Gravitational wave modeling for non-circular binary black holes within the effective one-body approach

Andrea Placidi

Work in collaboration with:

Simone Albanesi, Sebastiano Bernuzzi, Gianluca Grignani, Troels Harmark, Alessandro Nagar, Marta Orselli

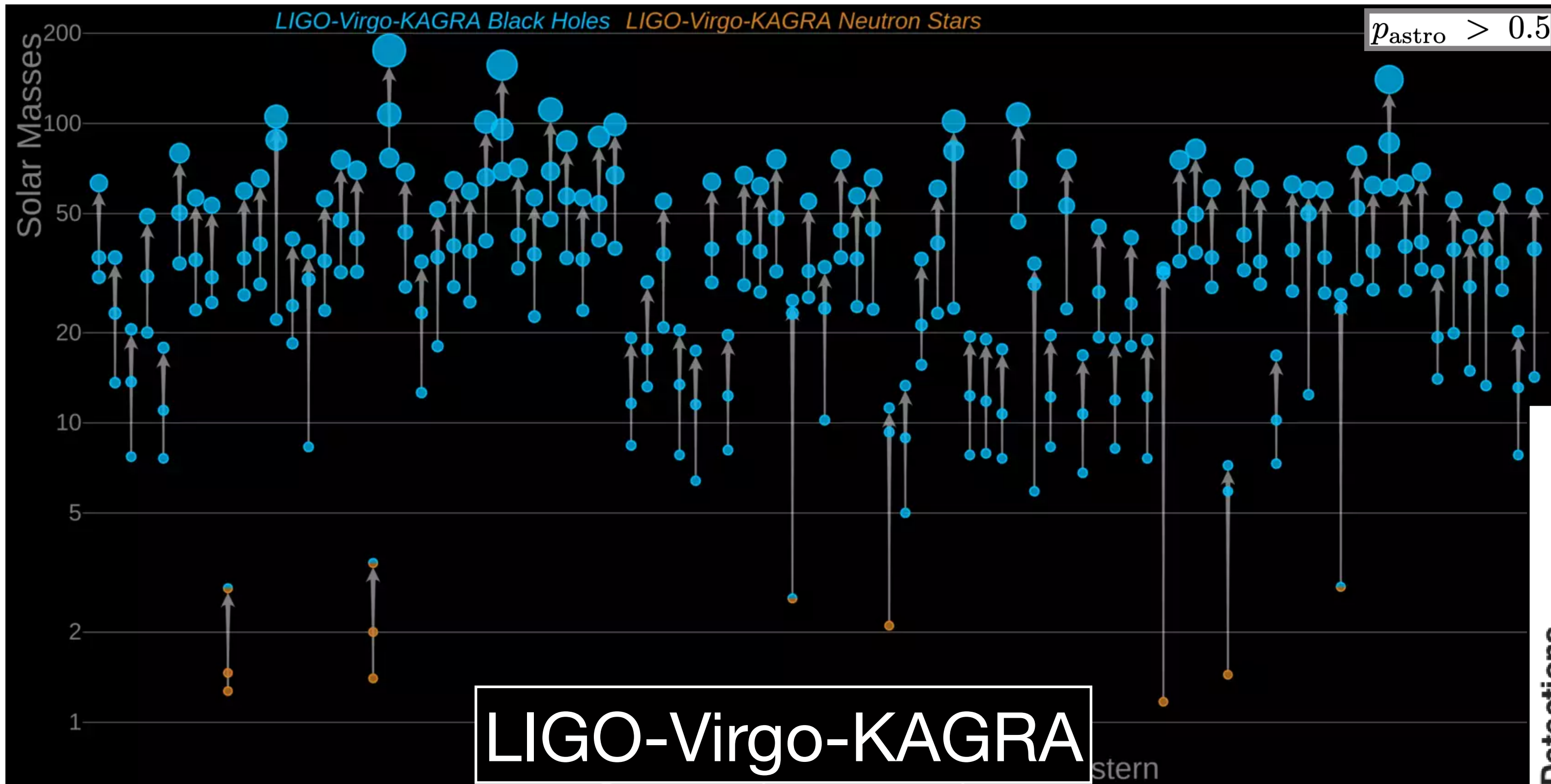
More details in the references:

arXiv:2112.05448 *Phys. Rev. D* 105 (2022) 10, 104030

arXiv:2202.10063 *Phys. Rev. D* 105 (2022) 10, 104031

arXiv:2203.16286 *Phys. Rev. D* 105 (2022) 12, L121503

Gravitational wave (GW) astronomy feats

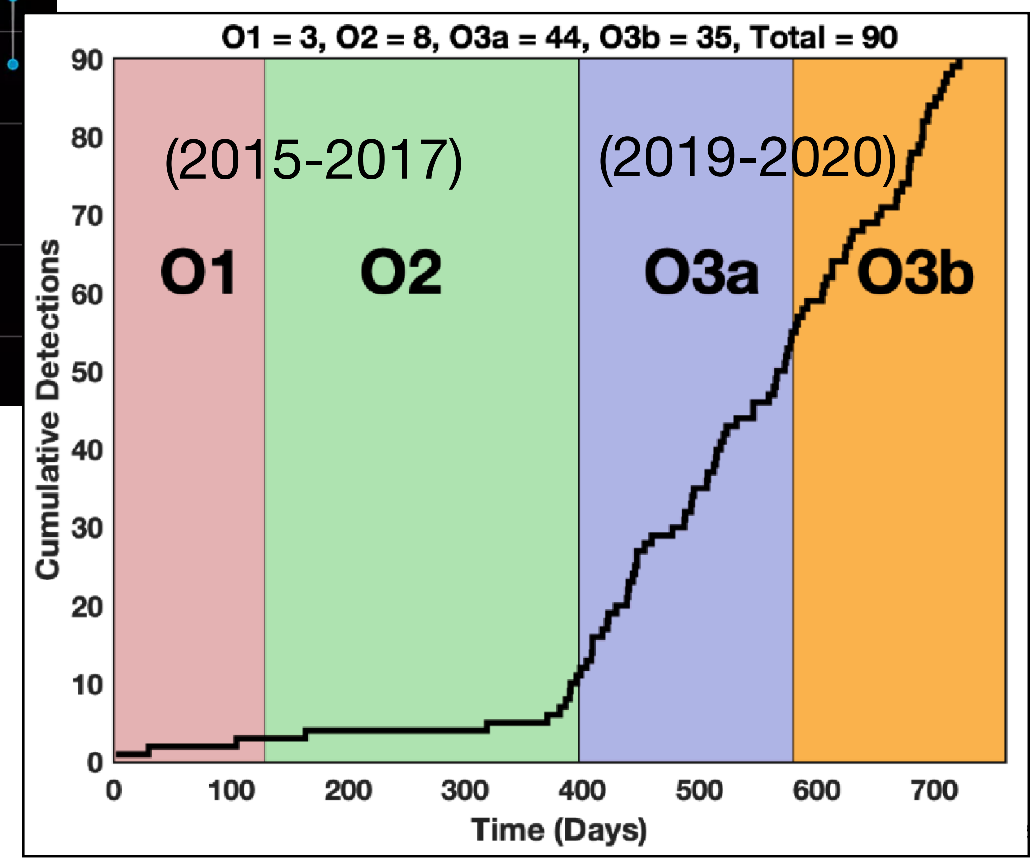


discovery date

Coalescing Compact Binaries (CCBs)

2 NS-NS +
3 NS-BH +
85 BH-BH

90 CCBs



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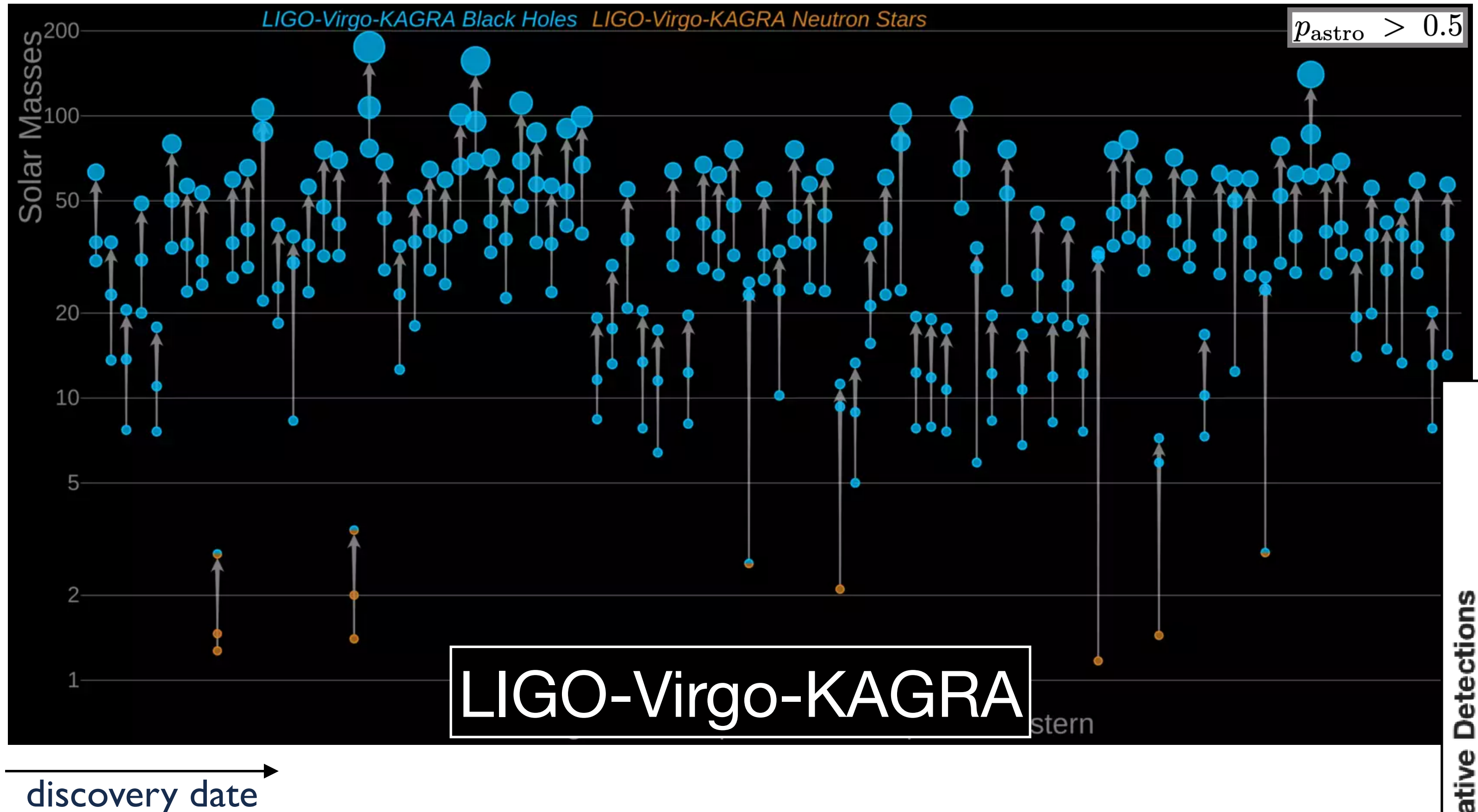

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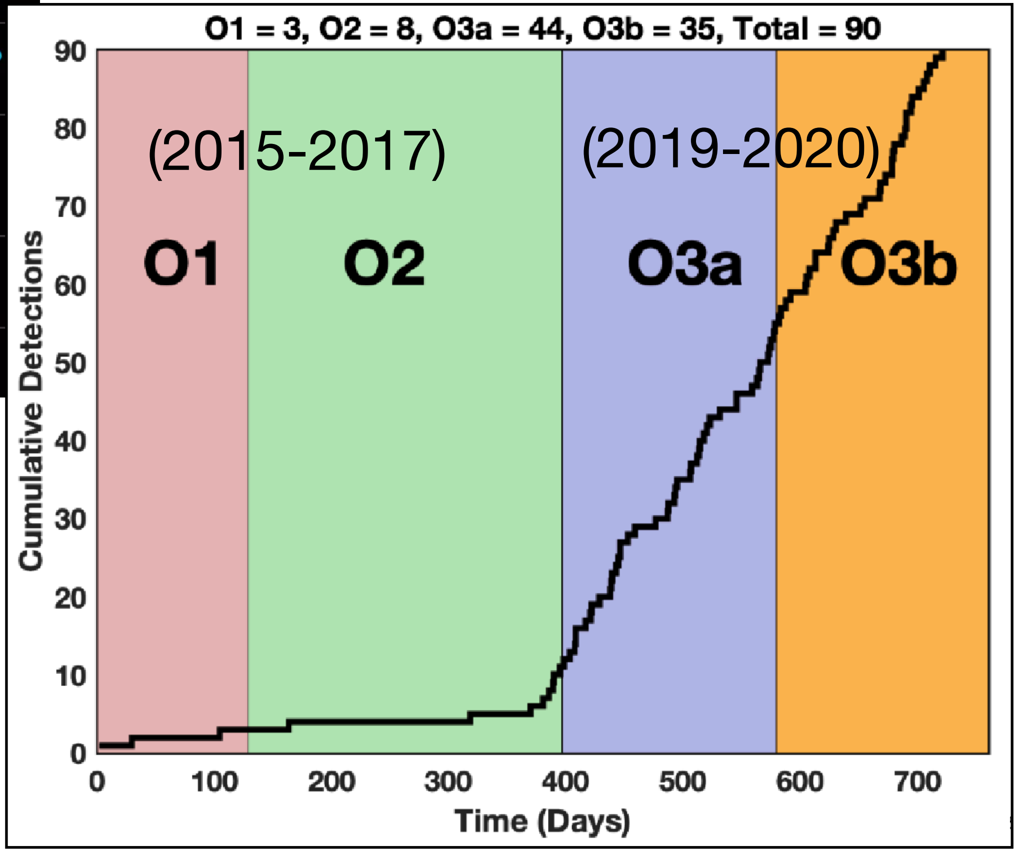

Many more to come!

O4 (24/05/2023) and other GW detectors

LIGO-India, Cosmic Explorer, Einstein Telescope, LISA, TianQuin, Taiji, Pulsar timing arrays

Analytical GW models

Analytical GW models

For any given astrophysical source, data analysis in GW astronomy requires the general prior knowledge of the respective GW signals

$$h(t,\theta) = F_+ h_+(t,\theta) + F_\times h_\times(t,\theta)$$

$h_+, h_\times \equiv$ **physical polarizations of the GW**

$\theta \equiv$ **set of parameters of the source**

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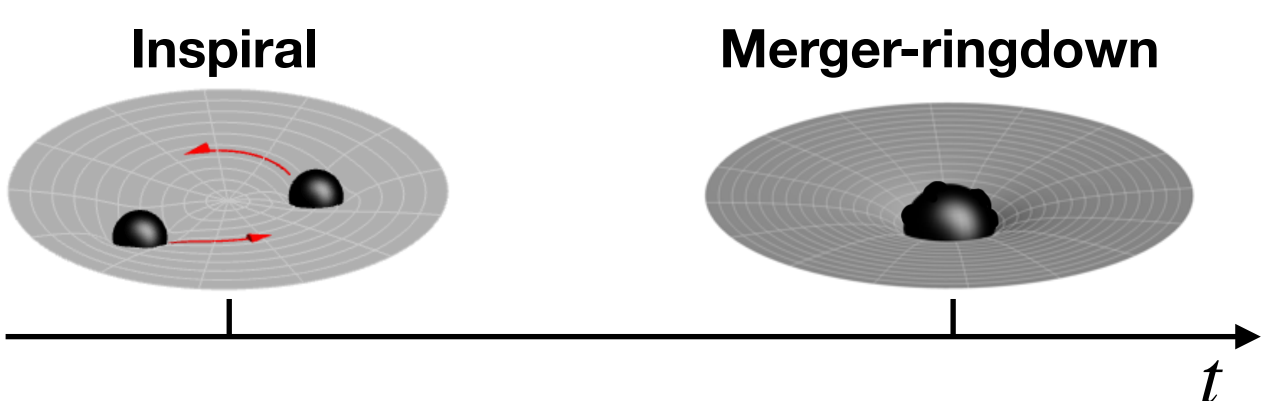
Primary GW sources: **coalescing compact binaries** of black holes (BH)

One needs:

CCB evolution



Post-Newtonian (PN) expansion: weak field and small internal velocity

BH perturbation theory: vibrational modes of the remnant BH (QNMs)

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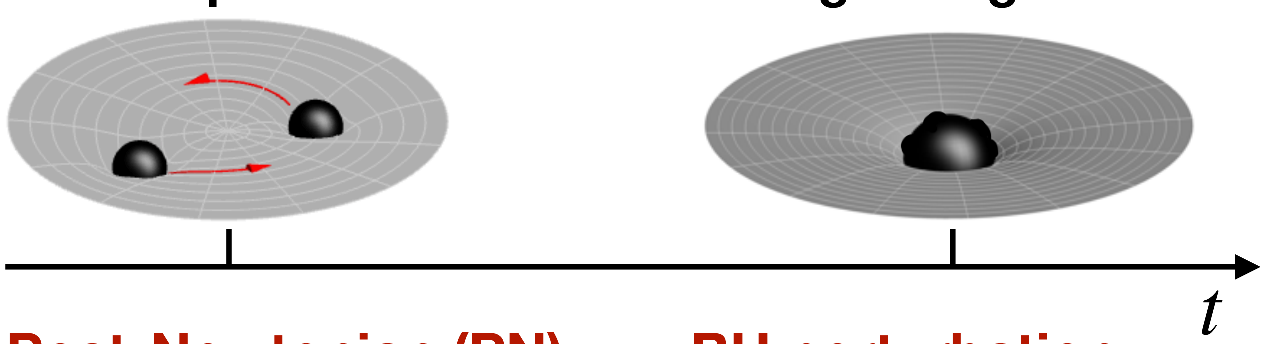
Inspiral **Merger-ringdown**



