.

Hadronic corrections to Bhabha scattering

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based on work with:

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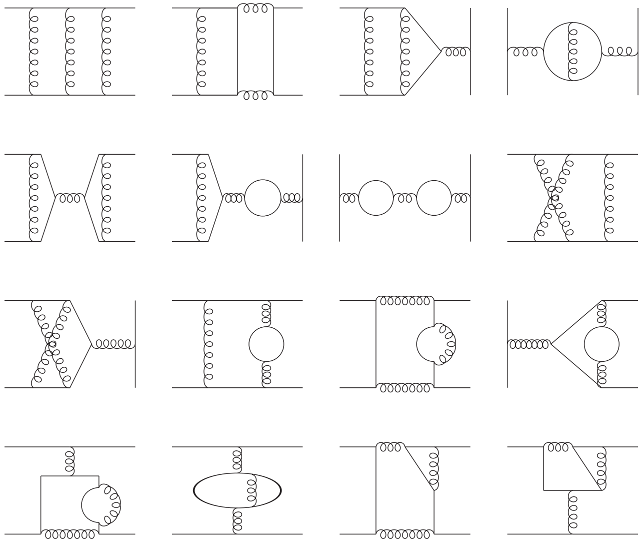
- **Introduction: Two-loop corrections to Bhabha Scattering**
- **Leptonic contributions with** $m_e^2 << m_f^2 << s, t$
  ACGR: NPB 786 (2007) [arXiv:0704.2400]
- **Leptonic contributions with** $m_e^2 << m_f^2, s, t$
  ACGR: APP B38 (2007) [arXiv:0710.5111] $\rightarrow$ **see also talk by Roberto Bonziani**
- **Hadronic contributions**
  ACGR: PRL 100 (2008) [arXiv:0711.53847]
- **Summary**

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.

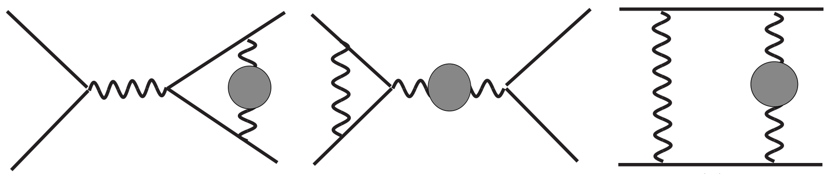


The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Adapted from Z.Bern, LoopFest 2002

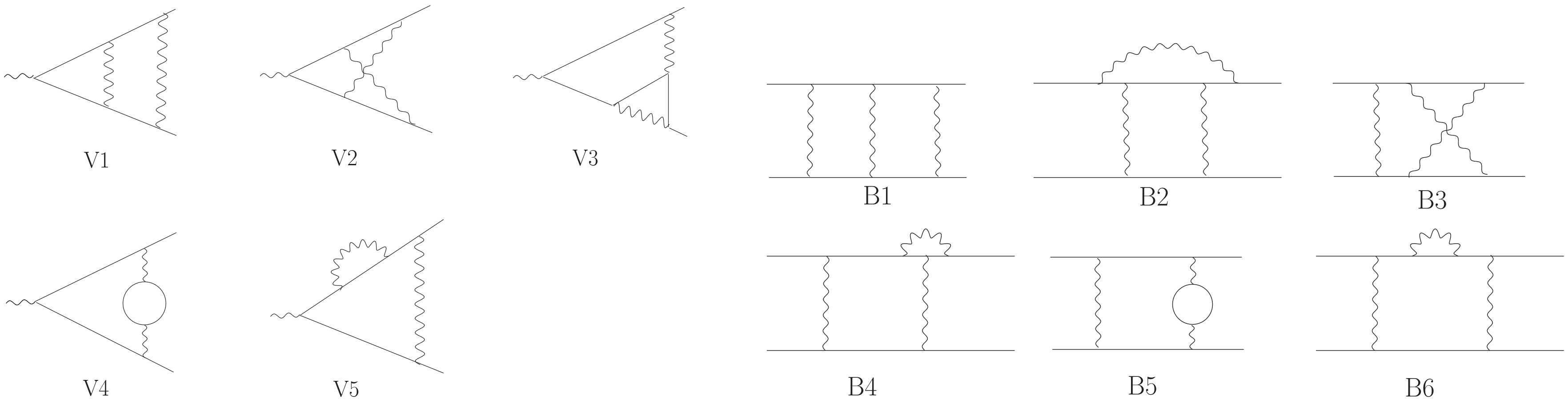
The diagrams with electrons and photons define an $n_f = 1$ problem.

But there are additional ones with heavier fermions.
So we have to investigate an $n_f = 2$ problem
For self-energies starting with 1-loops, and for vertices and boxes starting with 2-loops:



(a) (b) (c)

The technical target: photonic and fermionic $N_f = 1, 2$ topologies

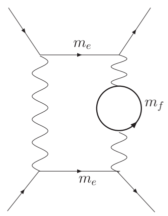


- **The unsolved problem, even in the limit $m_e^2 << s, t$:**
  The non-planar photonic 2-loop boxes B3
- **Finally the photonic corrections were derived from massless case by A. Penin . . .**
- **. . . and the $n_f = 1$ electron loops V4, B5 by R. Bonciani et al., and later also by ACGR**
- **We think we know how to do the non-planar boxes, but it is not easy**

