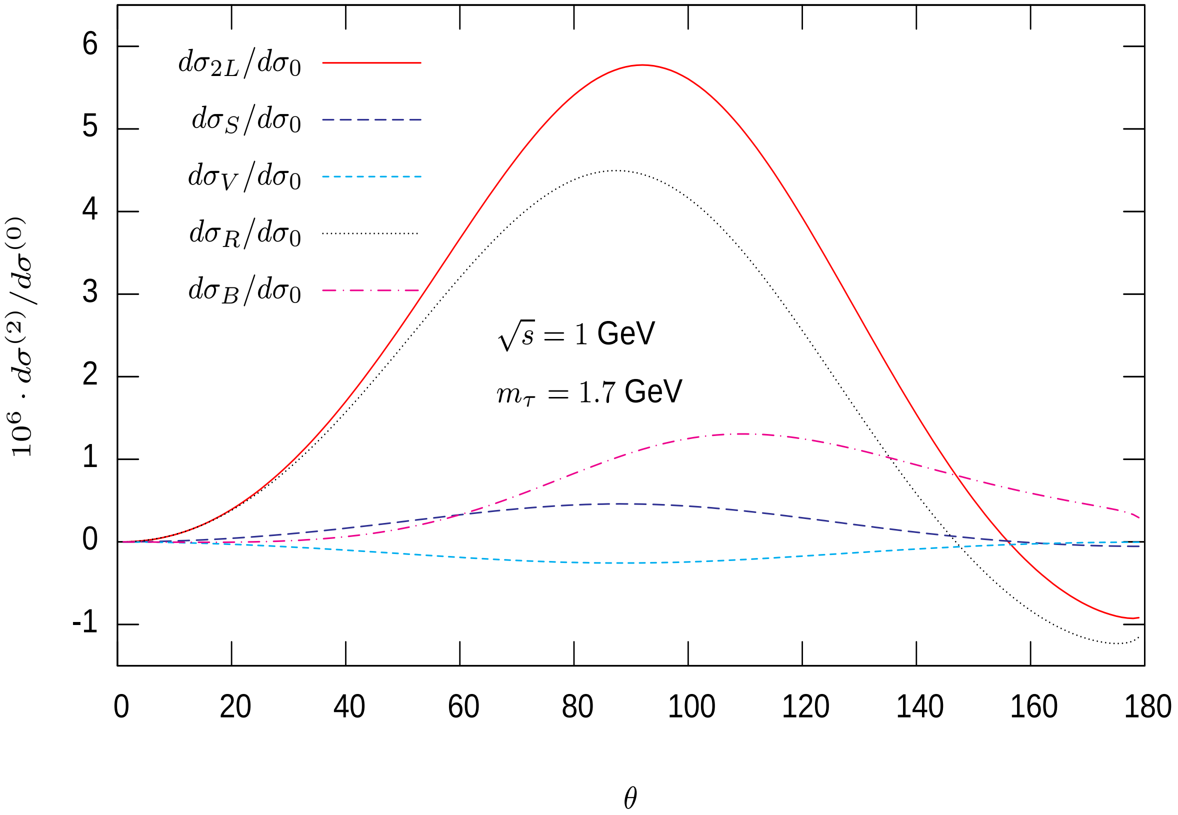
Two-loop leptonic corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 500$ GeV. The dash-dotted line represents the electron contribution including the soft-pair radiation. The dashed and dotted lines represent the contributions of muon and τ-lepton. The solid line is the sum of the three.



Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 500$ GeV due to a closed loop of top quark for $m_t = 170.9$ GeV.