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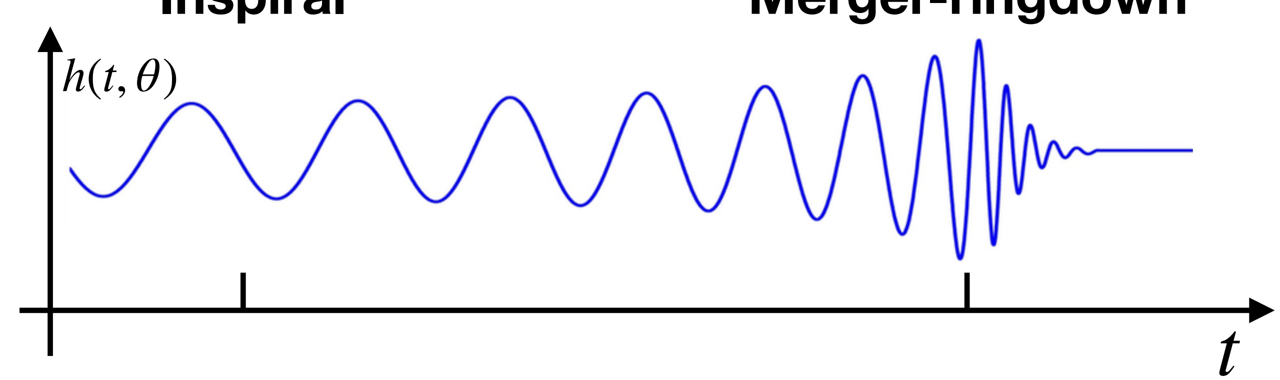
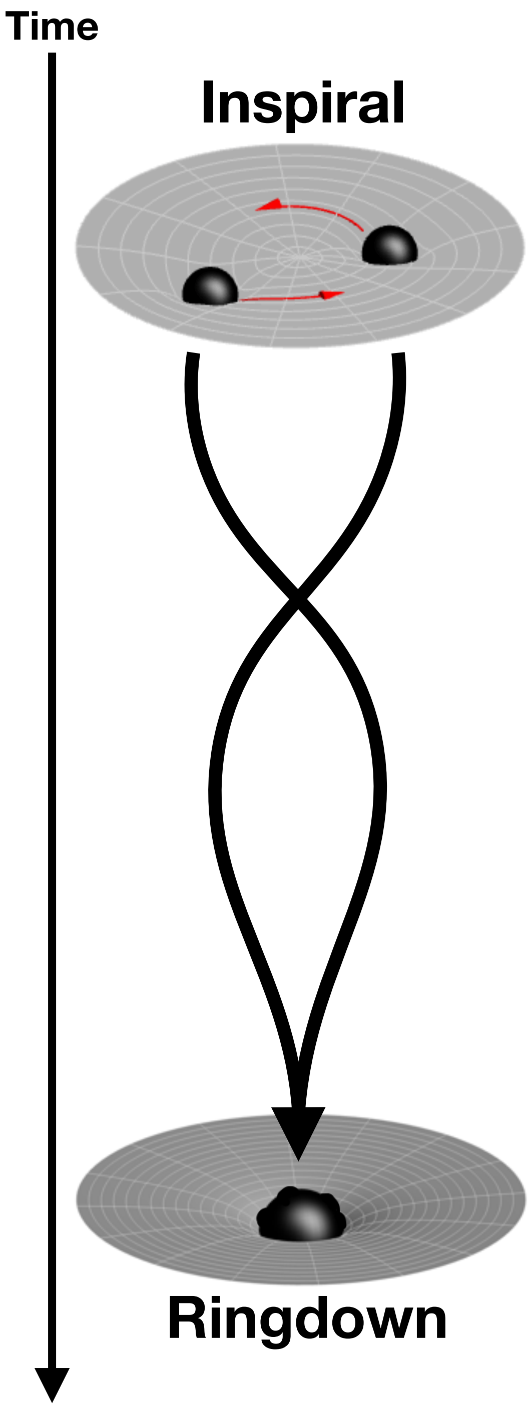
Methods to compute the respective GWs at infinity

Inspiral **Merger-ringdown**



Analytical GW models for CCBs



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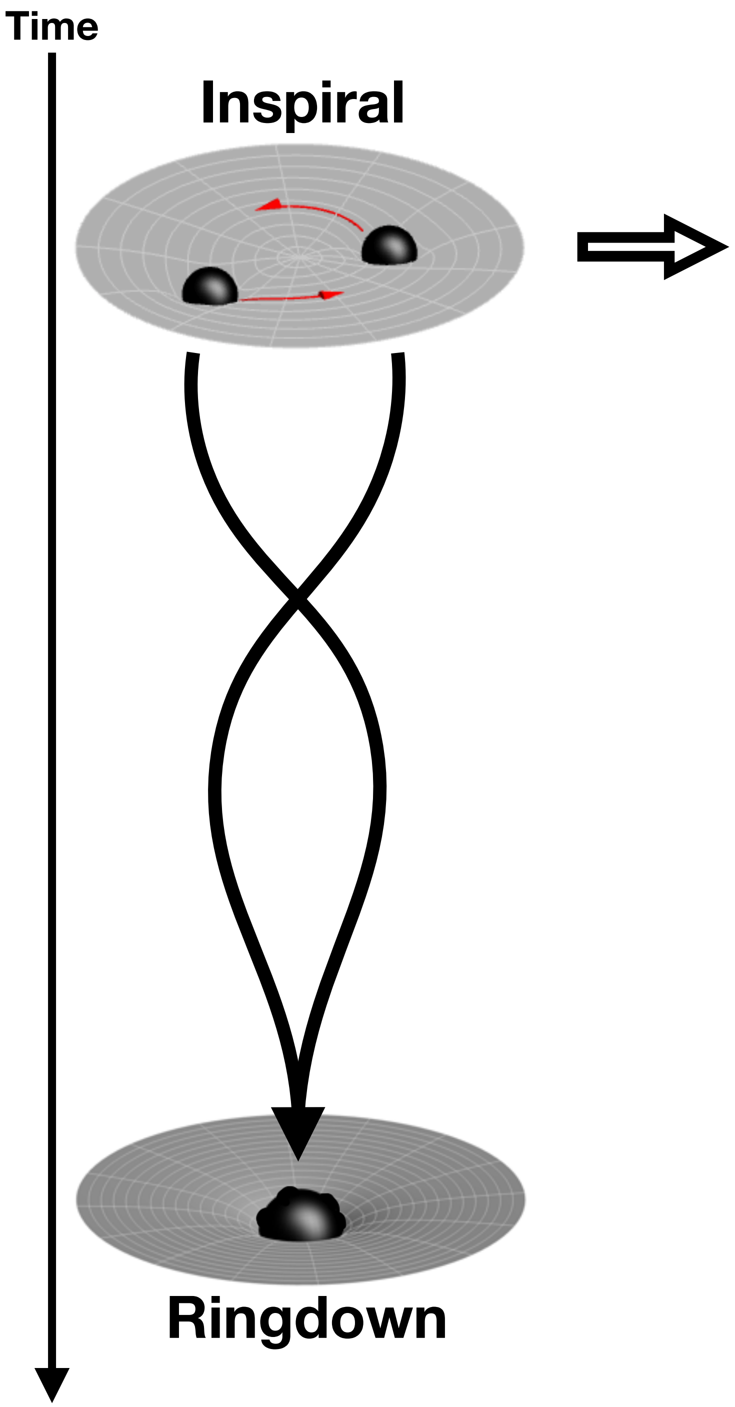
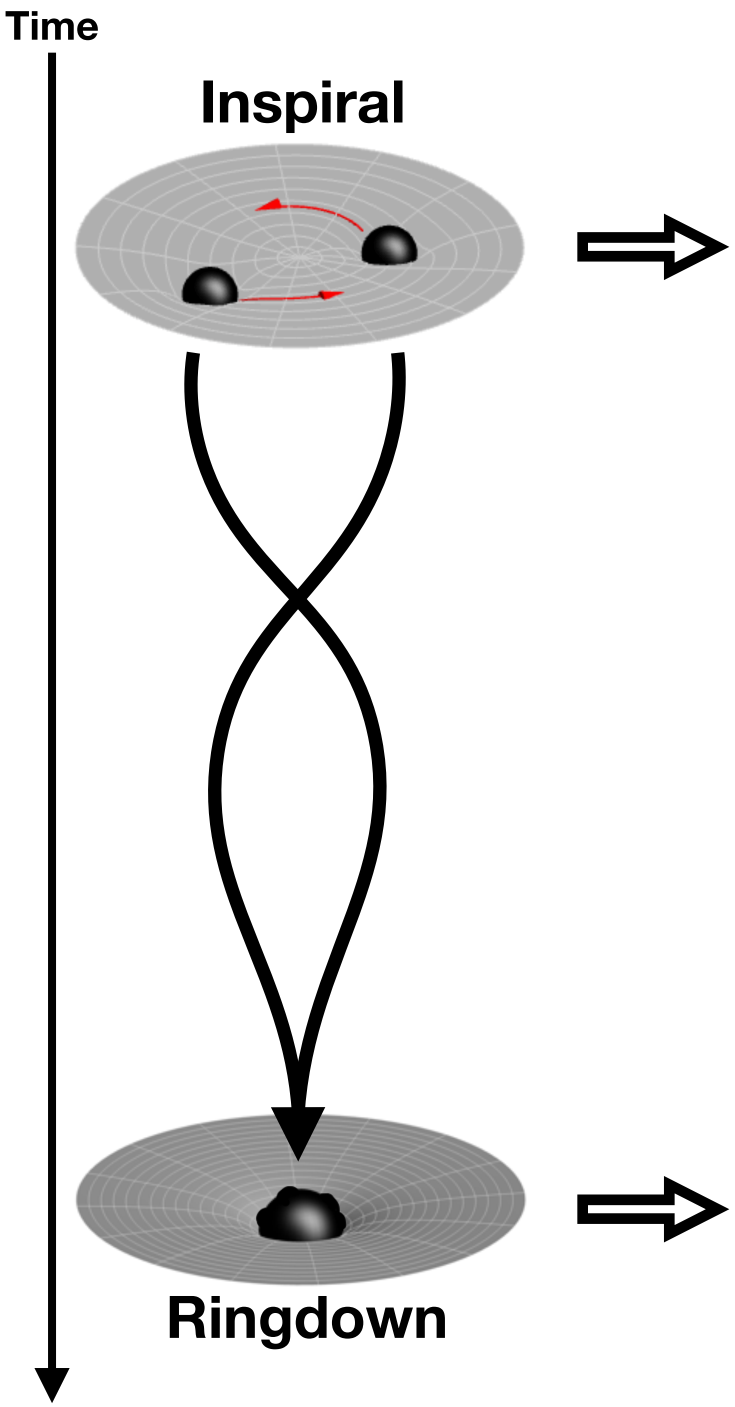


$h^{\mathrm{inspiral}}(t, \theta)$

[Blanchet-Damour 1989; Will-Wiseman 1996]

Post-Newtonian results from the GW generation formalism

Analytical GW models for CCBs



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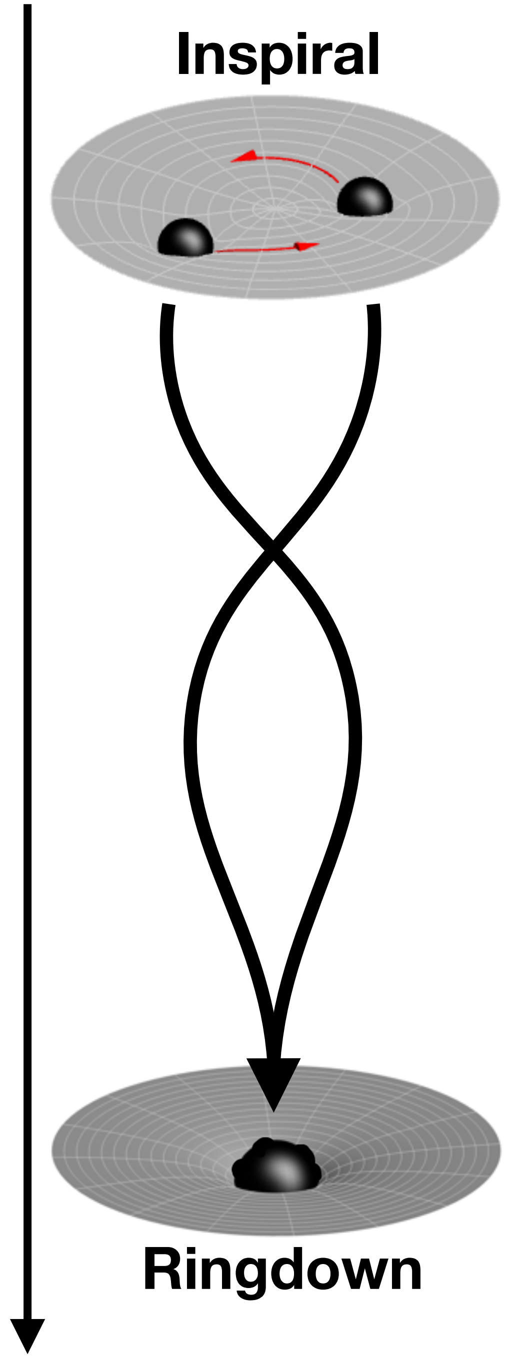
Superposition of the QNMs of the remnant BH

Analytical GW models for CCBs



[Blanchet-Damour 1989; Will-Wiseman 1996]

$h^{\text{inspiral}}(t, \theta)$ ⇒ **Post-Newtonian results from the GW generation formalism**

⇓

?

⇑

The PN approximation breaks down **before** the end of the inspiral phase!

$h^{\text{inspiral}}(t, \theta)$ is not reliable up to merger

[Cutler et al. 1993, Thorne et al. 1998]

[Regge–Wheeler 1957; Zerilli 1972]

$h^{\text{ringdown}}(t, \theta)$ ⇒ **Superposition of the QNMs of the remnant BH**

Effective one-body (EOB) approach

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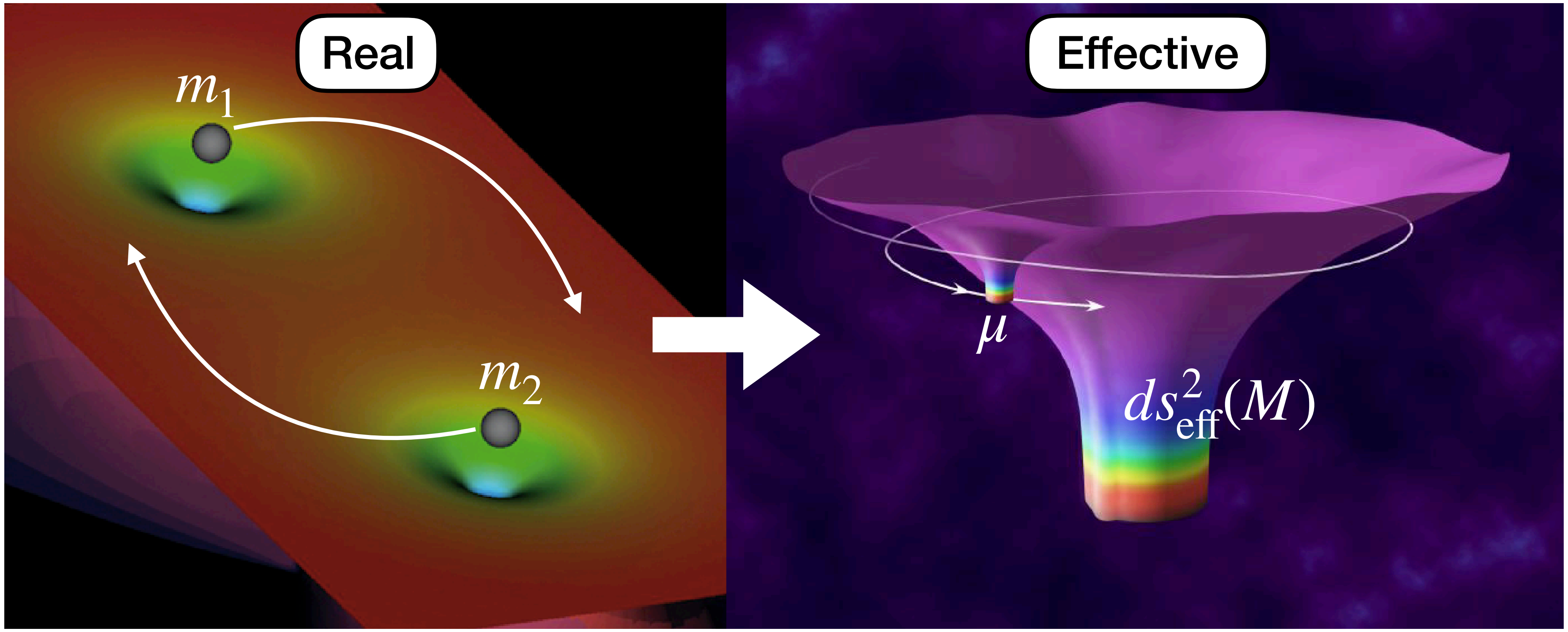
EOB basic idea:

[Buonanno-Damour '99]