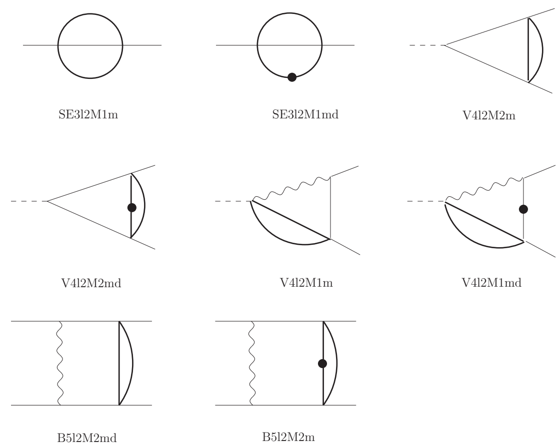
The $n_f = 2$ contributions have been determined in 2007

- Self-energies are not a two-masses-problem
- 2-vertices are known (for $m_e^2 = m_f^2$ and $m_e^2 << m_f^2$): G. Burgers PLB 164 (1885), Kniehl, Krawczyk, Kühn, Stuart PLB 209 (1988)
- What is really new: the 2-boxes with two different fermions involved

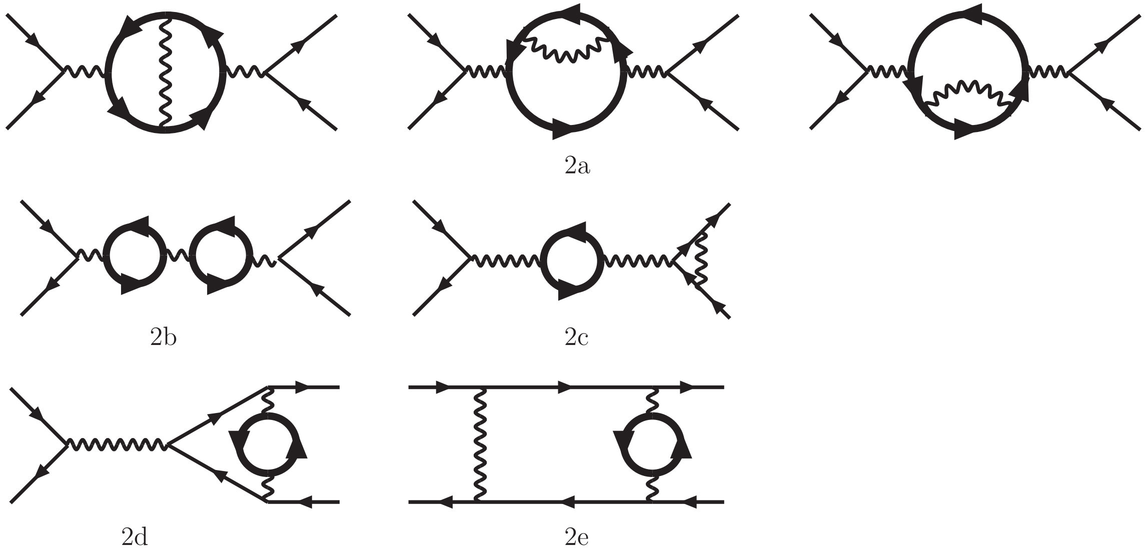
The 8 box-master integrals were identified in ACGR, PRD 71 (2005) [hep-ph/0412164]



$\longrightarrow$

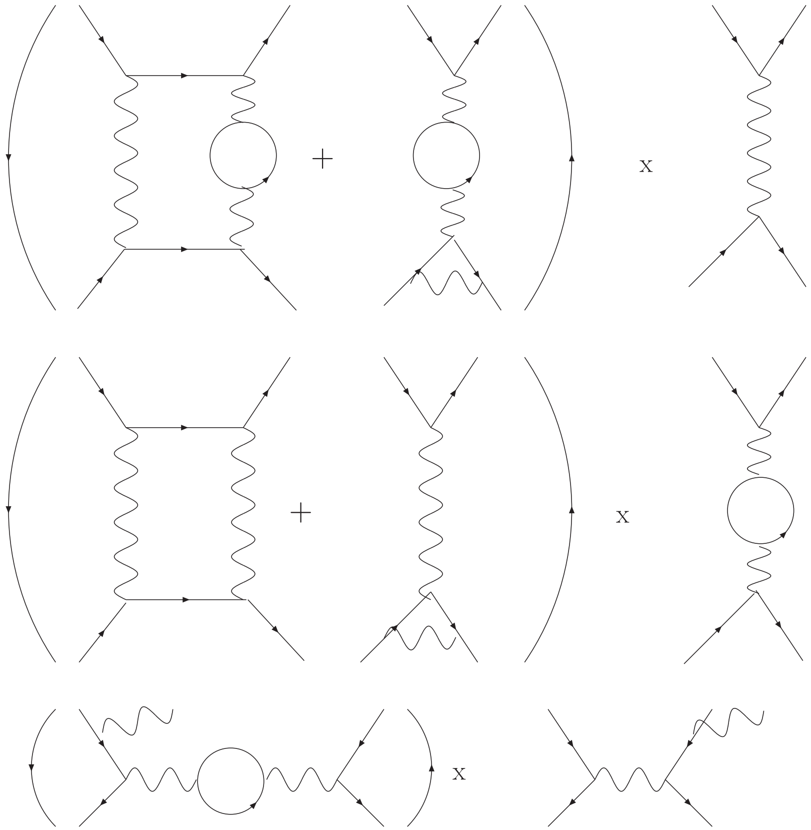


- $m_e^2 << m_f^2 << s, t$: Becher,Melnikov JHEP 6 (2007) and ACGR NPB 786 (2007)
- $m_e^2 << m_f^2, s, t$: ACGR 0710.5111 $->$ APP B38 (2007)
  and Bonciani,Ferroglia,Penin 0710.4775 (2007)
- $m_e^2 << m_{hadrons}^2, s, t$: ACGR 0711.3847 $->$ PRL 100 (2008) THIS TALK



Classes of Bhabha-scattering 2-loop diagrams containing at least one fermion loop.