Numerical Analysis

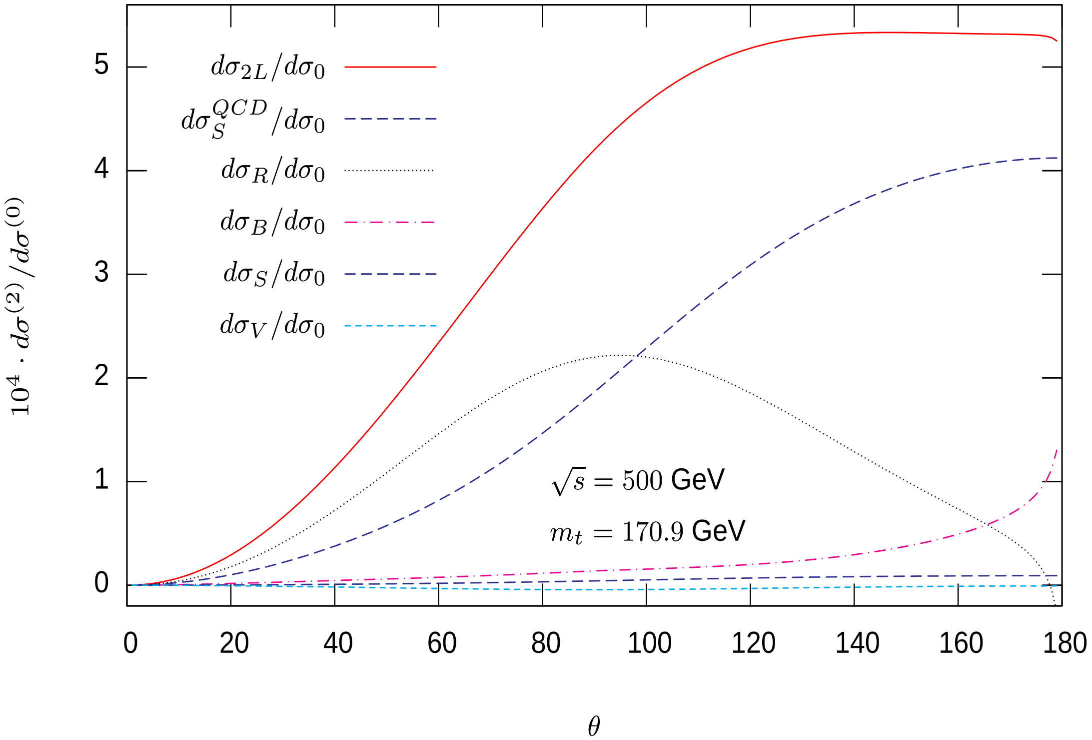
$\sqrt{s} = 500$ GeV Structure of the QED Corrections



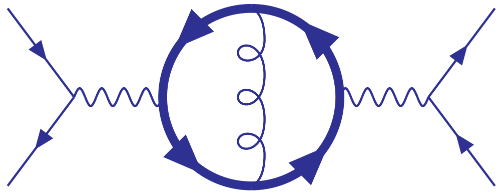
Self-energy ("S"), vertex ("V"), reducible plus one-loop times one-loop ("R"), and box ("B") contributions to the two-loop τ-lepton correction to the differential cross section of Bhabha scattering at $\sqrt{s} = 1$ GeV.

Numerical Analysis

$\sqrt{s} = 500$ GeV Including QCD Corrections



QED and QCD self-energy ("S"), vertex ("V"), reducible plus one-loop times one-loop ("R") and box ("B") contribution to the two-loop top-quark corrections to the differential cross section of Bhabha scattering at $\sqrt{s} = 500$ GeV.



$$\frac{d\sigma_2^V}{d\Omega}\Big|_{(2l,S)}^{QCD} = \frac{C_F}{Q_f^2}\,\frac{\alpha_s(m_f^2)}{\alpha}\,\frac{d\sigma_2^V}{d\Omega}\Big|_{(2l,S)}$$

$C_F = (N_c^2 - 1)/(2N_c)$ is the Casimir operator of the fundamental representation of the $SU(N_c)$ color group, and the strong coupling constant is evaluated at the scale $\mu = m_f$, using the NLO RG equation with the appropriate number of active quarks, starting from the input value $\alpha_S(M_Z) = 0.118$.

Summary and Outlook

- Bhabha scattering is among the “easiest” precesses to be studied in perturbation theory (it is basically “only” QED). This is the reason why its CS is known at the level of NNLO quantum corrections.
- In the past years, several groups contributed to the calculation of the CS. The state of the art includes
  - The complete NLO in the full Electroweak Standard Model
  - The full set of NNLO QED corrections ($\mathcal{O}(\alpha^4)$ and $\mathcal{O}(\alpha^3 \alpha_S)$) and hadronic effects for the process $e^+e^- \to e^+e^-$
- These corrections have been included already in several Monte Carlos that provide, at the moment, a very good precision. In the case of LABS at DAΦNE energies the CS is known at the level of better than 0.1%. Crucial is the showering.
- In order to complete the knowledge of the Bhabha scattering CS at the level of NNLO perturbative corrections (mostly for esthetic reasons), some pieces are still missing:
  - the soft-pair production contribution is known at the logarithmic level
  - the process $e^+e^- \to e^+e^- + \gamma$ (hard photon) enters in the MCs at the LO
  - the two-loop electroweak logarithmic corrections in four-fermion processes are studied (for instance for $e^+e^- \to \mu^+\mu^-$), but not included yet in the analysis of the Bhabha scattering (in the TeV region they range below the %).