$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$M \equiv m_1 + m_2$$

$$\nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

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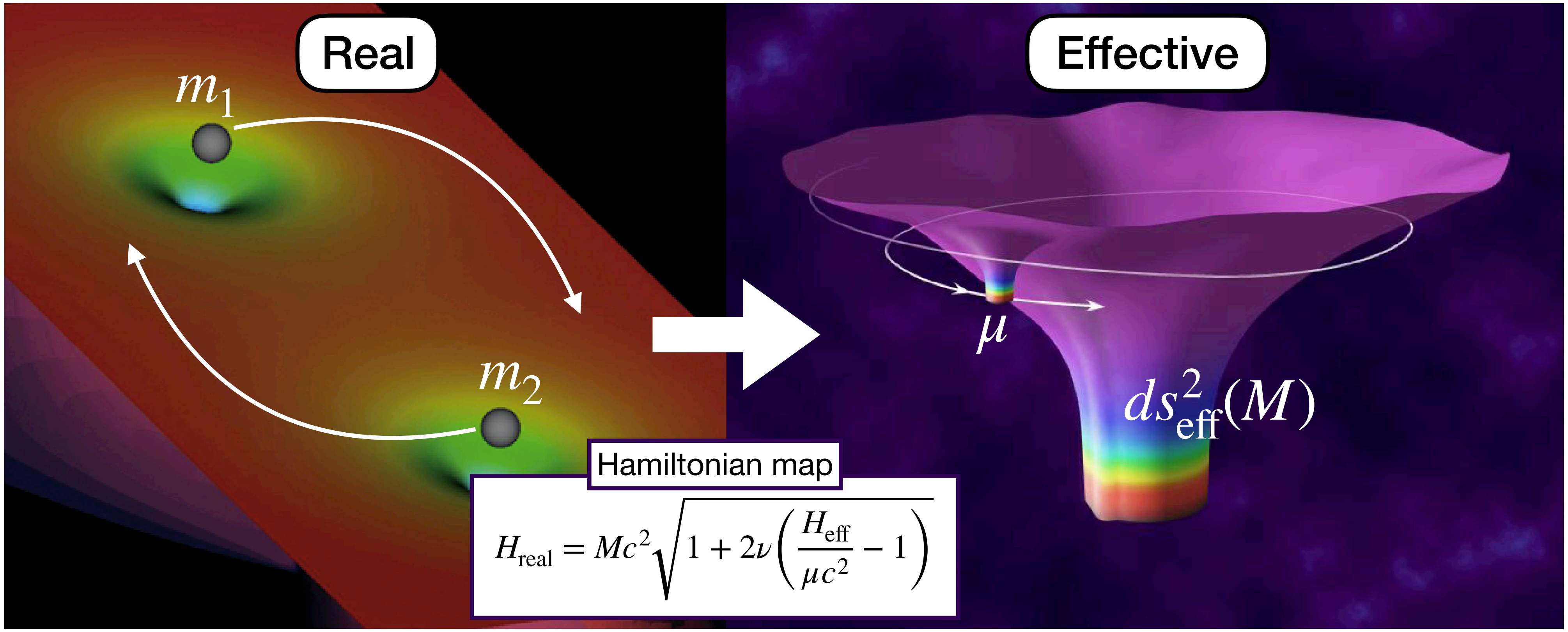

$$H_{\rm real} = Mc^2\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu c^2} - 1\right)}$$

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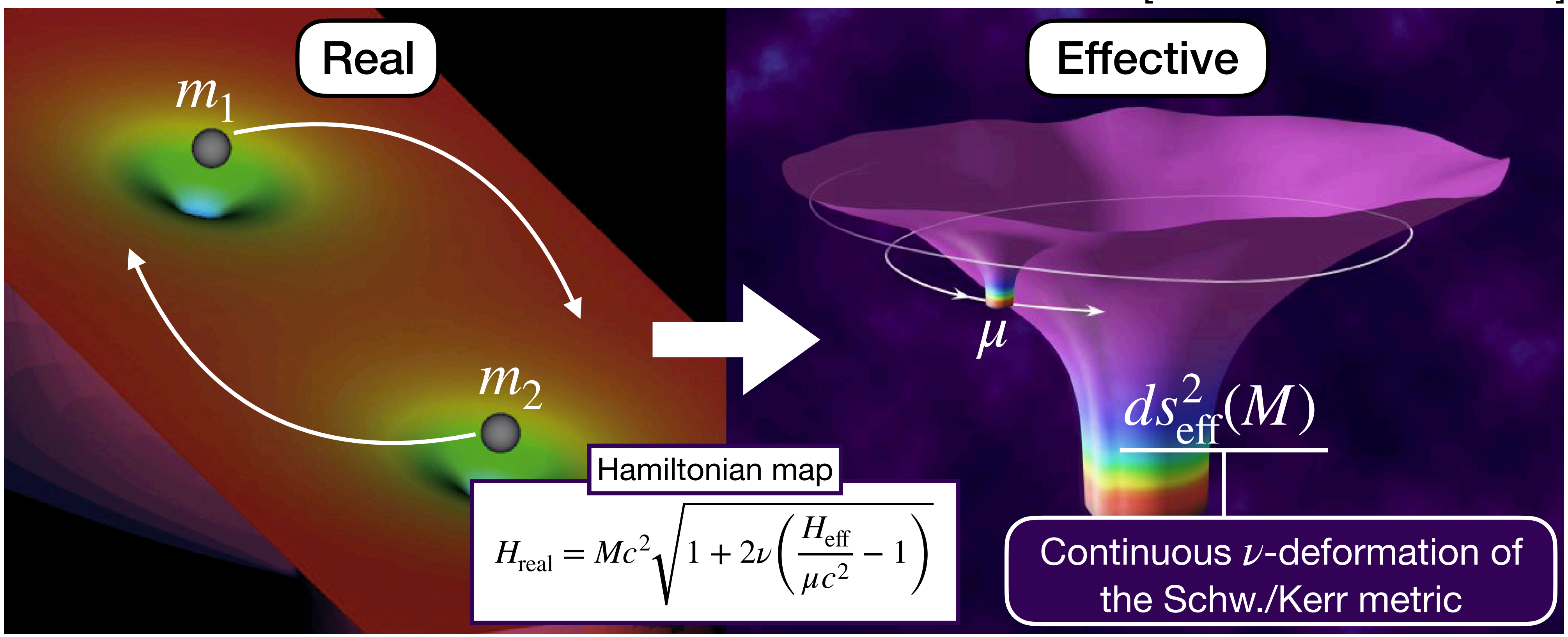

Hamiltonian map

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Example: non-spinning case at 3PN

$$ds^2_{\rm eff} \equiv A(r,\nu)\,c^2dt^2 + \frac{D(r,\nu)}{A(r,\nu)}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

$$A(r,\nu) = 1 - 2\frac{GM}{c^2 r} + 2\nu\left(\frac{GM}{c^2 r}\right)^3 + \nu\left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\left(\frac{GM}{c^2 r}\right)^4 \underset{\nu\to 0}{\longrightarrow} A_{\rm Schw}(r) = 1 - 2\frac{GM}{c^2 r}$$

$$D(r,\nu) = 1 - 6\nu\left(\frac{GM}{c^2 r}\right)^2 + 2(3\nu-26)\nu\left(\frac{GM}{c^2 r}\right)^3 \underset{\nu\to 0}{\longrightarrow} D_{\rm Schw}(r) = 1$$

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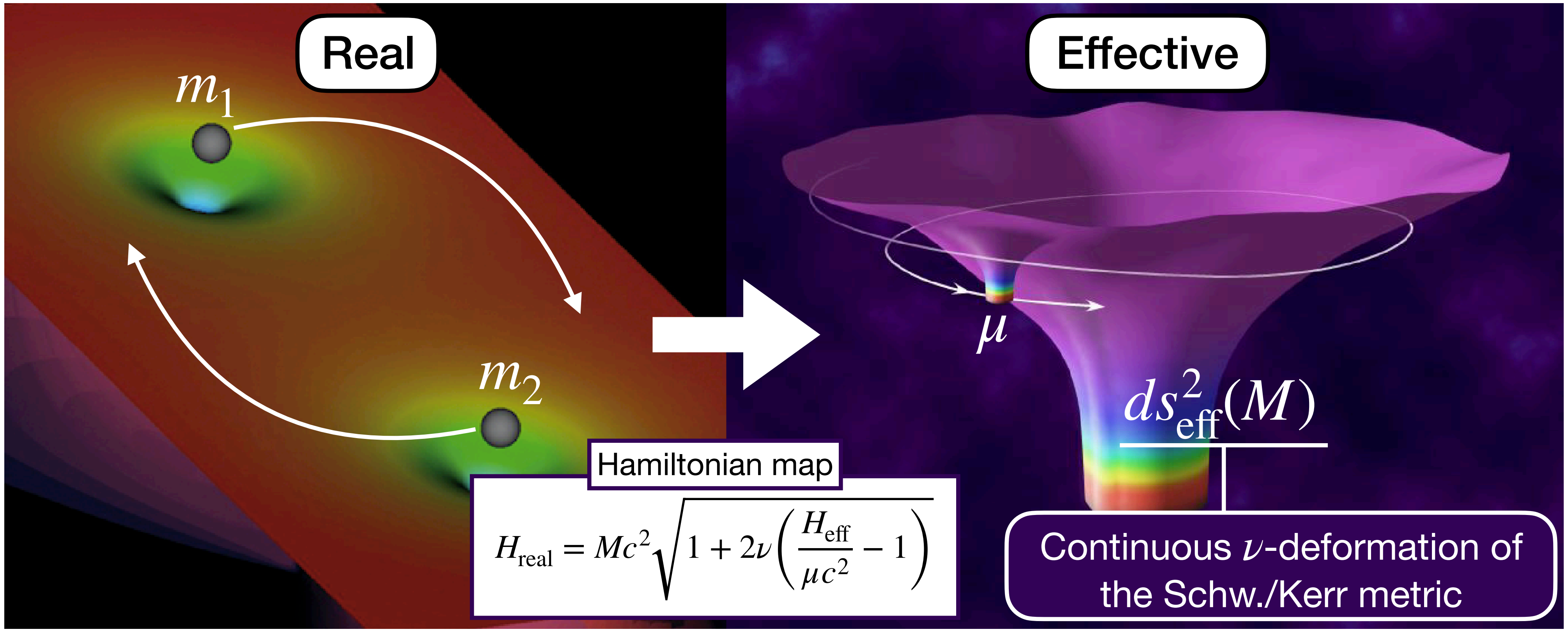

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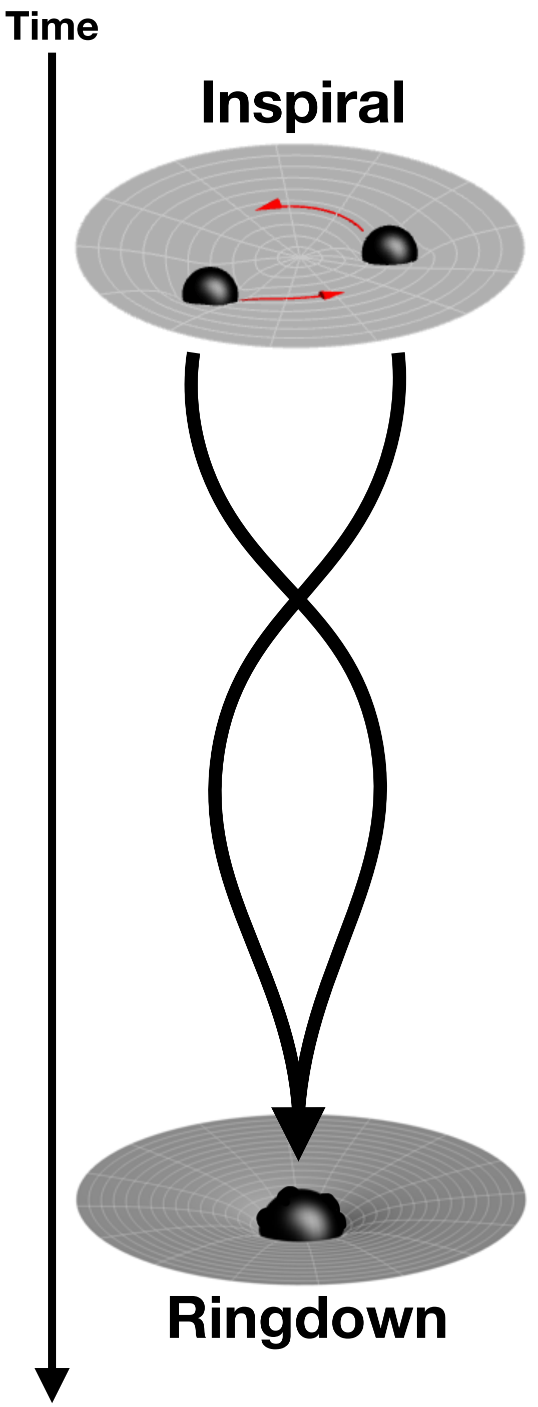
Effective dynamics

↓ ↑

Particle in motion around a BH

EOB waveform models



$h^{\text{inspiral}}(t, \theta)$

[Blanchet-Damour 1986; Will-Wiseman 1996]

Post-Newtonian results from the GW generation formalism

$h^{\text{ringdown}}(t, \theta)$

[Regge–Wheeler 1957; Zerilli 1972]

Superposition of the QNMs of the remnant BH

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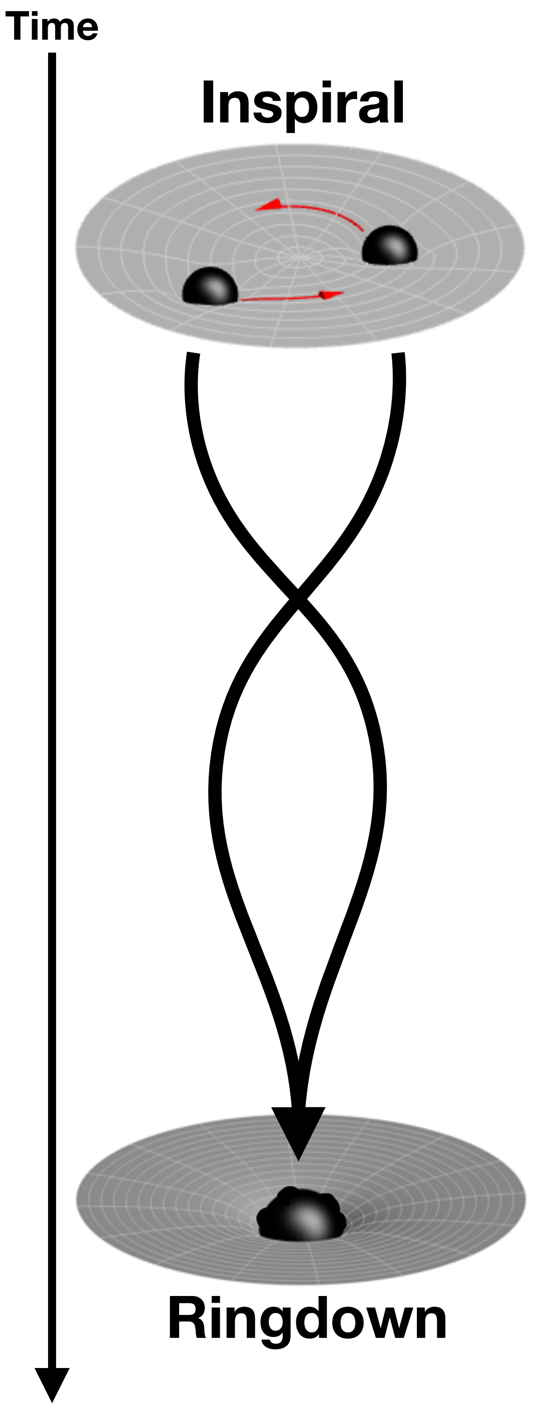

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$$h_{\text{EOB}}(t,\theta) = \theta(t - t_m) h^{\text{inspiral}}_{\text{EOB}}(t,\theta) + \theta(t_m - t) h^{\text{ringdown}}_{\text{EOB}}(t,\theta)$$

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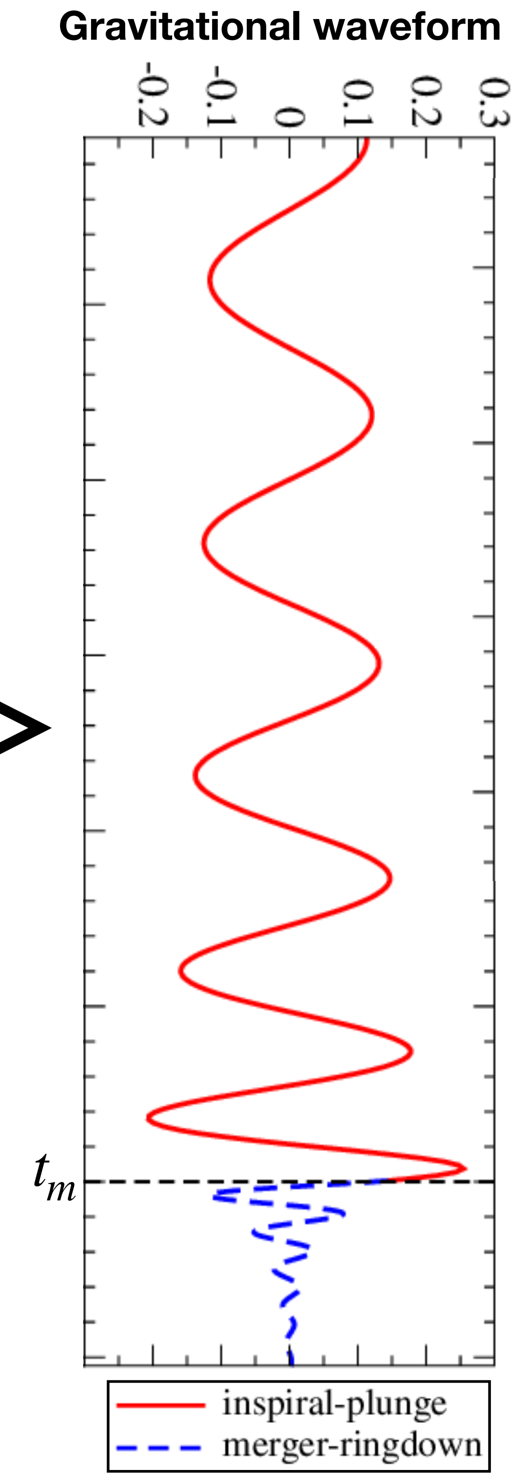