The 4 direct and 4 crossed fermionic 2-loop box diagrams have to be combined with other diagrams for an IR-finite contribution:



After combining the 2-loop terms with the loop-by-loop terms and with soft real corrections:

$$\frac{d\sigma^{\rm NNLO}}{d\Omega} + \frac{d\sigma_\gamma^{\rm NLO}}{d\Omega} = \frac{d\sigma^{\rm NNLO,e}}{d\Omega} + \sum_{f\neq e} Q_f^2 \frac{d\sigma^{\rm NNLO,f^2}}{d\Omega} + \sum_{f\neq e} Q_f^4 \frac{d\sigma^{\rm NNLO,f^4}}{d\Omega}$$
$$+ \sum_{f_1,f_2\neq e} Q_{f_1}^2 Q_{f_2}^2 \frac{d\sigma^{\rm NNLO,2f}}{d\Omega}.$$

The Box Corrections

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{2\mathrm{e}\times\mathrm{tree}}}{d\Omega} = \frac{\alpha^2}{2\,s}\left[\frac{1}{s}\,A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) + \frac{1}{t}\,A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t)\right]$$

Here the auxiliary functions can be conveniently expressed through three independent form factors $B^{(2)}_{\mathrm{I,f}}(x,y)$, where $\mathrm{i} = \mathrm{A,B,C}$,

$$\begin{aligned}
A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) &= F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{A},f}(s,t) + B^{(2)}_{\mathrm{B},f}(t,s) + B^{(2)}_{\mathrm{C},f}(u,t) - B^{(2)}_{\mathrm{B},f}(u,s)\right],\\
A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t) &= F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{B},f}(s,t) + B^{(2)}_{\mathrm{A},f}(t,s) - B^{(2)}_{\mathrm{B},f}(u,t) + B^{(2)}_{\mathrm{C},f}(u,s)\right].
\end{aligned}$$

The normalization factor is

$$F_\epsilon = \left(\frac{m_e^2\,\pi\,e^{\gamma_E}}{\mu^2}\right)^{-\epsilon}$$

.

How to evaluate the $N_f = 2$ diagrams?

We did it in 2 ways

- **Decompose the 2-loop integrals to master integrals, solve them.**
  Here: In the limit $m_e^2 << m_f^2 << s, t, u$
  This was done in hep-ph/07042400v2 $\longrightarrow$ ACGR, NPB 786 (2007)
- **Alternatively, rewrite the 2-loop integrals as dispersion integrals.**
  Decompose the loop integrals afterwards into master integrals
  The master integrals are simpler, of one-loop type, but the numerical dispersion integration remains then.

Advantages of the dispersion integrals:

- **get easily the range $m_e^2 << m_f^2, s, t, u$**
- **method applies also to hadronic insertions**

.

Dispersion Integrals

$$\frac{g_{\mu\nu}}{q^2+i\,\delta} \to \frac{g_{\mu\alpha}}{q^2+i\,\delta}\left(q^2\,g^{\alpha\beta}-q^\alpha\,q^\beta\right)\Pi_{\rm had}(q^2)\,\frac{g_{\beta\nu}}{q^2+i\,\delta},$$

the once-subtracted dispersion integral

$$\Pi_{\rm had}(q^2) = -\frac{q^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{dz}{z}\,\frac{\mathbf{Im}\,\Pi_{\rm had}(z)}{q^2-z+i\,\delta}.$$

Finally, one relates Im $\Pi_{\rm had}$ to the hadronic cross-section ratio $R_{\rm had}$,

$$\mathbf{Im}\Pi_{\rm had}(z) = -\frac{\alpha}{3}R_{\rm had}(z) = -\frac{\alpha}{3}\,\frac{\sigma_{e^+e^-\to{\rm hadrons}}(z)}{(4\pi\alpha^2)/(3z)},$$

For heavy fermion insertions, we have instead of $R_{\rm had}(z)$:

$$R_f(z) = Q_f^2 C_f(1+2m_f^2/z)\sqrt{1-4m_f^2/z},$$

Replacing the $\Pi_{\rm had}(q^2)$ in a vertex or in box diagram by the z-dispersion integral and exchanging the $\int d^4k$ with the $\int dz$ creates one-loop diagrams with a subsequent z-integration.

.

The kernel functions for the dispersion integrals

$$\Delta\alpha(x) = \Delta\alpha^{(5)}_{\rm had}(x) + \Pi_e(x) + \sum_{f=\mu,\tau,t} \Pi_f(x)$$

$$\Delta\alpha^{(5)}_{\rm had}(x) = \frac{\alpha}{\pi}\,\frac{x}{3} \int_{4m_\pi^2}^{\infty} dz\, \frac{R^{(5)}_{\rm had}(z)}{z}\, \frac{1}{x - z + i\,\delta}$$

$$\begin{aligned}
V_2(x) &= V_{2e}(x) + V_{2\rm rest}(x) \\
V_{2\rm rest}(x) &= \int_{4M^2}^{\infty} dz\, \frac{R(z)}{z}\, K_V(x+i\delta; z) \\
K_V(x;z) &= \frac{1}{3}\left\{ -\frac{7}{8} - \frac{z}{2\,x} + \left(\frac{3}{4} + \frac{z}{2\,x}\right) \ln\left(-\frac{x}{z}\right) - \frac{1}{2}\left(1+\frac{z}{x}\right)^2 \left[\zeta_2 - \mathbf{Li}_2\left(1+\frac{x}{z}\right)\right]\right\}
\end{aligned}$$