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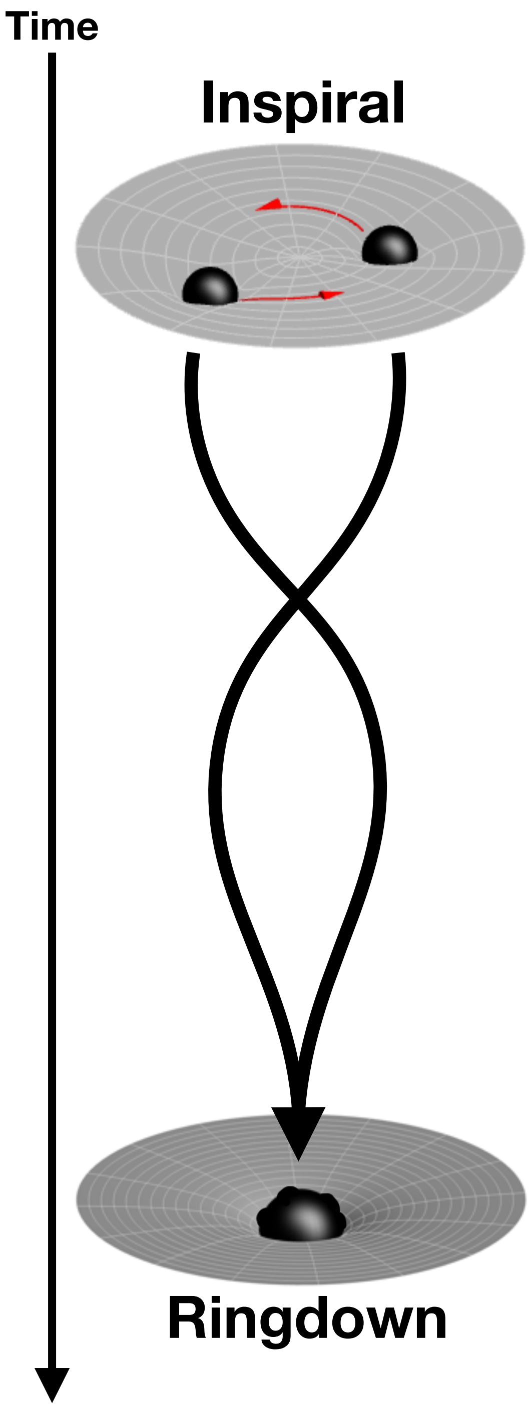

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TEOBResumS **SEOBNR**

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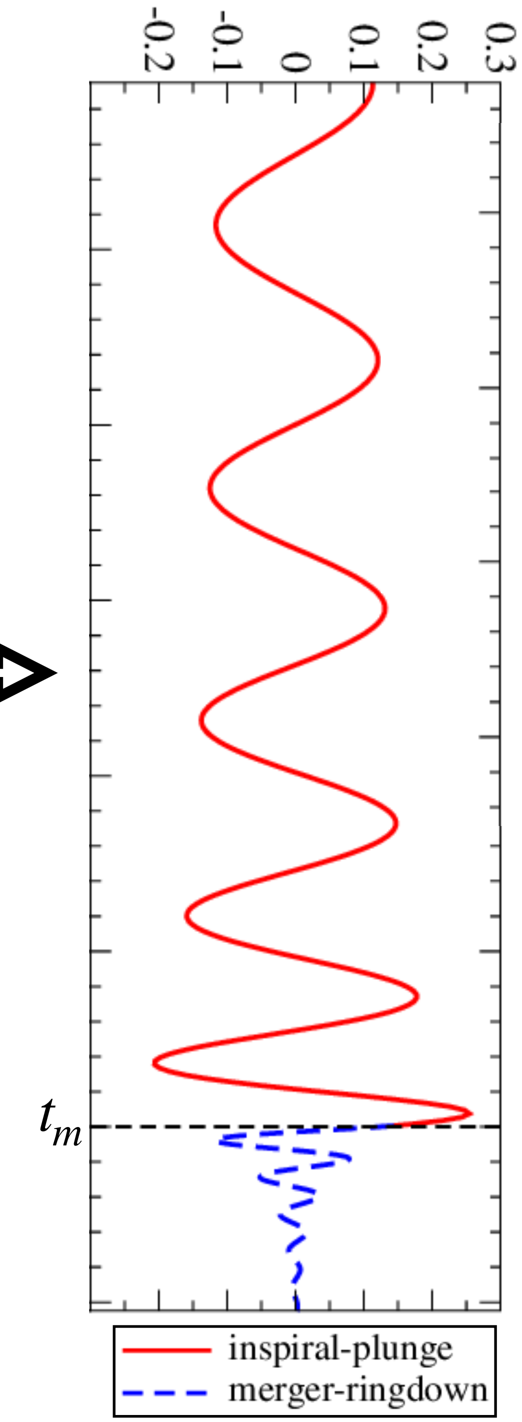

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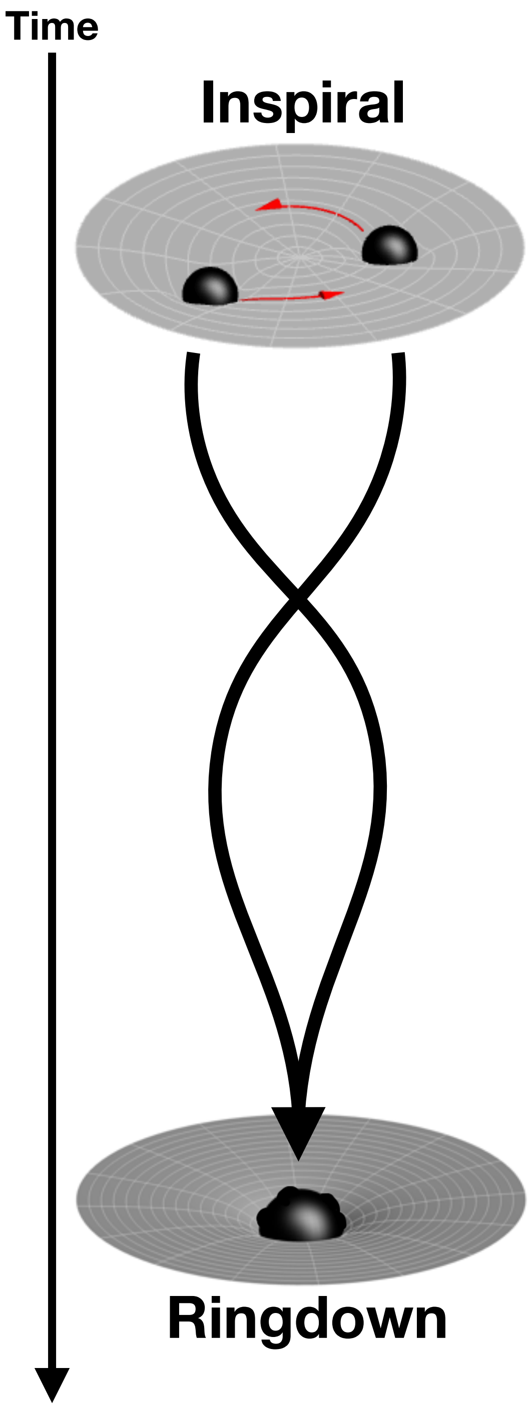

$h^{\rm inspiral}(t,\theta)$

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TEOBResumS

+ non-circular corrections

SEOBNR

erilli 1972]

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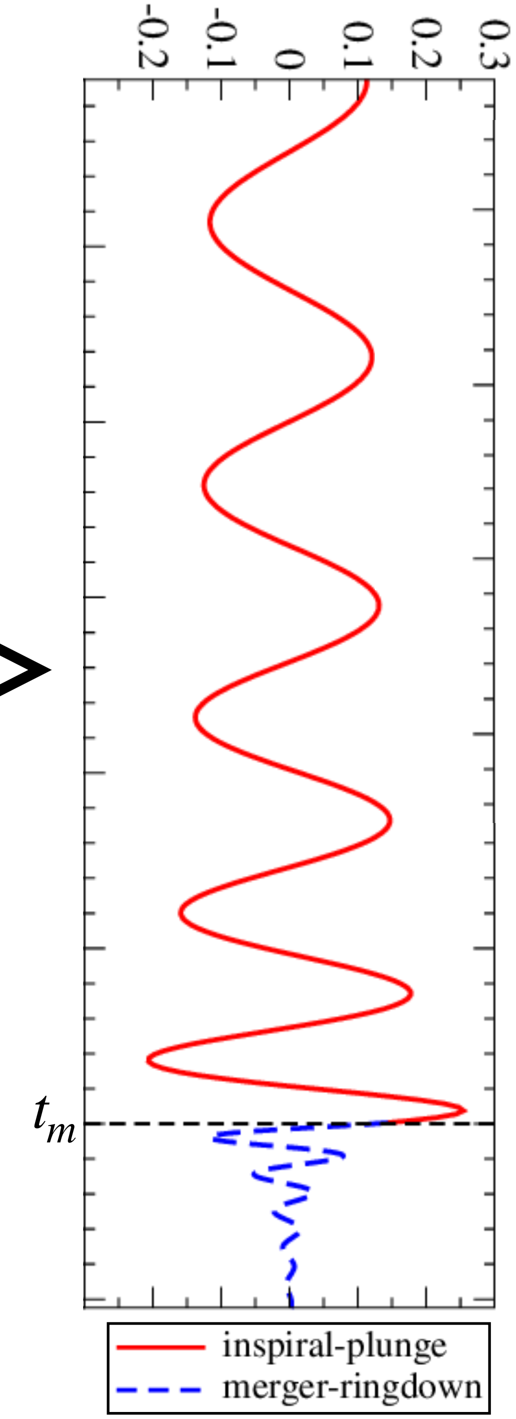

Non-circular EOB inspiral

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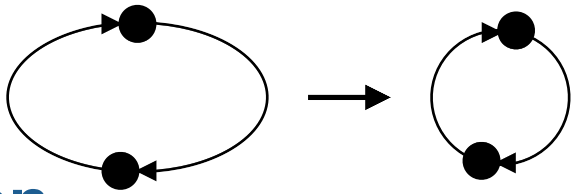
Isolated binaries circularize rapidly…  $\left(e \sim 10^{-6} \text{ at } f_{\rm orb} = 10\text{Hz}\right)$

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Non-circular EOB inspiral

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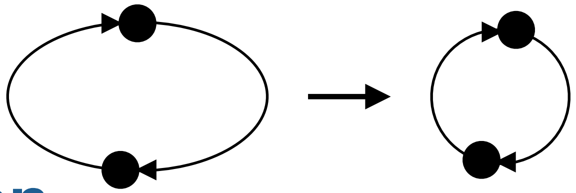

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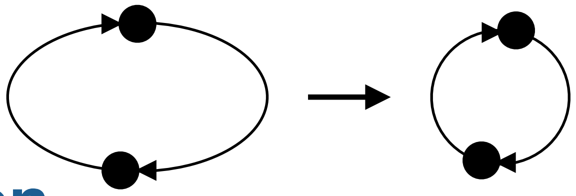

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$$h_+ - i h_\times = D_L^{-1} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} h_{\ell m}\, {}_{-2}Y_{\ell m}(\Theta, \Phi)$$

$$h_{\ell m} = h^{N_{\rm qc}}_{\ell m}\, \hat{h}^{N_{\rm nc}}_{\ell m}\, \hat{h}^{\rm qc}_{\ell m}$$

Newtonian factors

PN quasi-circular factor [Chiaramello-Nagar 2020]

TEOBResumS-DALI before our work

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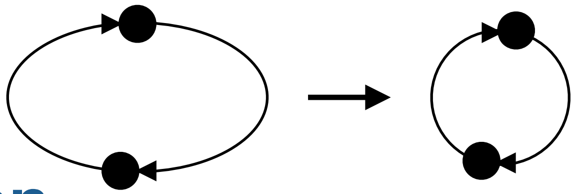

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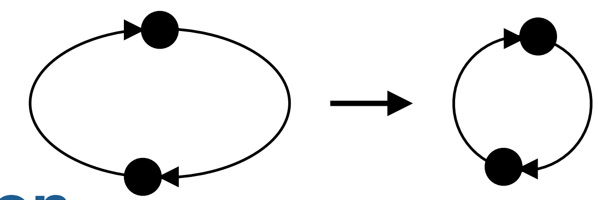

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TEOBResumS-DALI before our work

Non-circular extension of the PN sector with an extra PN non-circular factor for each spherical mode

Obtained by:
- Translating in EOB variables generic-planar-orbit PN results for the spherical modes $h_{\ell m}$
- Factoring out Newtonian and circular contributions

$$h_{\ell m} = h^{N_{\rm qc}}_{\ell m}\, \hat{h}^{N_{\rm nc}}_{\ell m}\, \hat{h}^{\rm qc}_{\ell m}\, \hat{h}^{\rm nc}_{\ell m}$$

TEOBResumS-DALI with our 2PN non-circular extension

Current results

With a first version of our non-circular factors $\hat{h}^{\rm nc}_{\ell m}$ we succeeded in improving how TEOBResumS-DALI deals with non-circularized binaries:

- Increased analytical/numerical agreement of the waveform phase
- More accurate fluxes of energy and angular momentum at infinity

$$\dot{E} = \frac{1}{16\pi} \sum_{\ell=2}^{\ell_{\rm max}} \sum_{m=-\ell}^{\ell} |\dot{h}_{\ell m}|^2 \qquad \dot{J} = -\frac{1}{16\pi} \sum_{\ell=2}^{\ell_{\rm max}} \sum_{m=-\ell}^{\ell} m \Im(\dot{h}_{\ell m} h^*_{\ell m})$$

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Future projects in this direction

- 2.5PN noncircular factors (with the inclusion of oscillatory memory effects) in preparation, almost ready
- Additional noncircular waveform information for spinning binaries

Thanks for your attention!


Illustration by Aleks Sennwald

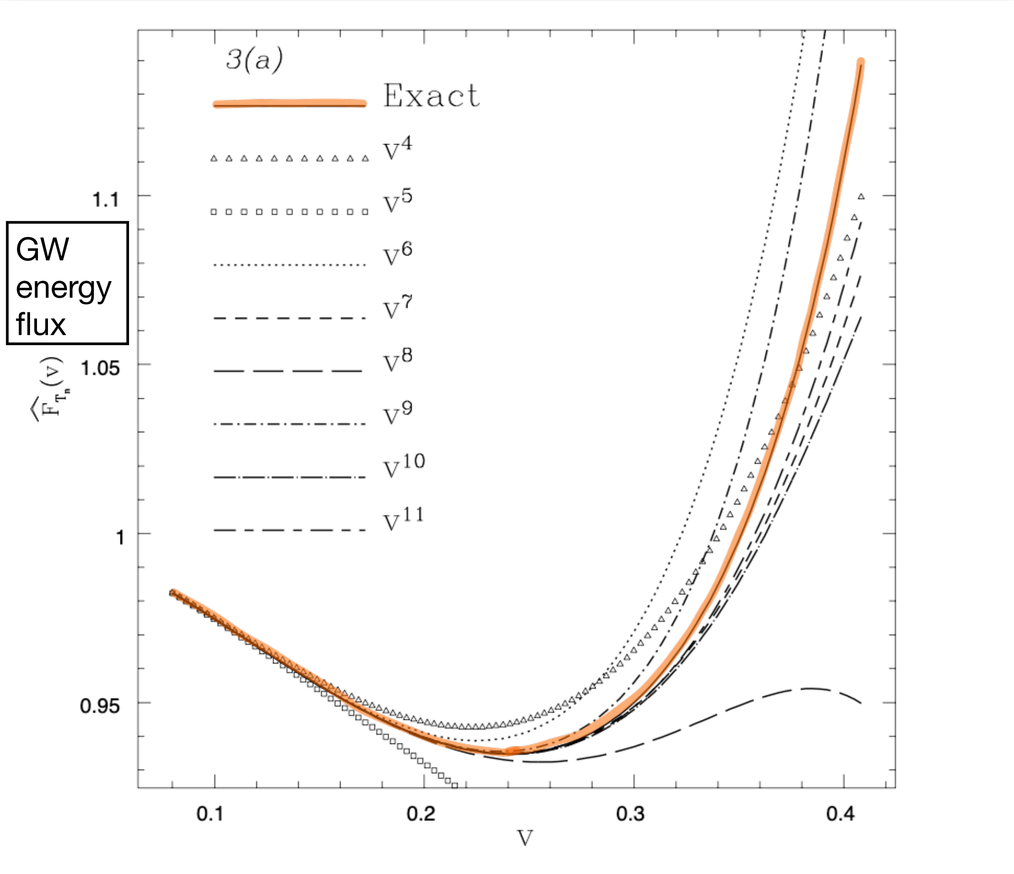
EXTRA SLIDES

Bad convergence of the PN series

Bad convergence of the PN series

- [Cutler et al. ’93]: Slow convergence of the PN series
- [Thorne-Brady-Creighton ’98]: PN results can’t be used in the strong field regime, in the last ~10 orbits of the inspiral they are unreliable
- [Damour-Iyer-Sathyaprakash ’98]: particle around a Schwarzschild BH

The PN series is badly convergent and erratic…



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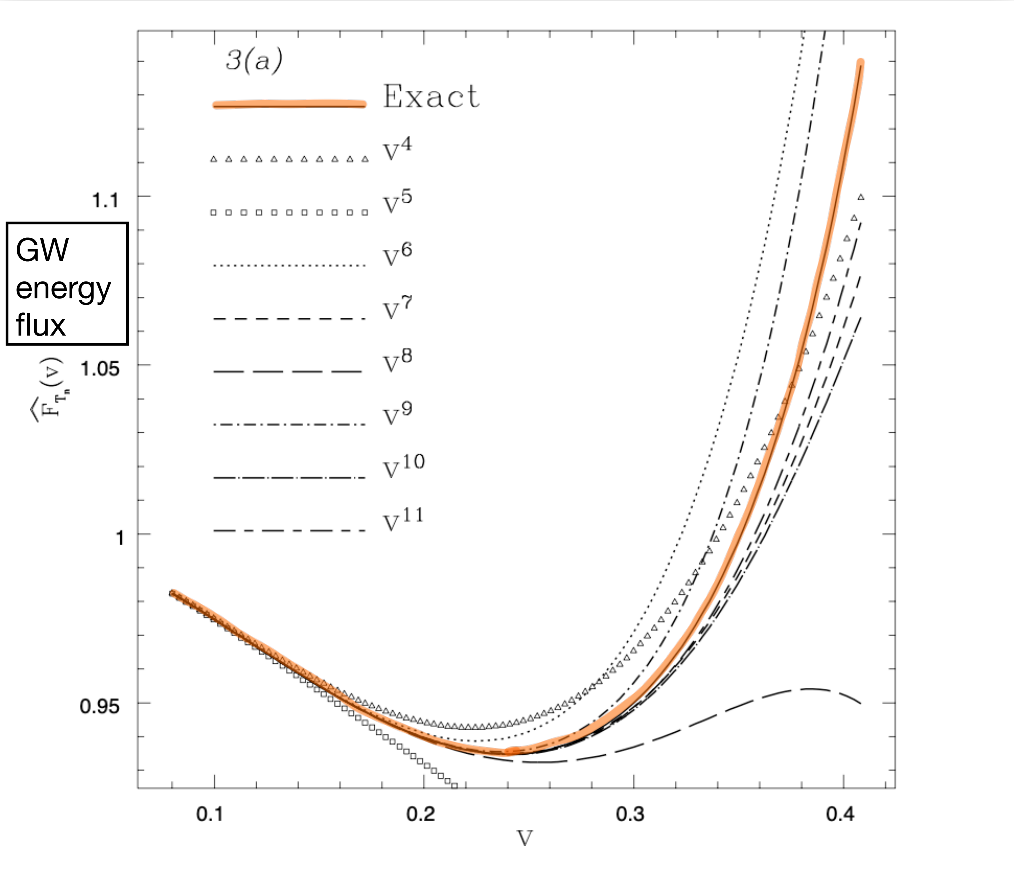
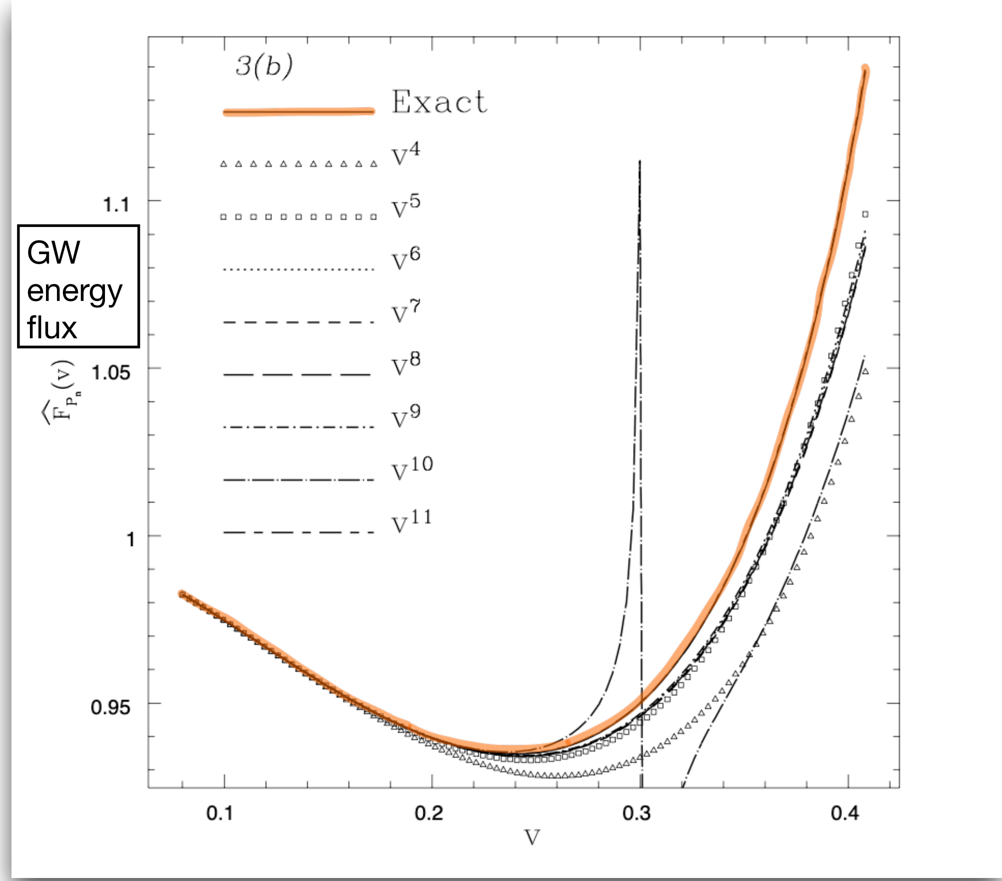
…but after proper resummations there is a substantial improvement!



Resummation with Padé approximants



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Dictionary between the two dynamics (no spin for simplicity):

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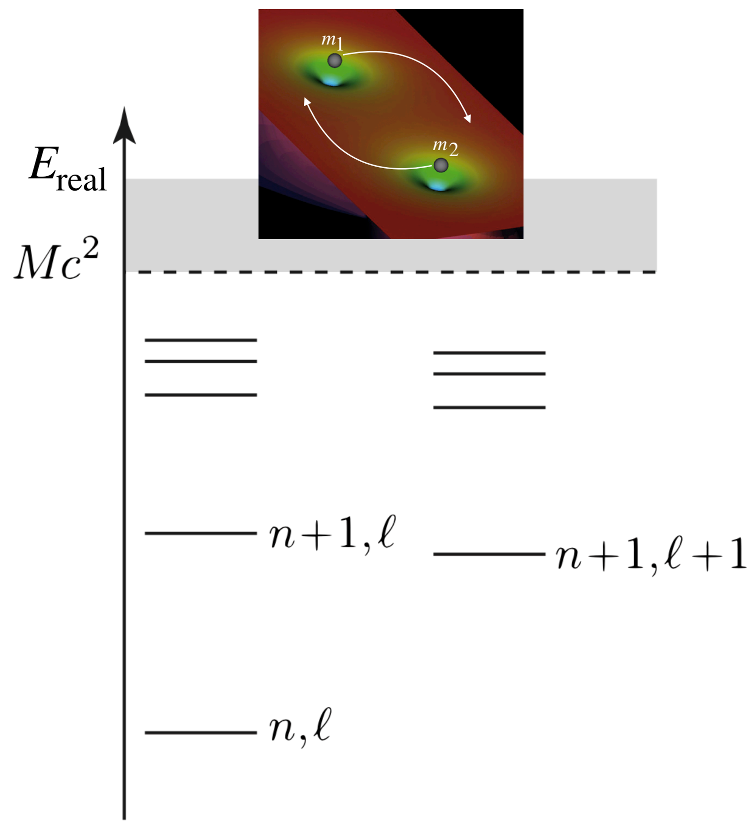

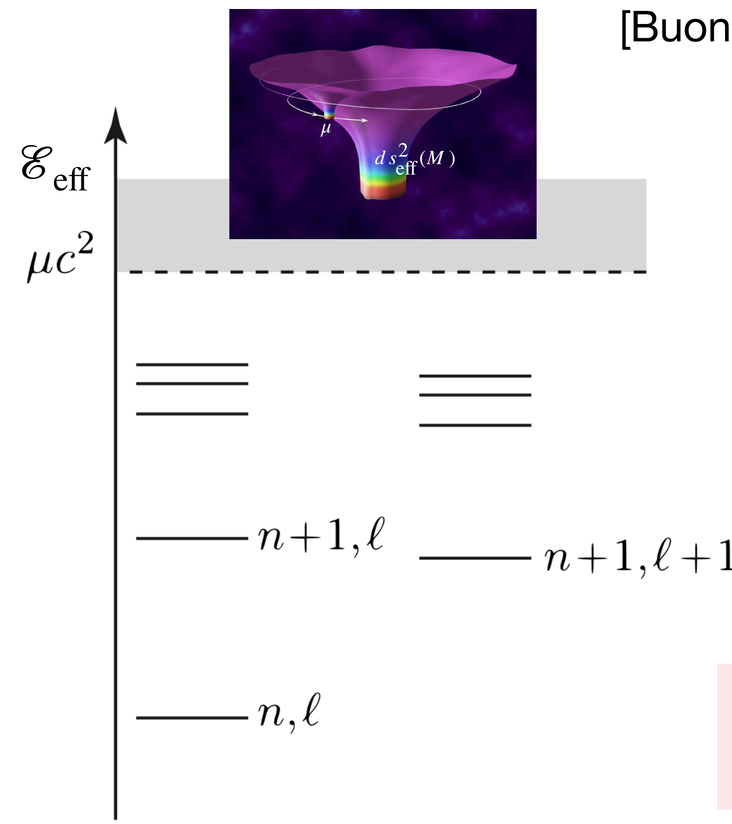

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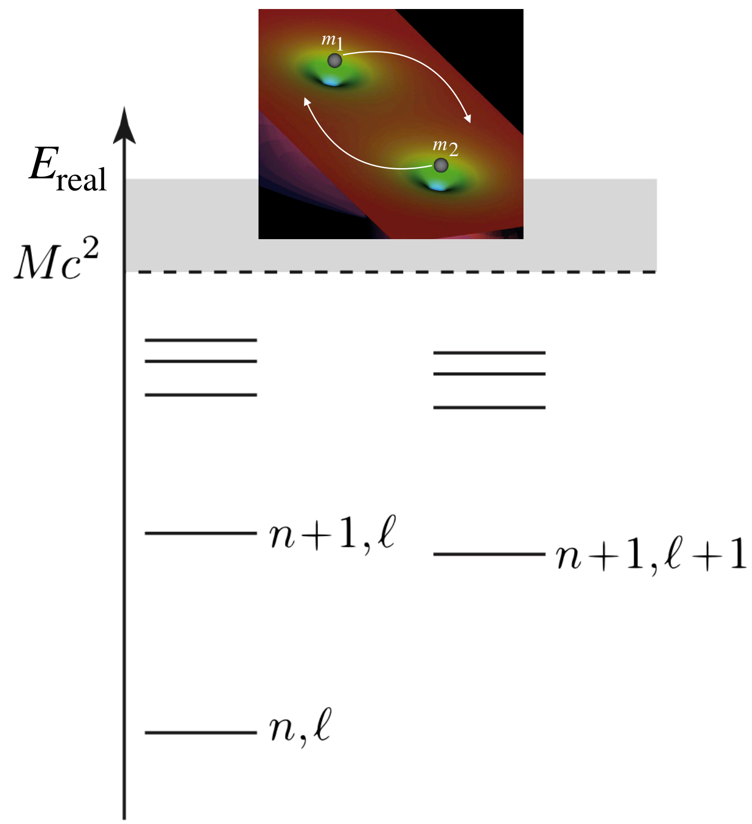

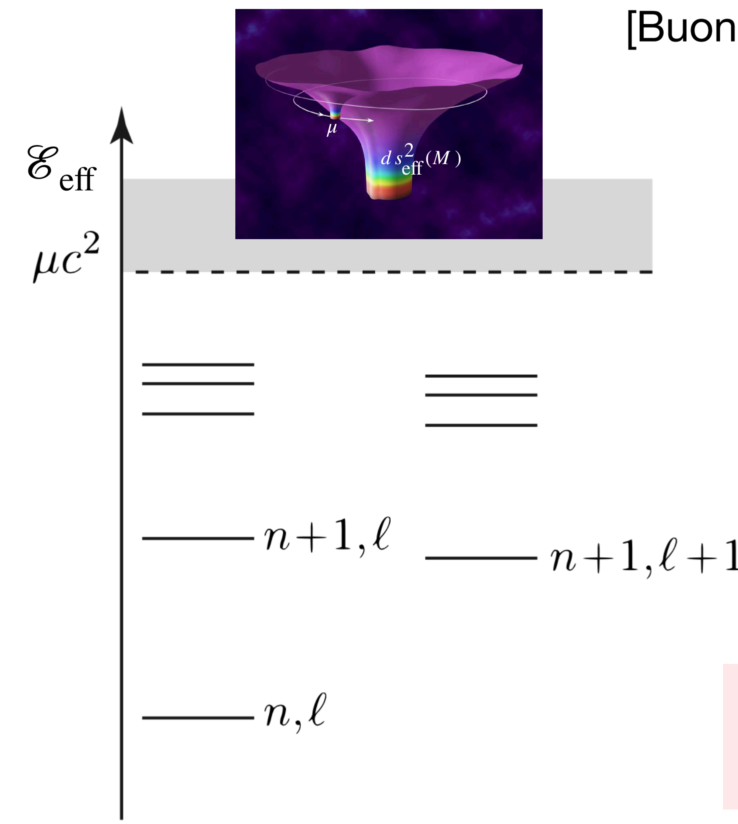

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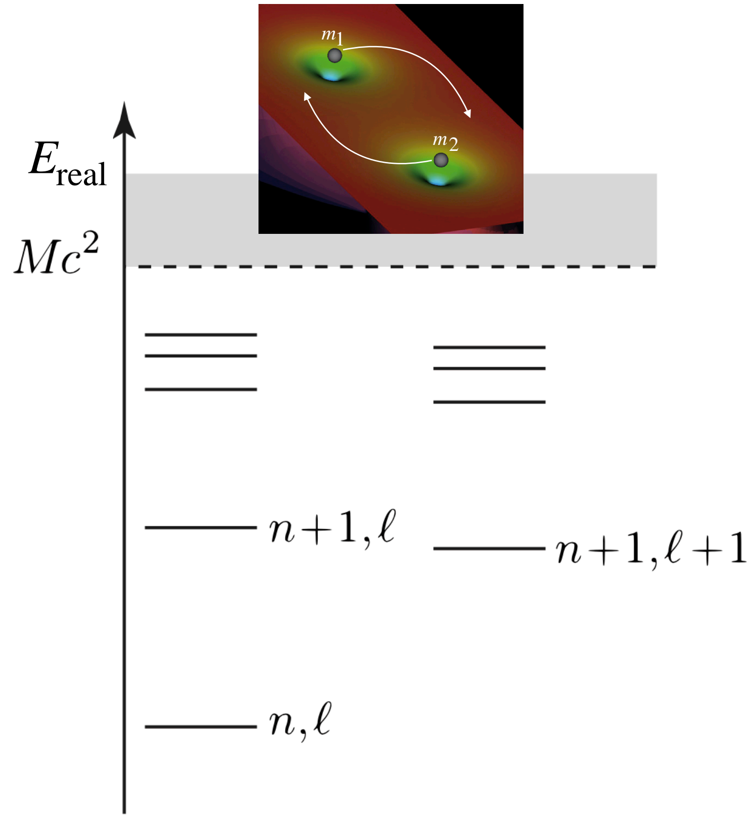

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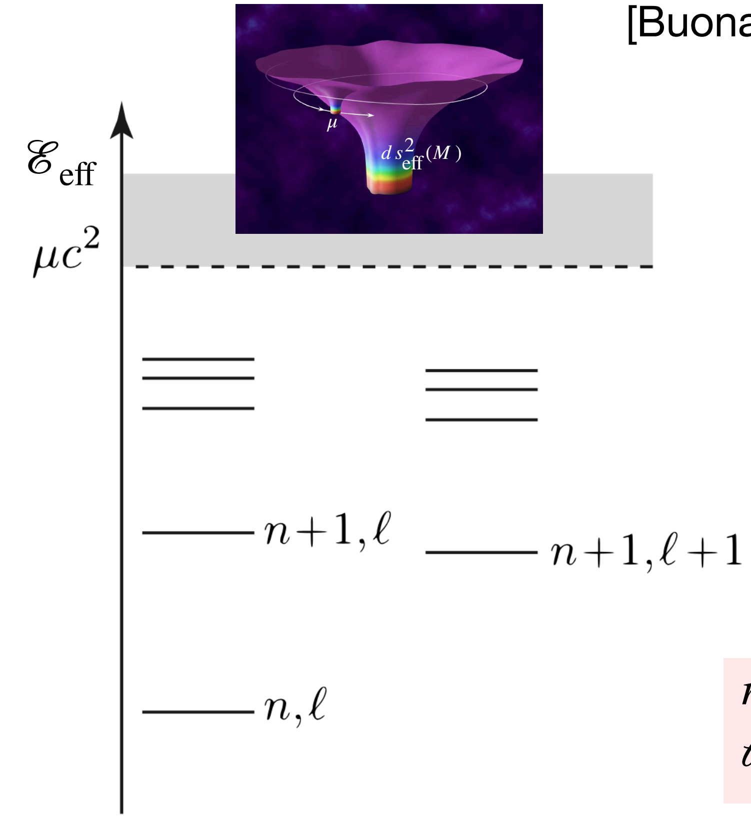