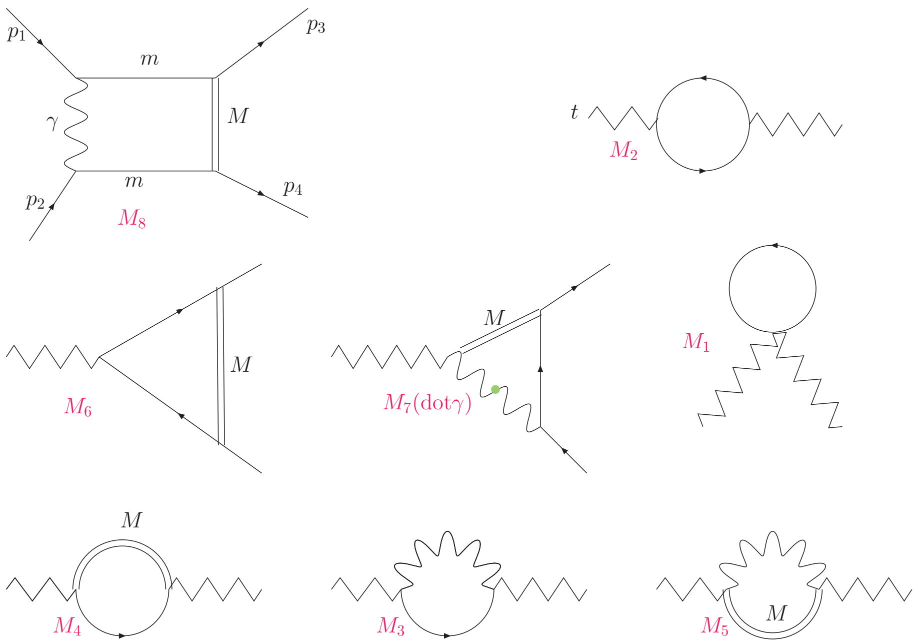
$$B_i(x,y) = \int_{4M^2}^{\infty} dz\, \frac{R(z)}{z}\, K_{box,i}(x+i\delta, y+i\delta; z)$$

The $K_{box,i}(x,y;z)$ are determined as linear combinations of one-loop integrals with mass $z = M^2$.

.

The dispersion master integrals for the $N_f = 2$ contributions

There are three box kernel functions, depending on m_e, m_f, s, t.with $m_e^2 << z = m_f^2, s, t$.
They are IR-divergent.
The eight master integrals for the 2-loop boxes are:



The Box Corrections (repeated here from above)

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{2e\times\mathrm{tree}}}{d\Omega} = \frac{\alpha^2}{2\,s}\left[\frac{1}{s}\,A_1^{2e\times\mathrm{tree}}(s,t) + \frac{1}{t}\,A_2^{2e\times\mathrm{tree}}(s,t)\right]$$

Here the auxiliary functions can be conveniently expressed through three independent form factors $B^{(2)}_{\mathrm{I,f}}(x,y)$, where $\mathrm{i} = \mathrm{A,B,C}$,

$$A_1^{2e\times\mathrm{tree}}(s,t) = F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{A},f}(s,t) + B^{(2)}_{\mathrm{B},f}(t,s) + B^{(2)}_{\mathrm{C},f}(u,t) - B^{(2)}_{\mathrm{B},f}(u,s)\right],$$

$$A_2^{2e\times\mathrm{tree}}(s,t) = F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{B},f}(s,t) + B^{(2)}_{\mathrm{A},f}(t,s) - B^{(2)}_{\mathrm{B},f}(u,t) + B^{(2)}_{\mathrm{C},f}(u,s)\right].$$

The normalization factor is

$$F_\epsilon = \left(\frac{m_e^2\,\pi\,e^{\gamma_E}}{\mu^2}\right)^{-\epsilon}$$

Look e.g. at $B^{(2)}_{\mathrm{C},f}(t,s)$ for hadrons:

$$B^{(2)}_{\mathrm{C},had}(t,s) = \int_{4M_\pi^2}^{\infty}\frac{dz}{z}R_{had}(z)K_C(s,t,z)$$

And similarly for leptons:

$$
\begin{aligned}
4M_\pi^2 &\longrightarrow 4m_l^2 \\
R_{had}(z) &\longrightarrow R_{lep}(z) \sim \sqrt{1-\frac{4m_l^2}{z}}\left(1+\frac{2m_l^2}{z}\right)+\epsilon R_{lep}^{\epsilon}(z)
\end{aligned}
$$

Get:

$$
B^{(2)}_{\mathrm{C},lep}(t,s) = \int_{4m_l^2}^{\infty}\frac{dz}{z}R_{lep}(z)K_C(s,t,z)
$$

$$
\begin{aligned}
K_C(x,y;z) &= F_\epsilon\sum_{i=1}^{8}c_{Ci}M_i(s,t,z) \\
&= \frac{1}{3\,m_e^2\,(y-z)}\Bigg\{2\,\frac{F_\epsilon}{\epsilon}\,x^2\,L_x+4\,\zeta_2\,x^2\left(\frac{z}{y}-2\right)-2\,\left(x^2+y^2+x\,y\right)\,L_x \\
&+ x^2\left(\frac{z}{y}-1\right)L_y+2\,x^2\left(\frac{z}{y}-1\right)L_y^2+4\,x^2\,L_x\,L_y+x^2\left(\frac{z}{y}-1\right)\,\ln\!\left(\frac{z}{m_e^2}\right) \\
&- 2\,x^2\left(\frac{z}{y}-\frac{1}{2}\right)\,\ln^2\!\left(\frac{z}{m_e^2}\right)+4\,x^2\left(\frac{z}{y}-1\right)\,\ln\!\left(\frac{z}{m_e^2}\right)\,\ln\!\left(1-\frac{z}{y}\right) \\
&+ 2\,x^2\,\ln\!\left(\frac{z}{m_e^2}\right)L_x-x^2\left(\frac{z}{y}+\frac{y}{z}-2\right)\,\ln\!\left(1-\frac{z}{y}\right)-4\,x^2\,\ln\!\left(1-\frac{z}{y}\right)L_x \\
&+ 4\,x^2\left(\frac{z}{y}-1\right)\mathrm{Li}_2\!\left(\frac{z}{y}\right)-2\,x^2\,\mathrm{Li}_2\!\left(1+\frac{z}{x}\right)\Bigg\}.
\end{aligned}
$$

The contributing masters are:

$$
\begin{aligned}
M_1 &= N \int d^D k \frac{1}{k^2 - m^2}, && (1)\\
M_2 &= N \int d^D k \frac{1}{(k^2 - m^2)[(k - p_1 - p_2)^2 - m^2]}, && (2)\\
M_3 &= N \int d^D k \frac{1}{k^2 (k - p_1 + p_3)^2}, && (3)\\
M_4 &= N \int d^D k \frac{1}{(k^2 - m^2)[(k - p_3)^2 - z]}, && (4)\\
M_5 &= N \int d^D k \frac{1}{(k^2 - z)(k - p_1 + p_3)^2}, && (5)\\
M_6 &= N \int d^D k \frac{1}{(k^2 - z)[(k + p_3)^2 - m^2][(k + p_3 - p_1 - p_2)^2 - m^2]}, && (6)\\
M_7 &= N \int d^D k \frac{1}{(k^2 - z)[(k + p_3)^2 - m^2](k + p_3 - p_1)^2}, && (7)\\
M_8 &= N \int d^D k \frac{1}{(k^2 - z)[(k + p_3)^2 - m^2](k + p_3 - p_1)^2[(k + p_3 - p_1 - p_2)^2 - m^2]}, && (8)
\end{aligned}
$$

where

$$
F_\epsilon = N = m^{2\epsilon} \frac{e^{\gamma} \epsilon}{i\pi^{2-\epsilon}}. \qquad (9)
$$

and e.g. the box integral $M_8 = Bo1$ is:

```
Bo1 = (4*ep*z2 + 2*Log[-(m^2/t)] - 4*ep*Log[me]*Log[-(m^2/t)] -
      4*ep*Log[1 - m^2/t]*Log[-(m^2/t)] + 3*ep*Log[-(m^2/t)]^2 -
      2*Log[-(me^2/t)] + 4*ep*Log[me]*Log[-(me^2/t)] +
      4*ep*Log[1 - m^2/t]*Log[-(me^2/t)] - 2*ep*Log[-(m^2/t)]*
       Log[-(me^2/t)] - ep*Log[-(me^2/t)]^2 + Log[-(m^2/s)]*
       (4*ep*Log[me] + 4*ep*Log[1 - m^2/t] - 2*(1 + ep*Log[-(m^2/t)] +
          ep*Log[-(me^2/t)])) + 2*ep*PolyLog[2, (m^2 + s)/s])/
     (2*ep*s*(m^2 - t))
```

$d\sigma$ / $d\Omega$ [nb] | $\sqrt{s}$ [GeV]	1	10	91	500
LO QED	46.6409	0.466409	0.00563228	0.000186564
LO `Zfitter`	46.643	0.468499	0.127292	0.0000854731
NNLO (e)	-0.230927	-0.00453987	-0.0000919387	-4.28105 $\cdot 10^{-6}$
NNLO ($e + \mu$) "	-0.256679	-0.00570942	-0.000122796	-5.90469 $\cdot 10^{-6}$
NNLO ($e + \mu + \tau$) "		-0.00586082	-0.000135449	-6.7059 $\cdot 10^{-6}$
NNLO ($e + \mu + \tau + t$) "				-6.6927 $\cdot 10^{-6}$
NNLO photonic	2.07476	0.0358755	0.000655126	0.0000284063
NNLO IR e	-0.19927	-0.00359349	-0.0000672264	-2.95317$\cdot 10^{-6}$
NNLO IR μ (analytic)	-0.0314292	-0.00134635	-0.0000335037	-1.66781 $\cdot 10^{-6}$
NNLO IR μ (dispersion)	-0.0333538	-0.00134663	-0.0000335037	-1.66781 $\cdot 10^{-6}$
NNLO IR τ (analytic)		-0.00021027	-0.0000162977	-1.00877 $\cdot 10^{-6}$
NNLO IR τ (dispersion)		-0.000272634	-0.0000163119	-1.00878 $\cdot 10^{-6}$

Table 1: Numerical values for the NNLO corrections to the differential cross section respect to the solid angle. Results are expressed in nanobarns for a scattering angle $\theta = 90°$. Empty entries are related to cases where the high-energy approximation cannot be applied.