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$\mathscr{E}_{\rm eff}(N_{\rm eff}, J_{\rm eff}) = \mathscr{E}_{\rm eff}(n, \ell)$ $\quad$ $\theta = \pi/2$

obtained from: $\quad$ quartic **mass-shell deformations**

$$g^{\mu\nu}_{\rm eff}\, p_\mu p_\nu + \mu^2 c^2 + Q\big(p_\mu, \{z_i\}\big) = 0$$

$$p_\mu = \partial S_{\rm eff}/\partial x^\mu$$

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- Solve for S_r with $I_r = \frac{2}{2\pi}\int_{r_{\rm min}}^{r_{\rm max}} \frac{dS_r}{dr}\, dr$
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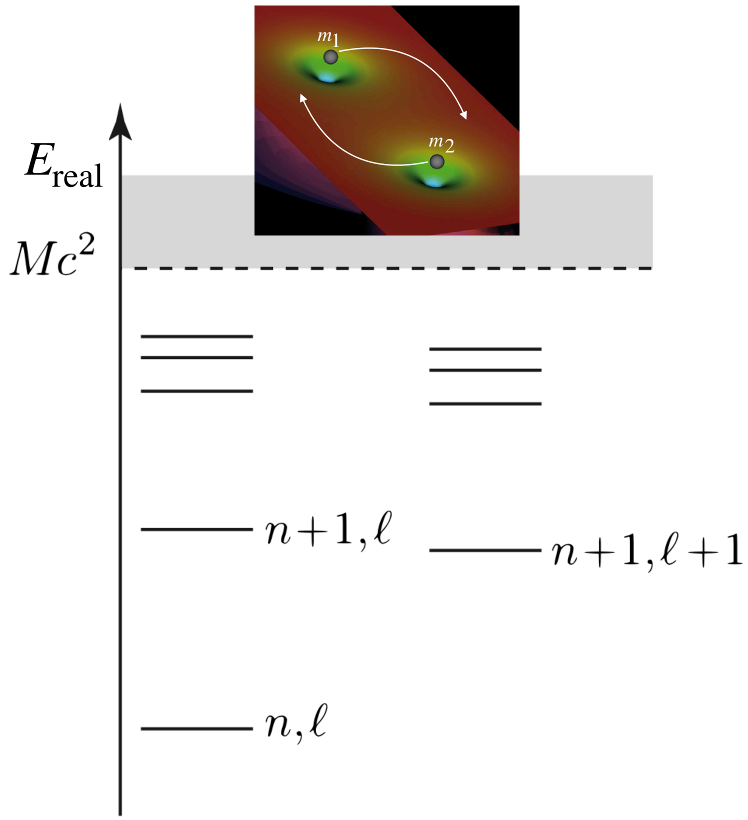

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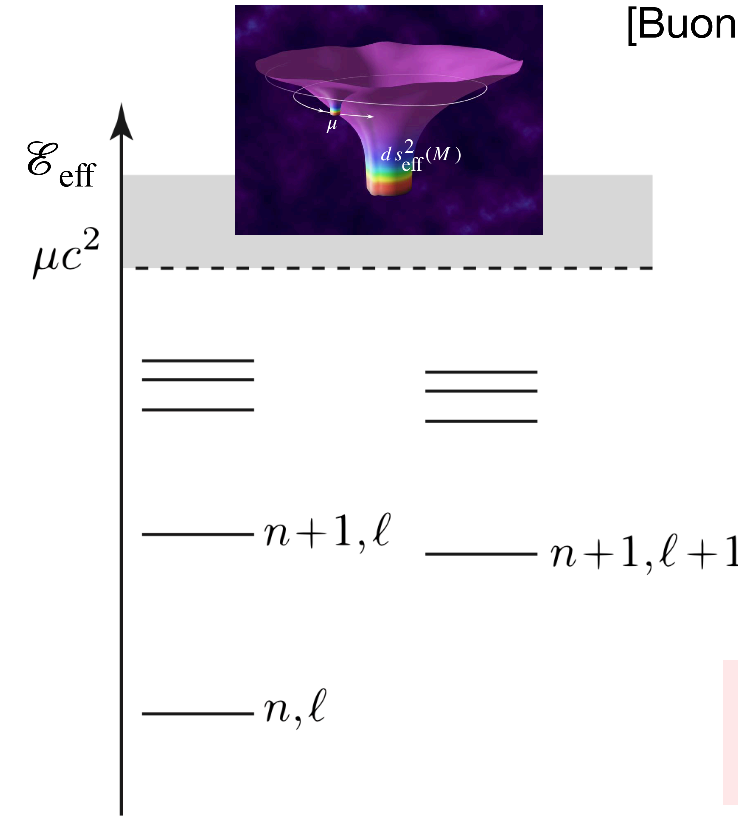

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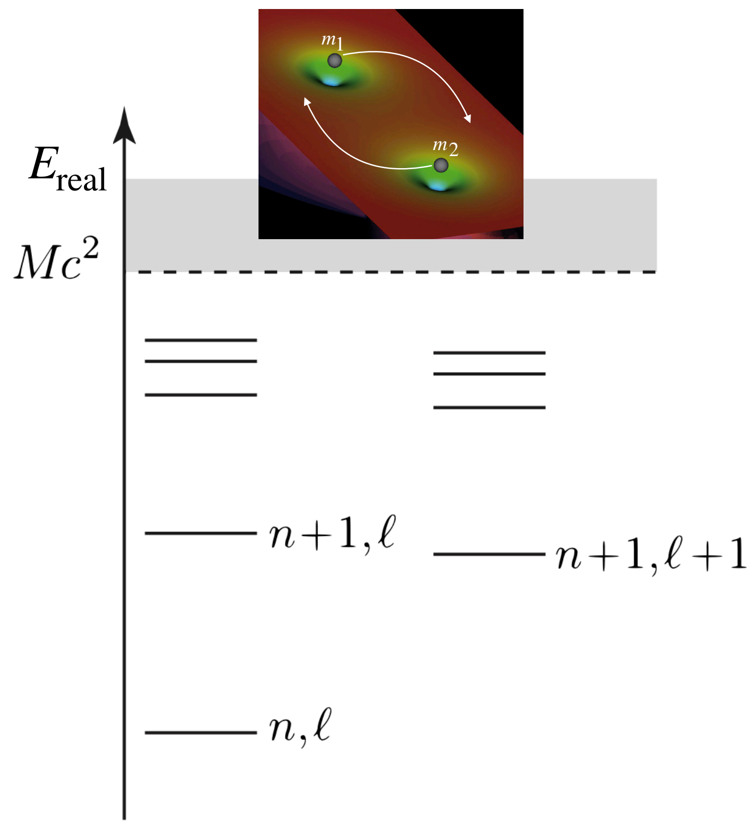

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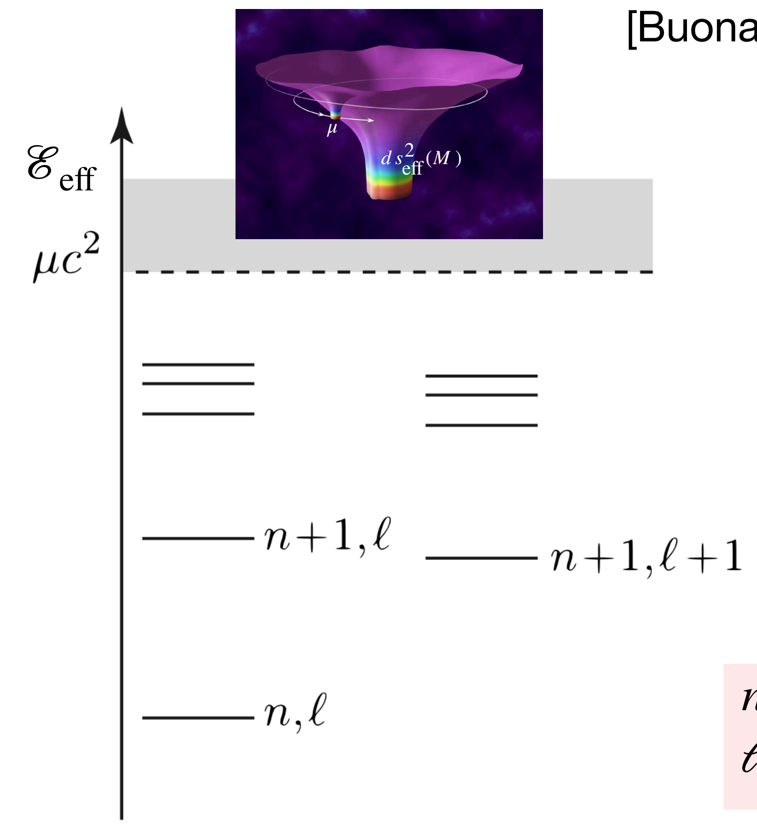

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$$\frac{\mathcal{E}^{\rm NR}_{\rm eff}}{\mu c^2} = \frac{E^{\rm NR}_{\rm real}}{\mu c^2}\left[1 + \alpha_1 \frac{E^{\rm NR}_{\rm real}}{\mu c^2} + \alpha_2 \left(\frac{E^{\rm NR}_{\rm real}}{\mu c^2}\right)^2 + \alpha_3 \left(\frac{E^{\rm NR}_{\rm real}}{\mu c^2}\right)^3 + \dots\right]$$

$\longrightarrow$ System of equations for the parameters $\tilde{a}_i$, $\tilde{b}_i$, z_i, and α_i (underconstrained system)

Importance of non-circularity

However…

- **Dynamical encounters** in dense stellar environments (globular clusters, galactic nuclei) and the **Lidov-Kozai mechanism** in compact triples may lead to CBC with **measurable eccentricity**

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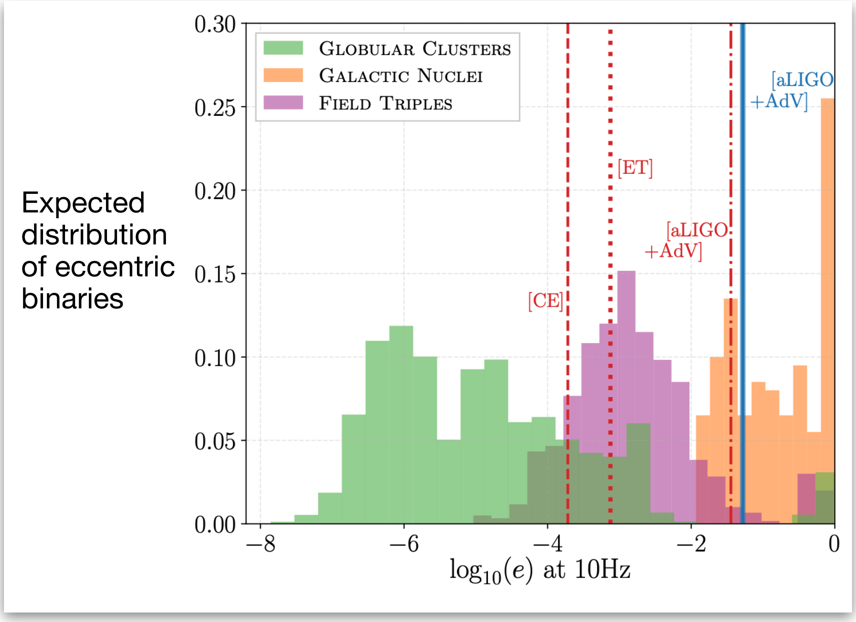

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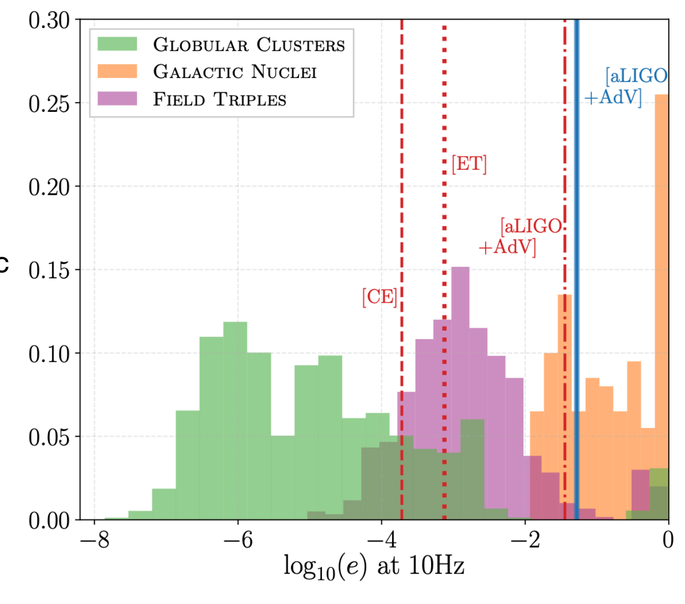
[Gamba et al. 2021]: GW190521 analysis in a dynamical capture scenario

Expected distribution of eccentric binaries



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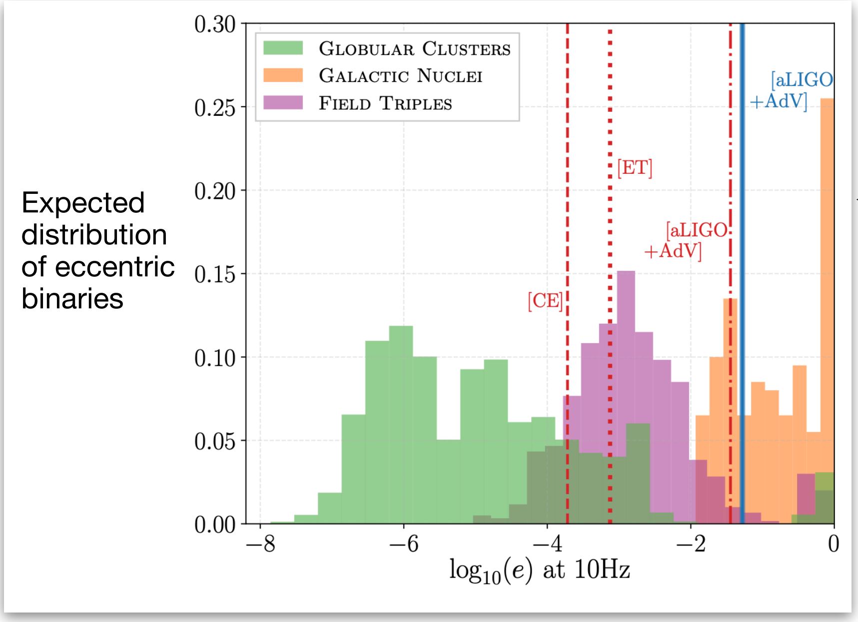

- **Neglecting eccentricity** can cause **systematic errors** in parameter inference [Favata 2014, Favata et al 2022] and induce **bias** in GR tests [Bhat et al 2022]

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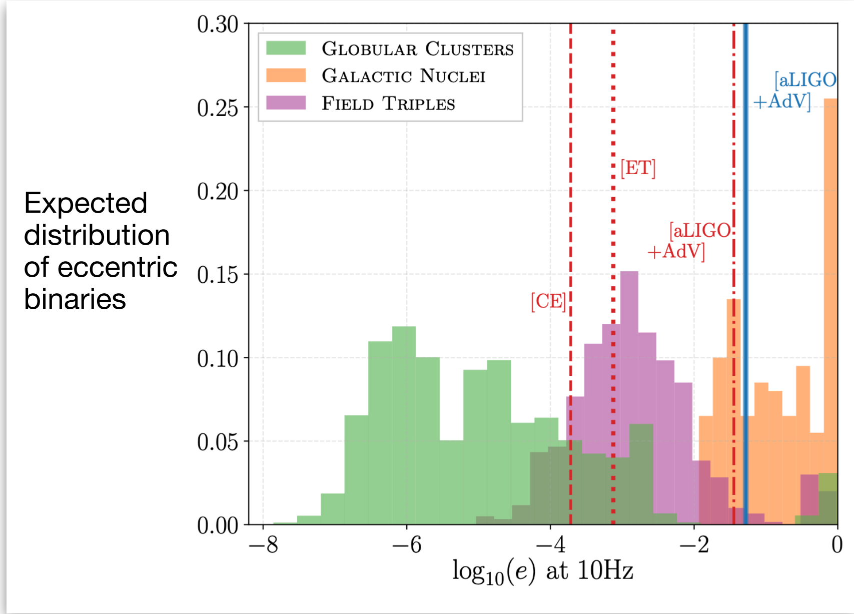

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The orbital eccentricity has a significant role in CBC waveform models!
EOB models need corresponding **non-circular corrections**

More details on the PN qc waveform factor

Native quasi-circular version of TEOBResumS:

$$h_+ - ih_\times = D_L^{-1} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} h_{\ell m} \, {}_{-2}Y_{\ell m}(\Theta, \Phi)$$

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Leading contribution of $h_{\ell m}$ in its PN expansion

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Residual PN information in factorized form

$$\hat{h}_{\ell m}^{\rm qc} = \hat{S}_{\rm eff} T_{\ell m} \times e^{i\delta_{\ell m}} f_{\ell m} \times \hat{h}_{\ell m}^{\rm NQC}$$

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Effective source

$$\hat{S}_{\rm eff} = \begin{cases} \hat{H}_{\rm eff} & \text{even } \ell + m \\ p_\varphi / r^2 \dot{\varphi} & \text{odd } \ell + m \end{cases}$$

Resummed tail logarithms

$$T_{\ell m} = \frac{\Gamma\left(\ell + 1 - 2i\hat{\hat{k}}\right)}{\Gamma\left(\ell + 1\right)} e^{\pi \hat{\hat{k}}} e^{2i\hat{\hat{k}} \log(2kr_0)}$$

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Residual phase

Residual amplitude

$$f_{\ell m} \rightarrow (\rho_{\ell m})^{\ell}$$
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Next-to-Quasi-Circular factor

Non-circular Newtonian factor

First non-circular extension:

$$h_+ - ih_\times = D_L^{-1} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} h_{\ell m} \, {}_{-2}Y_{\ell m}(\Theta, \Phi)$$

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[Chiaramello-Nagar 2020]

$$h_{\ell m} = h^{N_{\rm qc}}_{\ell m} \, \hat{h}^{N_{\rm nc}}_{\ell m} \, \hat{h}^{\rm qc}_{\ell m}$$

Quasi-circular approximation relaxed in the **Newtonian sector**:

$$h^N_{\ell m} \equiv \begin{cases} \dfrac{d^\ell}{dt^\ell}\left(r^\ell e^{-im\varphi}\right) & \text{even } \ell + m \\ \dfrac{d^\ell}{dt^\ell}\left(r^{\ell+1}\dot{\varphi} e^{-im\varphi}\right) & \text{odd } \ell + m \end{cases}$$

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Example - Newtonian factors of h_{22}:

$$h_{22}^{N_{\rm qc}} = -8\sqrt{\frac{\pi}{5}}(r\dot{\varphi})^2 e^{-2i\varphi}$$

$$\hat{h}_{22}^{N_{\rm nc}} = 1 - \frac{\ddot{r}}{2r\dot{\varphi}^2} - \frac{\dot{r}^2}{2(r\dot{\varphi})^2} + i\left(\frac{2\dot{r}}{r\dot{\varphi}} + \frac{\ddot{\varphi}}{2\dot{\varphi}^2}\right)$$

Note:
the time derivatives of the EOB variables (besides the orbital frequency $\Omega \equiv \dot{\varphi}$) **resum non-circular contribution at every PN order**

Non-circular Newtonian factor

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Non-circular PN information is still missing!