Using R_{had}

This is a topic by itself, because R_{had} is basically unpublished.

```
N.N.1:
Fuer R(s) mit Fehlern, Kontinuum + Resonanzen haben wir nur unsere interne
Arbeitsversion.

N.N.2:
This procedure is a follow up of complicated programs, which unfortunately
do not exist in a really user-friendly form.

N.N.3:
I understand that for your problem it is probably too
cumbersome (and time-consuming) to use the data.

N.N.4:
es hat etwas gedauert, bis ich in meinen alten Verzeichnissen auf einer
1994er Vax am MPI fuendig geworden bin.
```

So, finally, we might reproduce the old estimates given for the vertex dispersion relation in Kniehl, Krawczyk, Kühn, Stuart (1988) $\longrightarrow$ finally we have numerics, but with larger errors than necessary

Final formula and results

We distinguish 3 different categories of 2-loop contributions:

- **Running** α
- **the irreducible 2-loop vertices**
- **the 'rest': irreducible vertices and boxes plus 2-loop boxes**

$$
\begin{aligned}
\frac{d\overline{\sigma}}{d\Omega} &= c\int_{4M_\pi^2}^{\infty} dz \frac{R_{\mathrm{had}}(z)}{z}\frac{1}{t-z}F_1(z) \\
&+ \quad c\int_{4M_\pi^2}^{\infty} \frac{dz}{z\,(s-z)}\Big\{R_{\mathrm{had}}(z)\Big[F_2(z)+F_3(z)\ln\Big|1-\frac{z}{s}\Big|\Big] \\
&- \quad R_{\mathrm{h}}(s)\Big[F_2(s)+F_3(s)\ln\Big|1-\frac{z}{s}\Big|\Big]\Big\} \\
&+ \quad c\,\frac{R_{\mathrm{h}}(s)}{s}\Big\{F_2(s)\ln\Big(\frac{s}{4M_\pi^2}-1\Big)-6\zeta_2 F_a(s) \\
&+ \quad F_3(s)\Big[2\zeta_2+\frac{1}{2}\ln^2\Big(\frac{s}{4M_\pi^2}-1\Big)+\mathbf{Li}_2\Big(1-\frac{s}{4M_\pi^2}\Big)\Big]\Big\},
\end{aligned}
\tag{10}
$$

with $c=\alpha^4/(\pi^2 s)$ **and** $R_{\mathrm{h}}(s)=\theta(s-4M_\pi^2)\ R_{\mathrm{had}}(s)$**.**

$$
\begin{aligned}
F_1(z) \quad = \quad & \frac{1}{3}\Big\{ 9\,\bar{c}(s,t)\ln\Big(\frac{s}{m_e^2}\Big) + \Big[-z^2\Big(\frac{1}{s}+\frac{2}{t}+2\,\frac{s}{t^2}\Big) + z\,\Big(4+4\,\frac{s}{t}+2\,\frac{t}{s}\Big) + \frac{1}{2}\,\frac{t^2}{s} + 6\,\frac{s^2}{t} \\
+ \quad & 5\,s + 4\,t\Big]\ \ln\Big(-\frac{t}{s}\Big) + s\,\Big(-\frac{z}{t}+\frac{3}{2}\Big)\ \ln\Big(1+\frac{t}{s}\Big) + \Big[\frac{1}{2}\,\frac{z^2}{s} + 2\,z\,\Big(1+\frac{s}{t}\Big) - \frac{11}{4}\,s - 2\,t\Big]\ \ln^2\Big(-\frac{t}{s}\Big) \\
- \quad & \Big[\frac{1}{2}\,\frac{z^2}{t} - z\,\Big(1+\frac{s}{t}\Big) + \frac{t^2}{s} + 2\,\frac{s^2}{t} + \frac{9}{2}\,s + \frac{15}{4}\,t\Big]\ \ln^2\Big(1+\frac{t}{s}\Big) + \Big[\frac{z^2}{t} - 2\,z\,\Big(1+\frac{s}{t}\Big) + 2\,\frac{s^2}{t} + 5\,s + \frac{5}{2}\,t\Big] \\
\times \quad & \ln\Big(-\frac{t}{s}\Big)\ \ln\Big(1+\frac{t}{s}\Big) - 4\,\Big[\frac{t^2}{s} + 2\,\frac{s^2}{t} + 3\,\Big(s+t\Big)\Big]\ \Big[1+\mathsf{Li}_2\Big(-\frac{t}{s}\Big)\Big] - \Big[\frac{t^2}{s} + 2\,\frac{s^2}{t} + 3\,\Big(s+t\Big)\Big]\ \ln\Big(\frac{z}{s}\Big)\ \ln\Big(1+\frac{t}{s}\Big) \\
- \quad & \Big[2\,\frac{z^2}{t} - 4\,z\,\Big(1+\frac{s}{t}\Big) - 4\,\frac{t^2}{s} - 2\,\frac{s^2}{t} + s - \frac{11}{2}\,t\Big]\,\zeta_2 + \Big[z^2\,\Big(\frac{1}{s} + 2\,\frac{s}{t^2} + \frac{2}{t}\Big) - z\,\Big(\frac{t}{s} + 2\frac{s}{t} + 2\Big)\Big]\ \ln\Big(\frac{z}{s}\Big) \\
- \quad & \Big[z^2\,\Big(\frac{1}{s}+\frac{1}{t}\Big) + 2\,z\,\Big(1+\frac{s}{t}\Big) + s + 2\,\frac{s^2}{t}\Big]\ \ln\Big(\frac{z}{s}\Big)\ \ln\Big(1+\frac{z}{s}\Big) + \Big[\frac{z^2}{s} + 4\,z\,\Big(1+\frac{s}{t}\Big) - \frac{t^2}{s} - 4\,\Big(s+t\Big)\Big] \\
\times \quad & \ln\Big(\frac{z}{s}\Big)\ln\Big(1-\frac{z}{t}\Big) - \Big[z^2\,\Big(\frac{1}{s} + 2\frac{s}{t^2} + \frac{2}{t}\Big) - 2\,z\,\Big(\frac{t}{s} + 2\,\frac{s}{t} + 2\Big) + \frac{t^2}{s} + 2\,\Big(s+t\Big)\Big]\ \ln\Big(1-\frac{z}{t}\Big) \\
+ \quad & \Big[\frac{z^2}{t} - 2\,z\,\Big(1+\frac{s}{t}\Big) + 2\,\frac{t^2}{s} + 8\,s + 4\,\frac{s^2}{t} + 7\,t\Big]\ \ln\Big(1-\frac{z}{t}\Big)\ \ln\Big(1+\frac{t}{s}\Big) + \Big[\frac{z^2}{s} + 4\,z\,\Big(1+\frac{s}{t}\Big) - \frac{t^2}{s} - 4\,\Big(s+t\Big)\Big] \\
\times \quad & \mathsf{Li}_2\,\Big(\frac{z}{t}\Big) - \Big[z^2\,\Big(\frac{1}{s}+\frac{1}{t}\Big) + 2\,z\,\Big(1+\frac{s}{t}\Big) + s + 2\,\frac{s^2}{t}\Big]\ \mathsf{Li}_2\,\Big(-\frac{z}{s}\Big) - \Big[\,\frac{z^2}{t} - 2\,z\,\Big(1+\frac{s}{t}\Big) + \frac{t^2}{s} + 5\,s + 2\,\frac{s^2}{t} + 4\,t\Big] \\
\times \quad & \mathsf{Li}_2\,\Big(1+\frac{z}{u}\Big)\Big\} + 4\,\bar{c}(s,t)\ln\Big(\frac{2\,\omega}{\sqrt{s}}\Big)\ \Big[\ln\Big(\frac{s}{m_e^2}\Big) + \ln\Big(-\frac{t}{s}\Big) - \ln\Big(1+\frac{t}{s}\Big) - 1\Big],
\end{aligned}
$$

and similarly for $F_2(z)$ and $F_3(z)$.

The $\int_{4M^2} dz F_i(z)$ gives from the lower integration bound the logarithmically enhanced terms $\ln(= M^2)^n$, e.g. from terms like $A(z)\ \ln(z/s)$ or from $B(z)\mathsf{Li}_2\Big(\frac{z}{s}\Big)$.

Some numerical results

We will now discuss the numerical net effects arising from the $N_f = 2$ vertex plus box diagrams (i.e. excluding the pure running coupling effects):

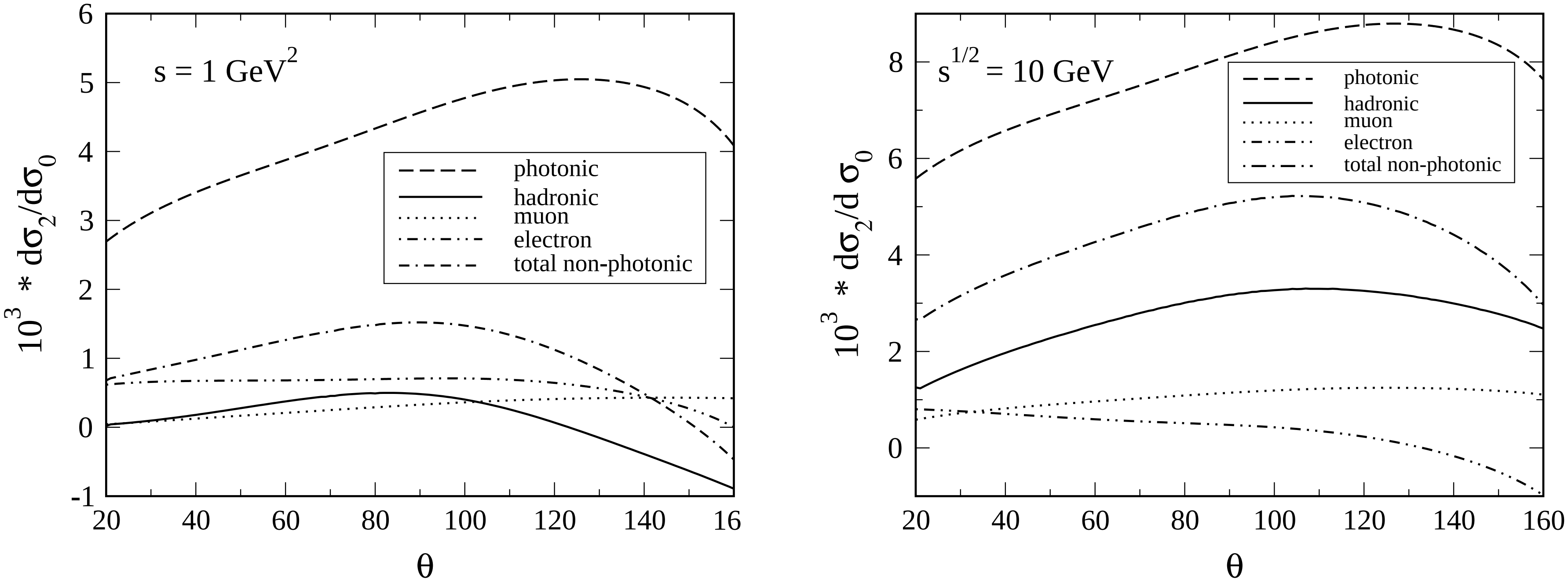
$$\frac{d\sigma_2}{d\Omega} = \frac{d\overline{\sigma}}{d\Omega} + \frac{d\sigma_v}{d\Omega},$$

with $d\overline{\sigma}/d\Omega$ from Eqn. (10). The expression for the irreducible vertex term $d\sigma_v/d\Omega$ derives directly from