More on our nc factors

Noncircular extension of the PN sector:

$$h_+ - i h_\times = D_L^{-1} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} h_{\ell m} \, {}_{-2}Y_{\ell m}(\Theta, \Phi)$$

$$h_{\ell m} = h_{\ell m}^{N_{\rm qc}} \hat{h}_{\ell m}^{N_{\rm nc}} \hat{h}_{\ell m}^{\rm qc} \hat{h}_{\ell m}^{\rm nc}$$

$\hat{h}_{\ell m}^{\rm nc} \rightarrow$ **Noncircular PN factor** (2PN accurate for now)

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Calculation procedure:

Starting generic-orbit waveform mode

$$h_{\ell m} = h_{\ell m}^{N} + \frac{1}{c^2} h_{\ell m}^{1{\rm PN}_{\rm inst}} + \frac{1}{c^3} h_{\ell m}^{1.5{\rm PN}_{\rm tail}} + \frac{1}{c^4} h_{\ell m}^{2{\rm PN}_{\rm inst}} + O\left(c^{-5}\right)$$

Obtained by **translating in EOB variables** the generic-orbit spherical modes $h_{\ell m}$ provided in [Mishra-Arun-Iyer 2015], [Boetzel et al. 2019], [Khalil et al. 2021]

More on our nc factors

Noncircular extension of the PN sector:

$$h_+ - i h_\times = D_L^{-1} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} h_{\ell m} \, {}_{-2}Y_{\ell m}(\Theta, \Phi)$$

$$h_{\ell m} = h_{\ell m}^{N_{\rm qc}} \hat{h}_{\ell m}^{N_{\rm nc}} \hat{h}_{\ell m}^{\rm qc} \hat{h}_{\ell m}^{\rm nc}$$

$\hat{h}_{\ell m}^{\rm nc}$ → **Noncircular PN factor** (2PN accurate for now)

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$$\hat{h}_{\ell m} \equiv \frac{h_{\ell m}}{h_{\ell m}^{N}}$$

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$$\hat{h}_{\ell m} \equiv \frac{h_{\ell m}}{h_{\ell m}^{N}} \xrightarrow{\text{Circular limit}} \hat{h}_{\ell m}^{\rm qc}$$

Circular limit: $p_r \rightarrow 0$ as well as all the time derivatives of the EOB variables, except for $\dot{\varphi}$

More on our nc factors

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Circular limit $\rightarrow \hat{h}^{\rm qc}_{\ell m}$

Circular limit: $p_r \rightarrow 0$ as well as all the time derivatives of the EOB variables, except for $\dot{\varphi}$

$$\hat{h}^{\rm nc}_{\ell m} \equiv T_{\rm 2PN}\left[\frac{\hat{h}_{\ell m}}{\hat{h}^{\rm qc}_{\ell m}}\right]$$

$T_{\rm 2PN} \rightarrow$ PN Taylor expansion truncated at the 2PN order

More on our nc factors

Noncircular extension of the PN sector:

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$\hat{h}_{\ell m}^{nc}$ → **Noncircular PN factor** (2PN accurate for now)

Internal structure:

$$\hat{h}_{\ell m}^{nc} \equiv T_{2PN}\left[\frac{\hat{h}_{\ell m}}{\hat{h}_{\ell m}^{qc}}\right] = 1 + \frac{1}{c^2}\hat{h}_{\ell m}^{1PN_{inst,nc}} + \frac{1}{c^3}\hat{h}_{\ell m}^{1.5PN_{tail,nc}} + \frac{1}{c^4}\hat{h}_{\ell m}^{2PN_{inst,nc}}$$

More on our nc factors

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Additional factorization

$$\hat{h}_{\ell m}^{\rm nc} = \hat{h}_{\ell m}^{\rm nc_{inst}} \hat{h}_{\ell m}^{\rm nc_{tail}}$$

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$$\hat{h}_{\ell m}^{nc_{tail}} \equiv 1 + \frac{1}{c^3}\hat{h}_{\ell m}^{1.5PN_{tail,nc}}$$

We developed two versions of our extra factor:

- [Placidi et al. 2022, Albanesi et al. 04/2022] → $\hat{h}_{\ell m}^{nc}[1]$
- [Albanesi et al. 06/2022] → $\hat{h}_{\ell m}^{nc}[2]$

Difference in the computation of the instantaneous factor $\hat{h}_{\ell m}^{nc_{inst}}$

The two version of our factors

	Spherical modes in **radiative multipoles**	In terms of their **symmetric trace free (STF)** counterparts ($L \equiv i_1, \ldots, i_\ell$)	In terms of the **STF multipoles of the source**
even $\ell + m$ →	$h_{\ell m} = -\dfrac{U_{\ell m}}{\sqrt{2}\, c^{\ell+2}}$ →	$U_{\ell m} = \dfrac{4}{\ell!}\sqrt{\dfrac{(\ell+1)(\ell+2)}{2\ell(\ell+1)}}\alpha^L_{\ell m} U_L$ →	$U_L = \dfrac{d^\ell}{dt^\ell} I_L + O(c^{-3})$
odd $\ell + m$ →	$h_{\ell m} = i\dfrac{V_{\ell m}}{\sqrt{2}\, c^{\ell+3}}$ →	$V_{\ell m} = -\dfrac{8}{\ell!}\sqrt{\dfrac{\ell(\ell+2)}{2(\ell-1)(\ell+1)}}\alpha^L_{\ell m} V_L$ →	$V_L = \dfrac{d^\ell}{dt^\ell} J_L + O(c^{-3})$

Nonlinearities (↕ $O(c^{-3})$ terms)

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Two versions of our noncircular PN factors:

- $\hat{h}^{\rm nc}_{\ell m}[1]$ → All the time derivatives of the EOB variables are removed using the 2PN-expanded equations of motion
- $\hat{h}^{\rm nc}_{\ell m}[2]$ → In the instantaneous part we keep them explicit → $\hat{h}^{\rm nc_{inst}}_{\ell m}$ is a PN generalization of $\hat{h}^{N_{\rm nc}}_{\ell m}$

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Example - Instantaneous noncircular PN factor for the mode h_{22}:

$h^{\rm inst}_{22} \sim U^{\rm inst}_{22} \sim U^{\rm inst}_{ij} = \ddot{I}_{ij} + O(c^{-5})$ → 2PN eqs. of motion → $h^{\rm inst}_{22}(r, \varphi, p_r, p_\varphi)$ —factorization→ $\hat{h}^{\rm nc_{inst}}_{22}$ of $\hat{h}^{\rm nc}_{22}[1]$

↳ $h^{\rm inst}_{22}(r, \dot{r}, \ddot{r}, \varphi, \dot{\varphi}, \ddot{\varphi}, p_r, \dot{p}_r, \ddot{p}_r, p_\varphi, \dot{p}_\varphi, \ddot{p}_\varphi)$ —factorization→ $\hat{h}^{\rm nc_{inst}}_{22}$ of $\hat{h}^{\rm nc}_{22}[2]$

Waveform results [1]: test-mass limit

Simple testing ground: GW of a test-mass plunging on a black hole ($\nu \to 0$)

Models:

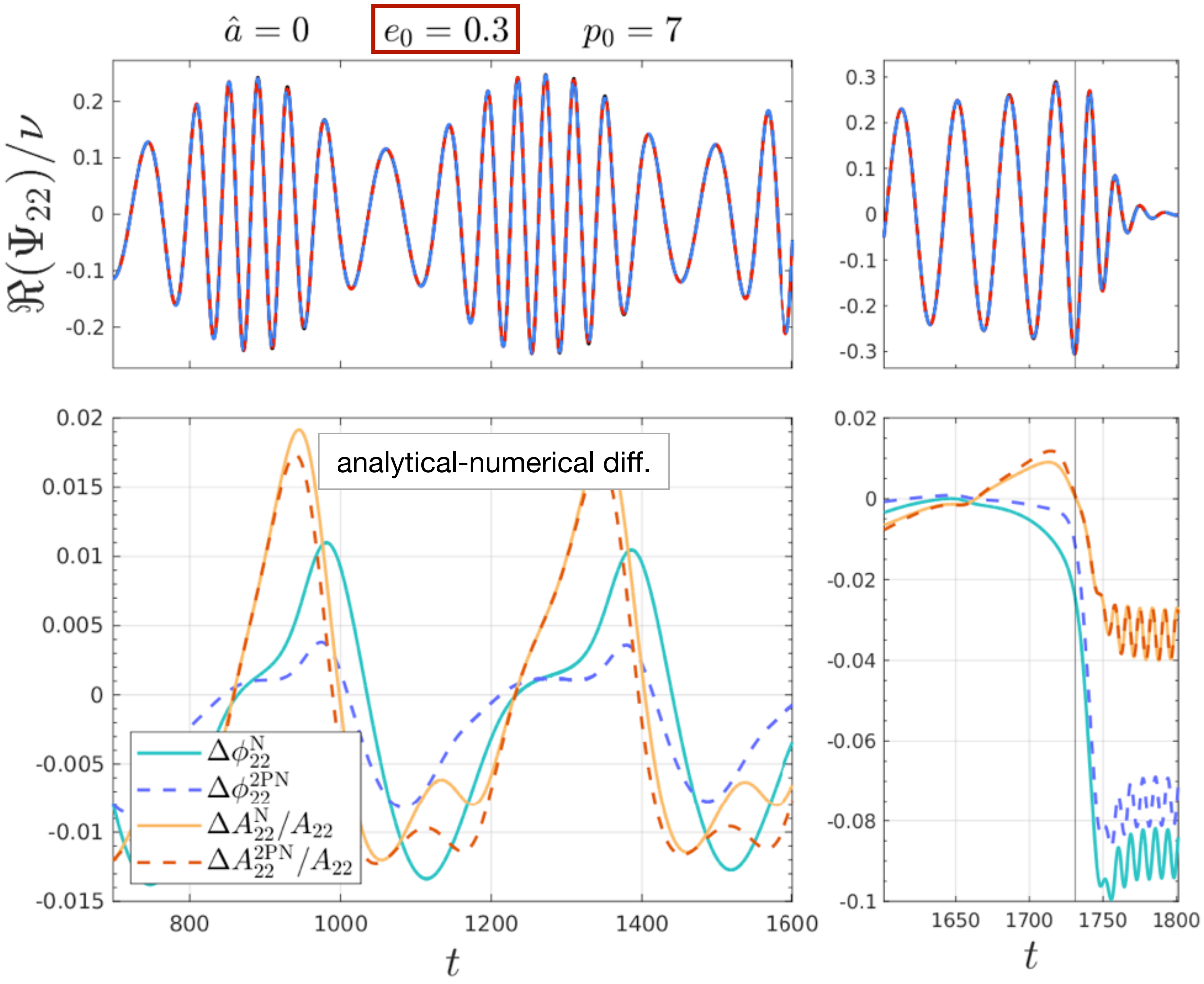
- N $\to$ $h_{\ell m} = h_{\ell m}^{N_{qc}} \hat{h}_{\ell m}^{N_{nc}} \hat{h}_{\ell m}^{qc}$
- 2PN $\to$ $h_{\ell m} = h_{\ell m}^{N_{qc}} \hat{h}_{\ell m}^{N_{nc}} \hat{h}_{\ell m}^{qc} \hat{h}_{\ell m}^{nc}[1]$

$\Psi_{\ell m} \equiv h_{\ell m}/\sqrt{(\ell+2)(\ell+1)\ell(\ell-1)}$

$\hat{a} \equiv$ BH spin
$e_0 \equiv$ initial eccentricity
$p_0 \equiv$ initial semilatus rectum

$$\phi_{\ell m} \equiv \arctan \frac{\Im(h_{\ell m})}{\Re(h_{\ell m})}$$

$$A_{\ell m} \equiv |h_{\ell m}|$$



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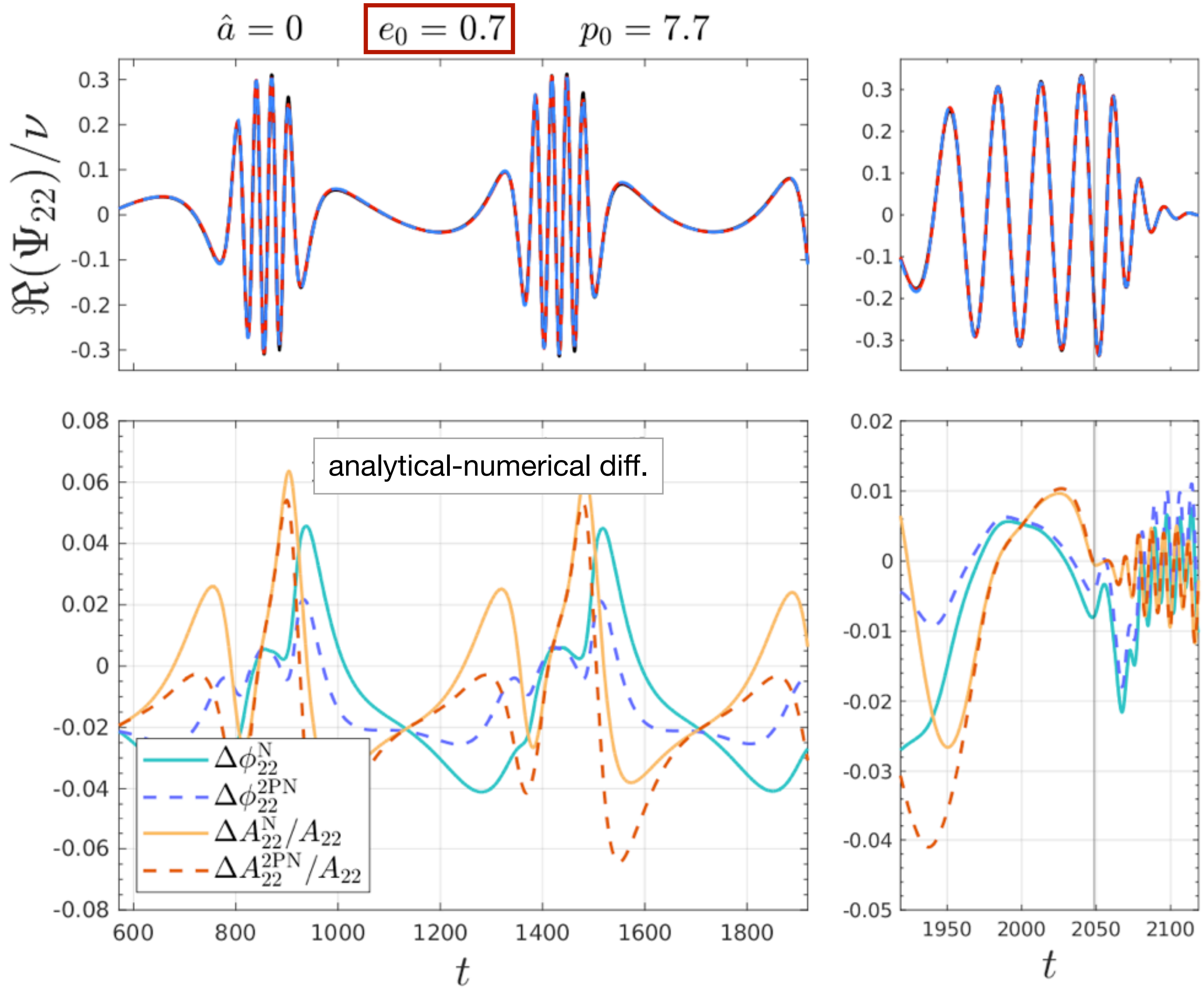