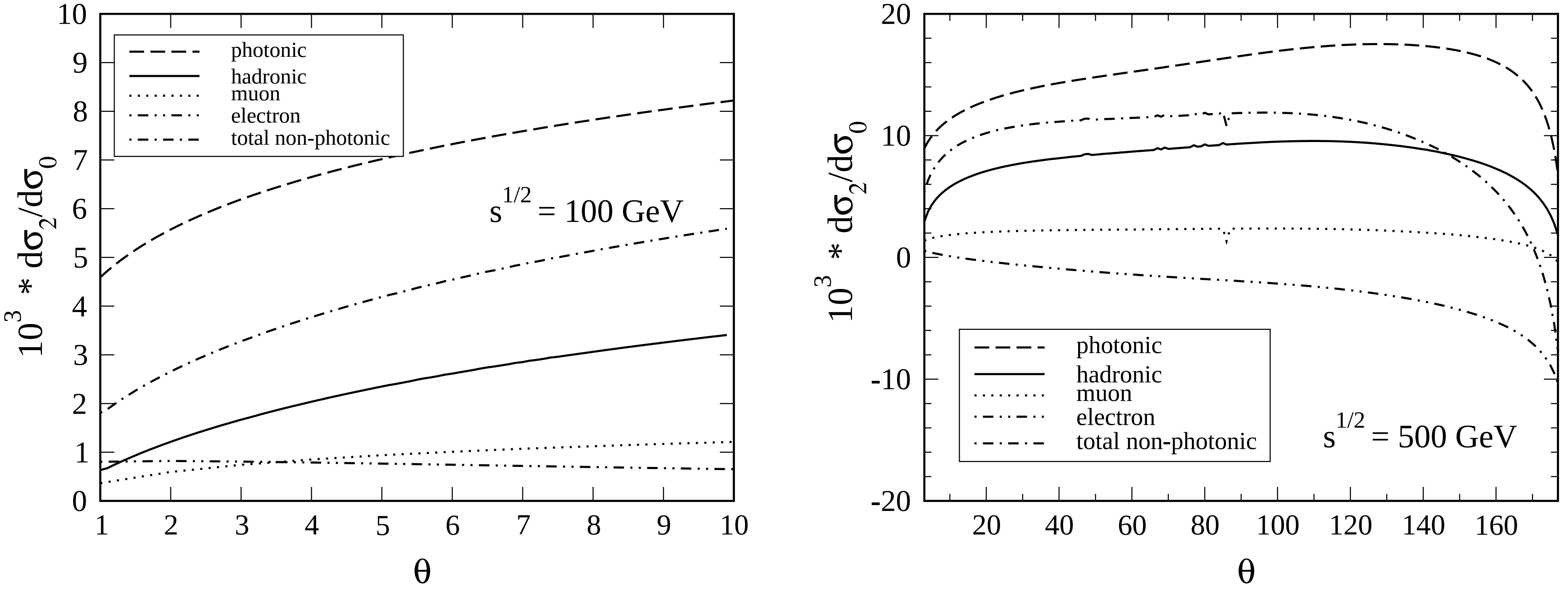
[Kniehl:1988id,webPage:2007x3]

. The $d\sigma_2/d\Omega$ is normalized to the pure photonic Bhabha Born cross section $d\sigma_0/d\Omega$:

$$\frac{d\sigma_0}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{s}{t} + 1 + \frac{t}{s}\right)^2.$$



Two-loop $N_f = 2$ vertex and box corrections $d\sigma_2$ to Bhabha scattering in units of $10^{-3} d\sigma_0$ at meson factories, $\sqrt{s} = 1$ GeV (a) and $\sqrt{s} = 10$ GeV (b).



Two-loop $N_f = 2$ vertex and box corrections $d\sigma_2$ to Bhabha scattering in units of $10^{-3} d\sigma_0$ at ILC energies of $\sqrt{s} = 100$ GeV (GigaZ option) and $\sqrt{s} = 500$ GeV.

Summary

- **We determine the $N_f = 2$ contributions to 2-loop Bhabha scattering, including the hadronic corrections**
- **They are small, but non-negligible at the scale 10^{-3} (→ No LEP influencing)**
- **Agreement for $m_e^2 << m_l^2 << s, t, u$ with:**
  "Two-loop QED corrections to Bhabha scattering"
  Thomas Becher, Kirill Melnikov, arXiv:0704.3582 [hep-ph], JHEP
- **Agreement for $m_e^2 << m_l^2, s, t, u$ with:**
  "Two-Loop Heavy-Flavor Contribution to Bhabha Scattering",
  Roberto Bonciani, Andrea Ferroglia, Sacha Penin, arXiv:0710.4775v3 [hep-ph]
- **To be evaluated yet:**
  $\longrightarrow$ 1-loop diagrams with real photon emission, interfering with real (Born) radiation, including 5-point functions
- **Also: $\longrightarrow$ Real pair production**
- **Both items were studied already by Andrei Arbuzov, Kuraev, Shaitchatdenov (1998, small photon mass)**