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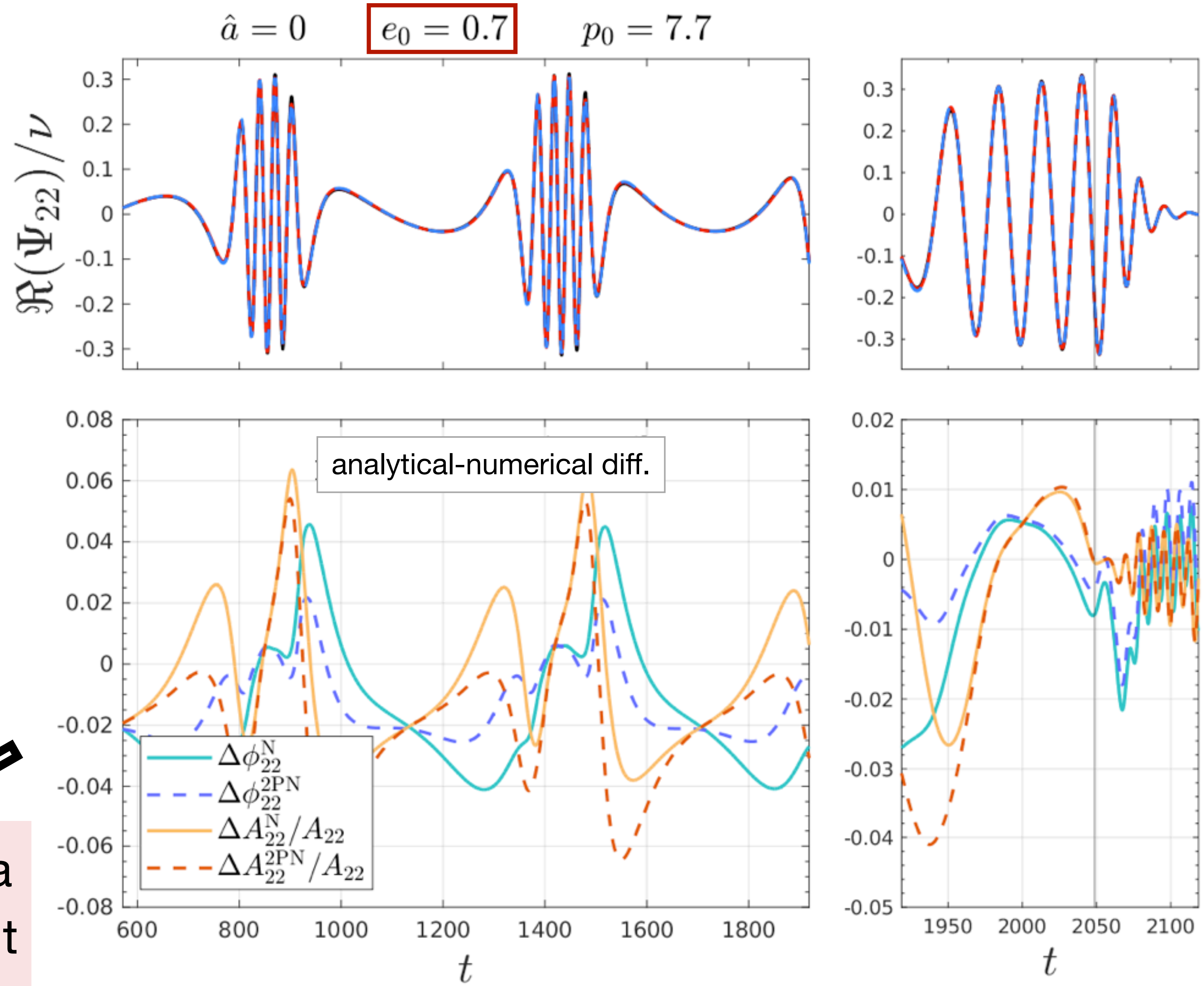
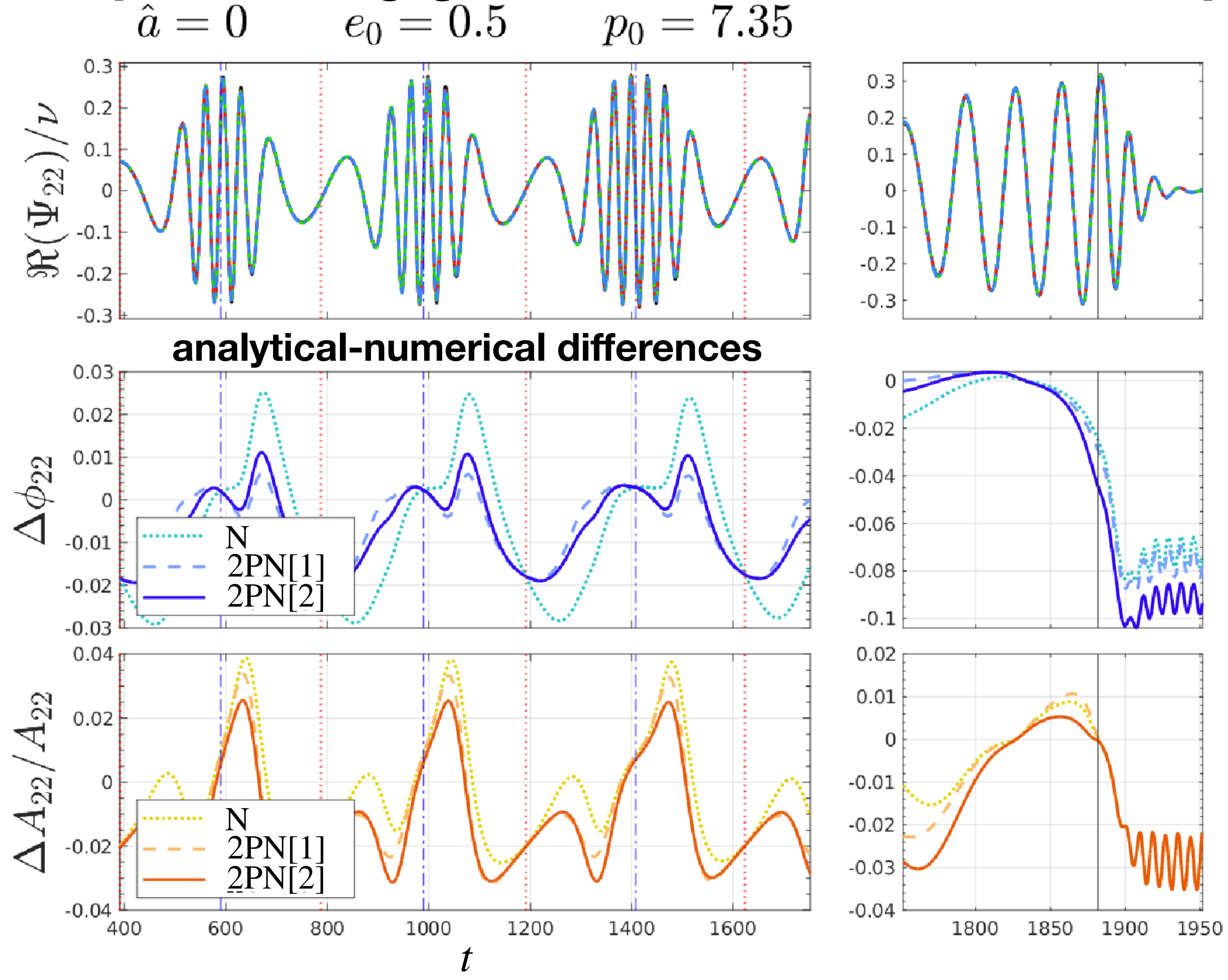


For **each initial eccentricity**, the extra $\hat{h}_{\ell m}^{nc}[1]$ factor **improves the phase** but has a marginal effect on the amplitude

Waveform results [2]: test-mass limit

Simple testing ground: GW of a test-mass plunging on a black hole ($\nu \to 0$)



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- 2PN[2] $\rightarrow$ $h_{\ell m} = h^{N_{\rm qc}}_{\ell m}\, \hat{h}^{N_{\rm nc}}_{\ell m}\, \hat{h}^{\rm qc}_{\ell m}\, \hat{h}^{\rm nc}_{\ell m}[2]$

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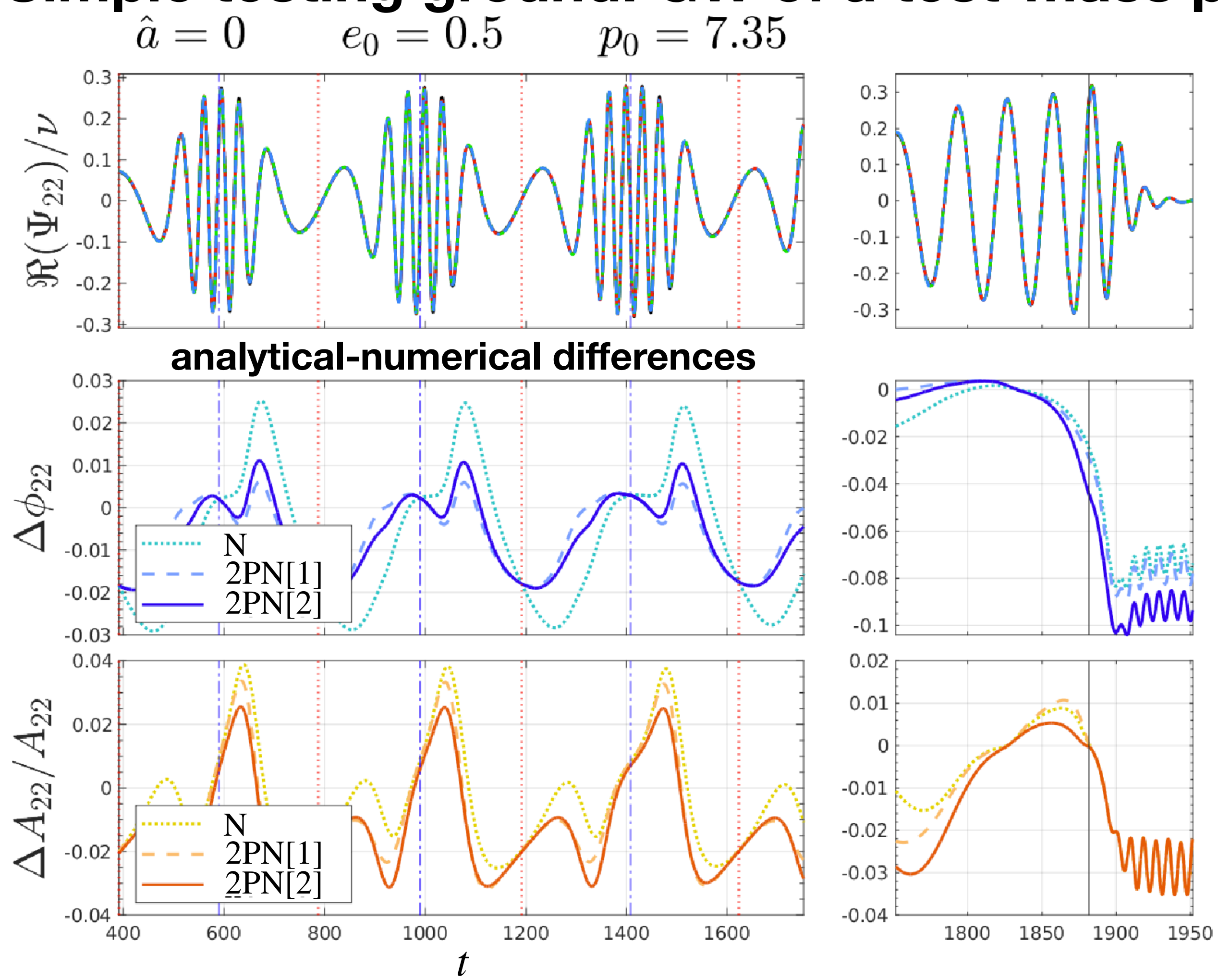
Simple testing ground: GW of a test-mass plunging on a black hole ($\nu \to 0$)



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$\Rightarrow$ $\hat{h}_{\ell m}^{\rm nc}[1]$ and $\hat{h}_{\ell m}^{\rm nc}[2]$ give similar phase corrections but $\hat{h}_{\ell m}^{\rm nc}[2]$ also yields a small but significant improvement at the level of the amplitude

Waveform results [2]: test-mass limit

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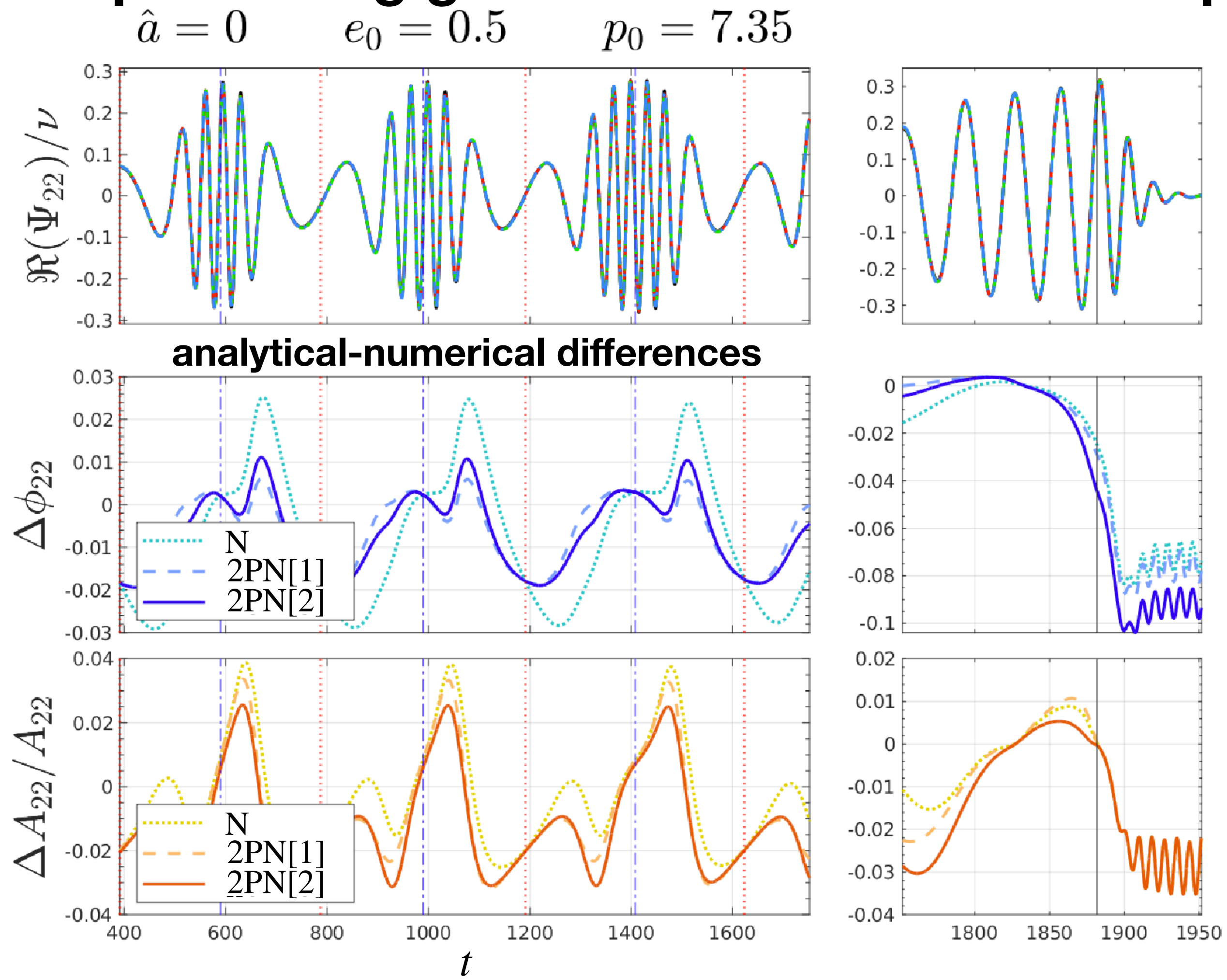

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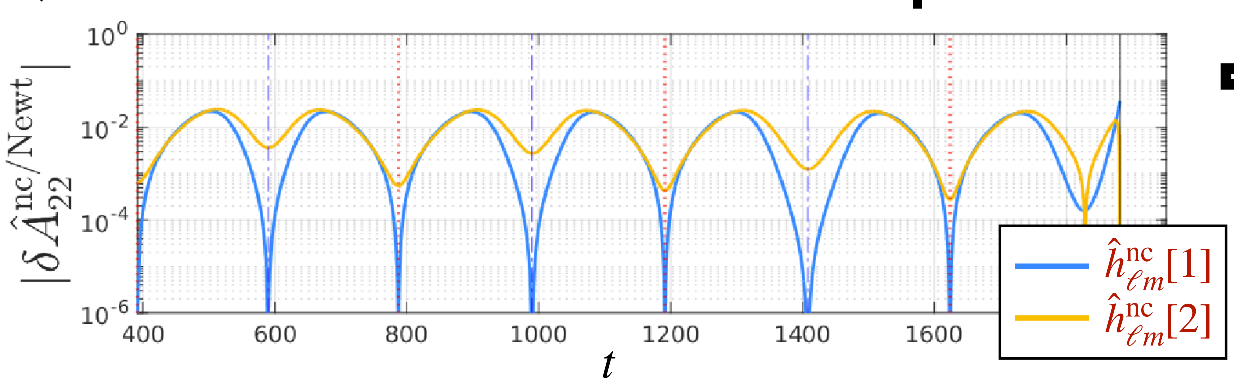
Qualitative difference in the amplitude corrections [1] and [2]:



$\Rightarrow$ As opposed to $\hat{h}_{\ell m}^{\rm nc}[1]$, $\hat{h}_{\ell m}^{\rm nc}[2]$ brings amplitude corrections that do not vanish at the apastra and periastra (vertical lines in the plot) of the orbital motion

Flux results for the geodesic motion [1]-[2]

Analytical/numerical relative differences averaged over a geodesic orbit with p=9

Models:

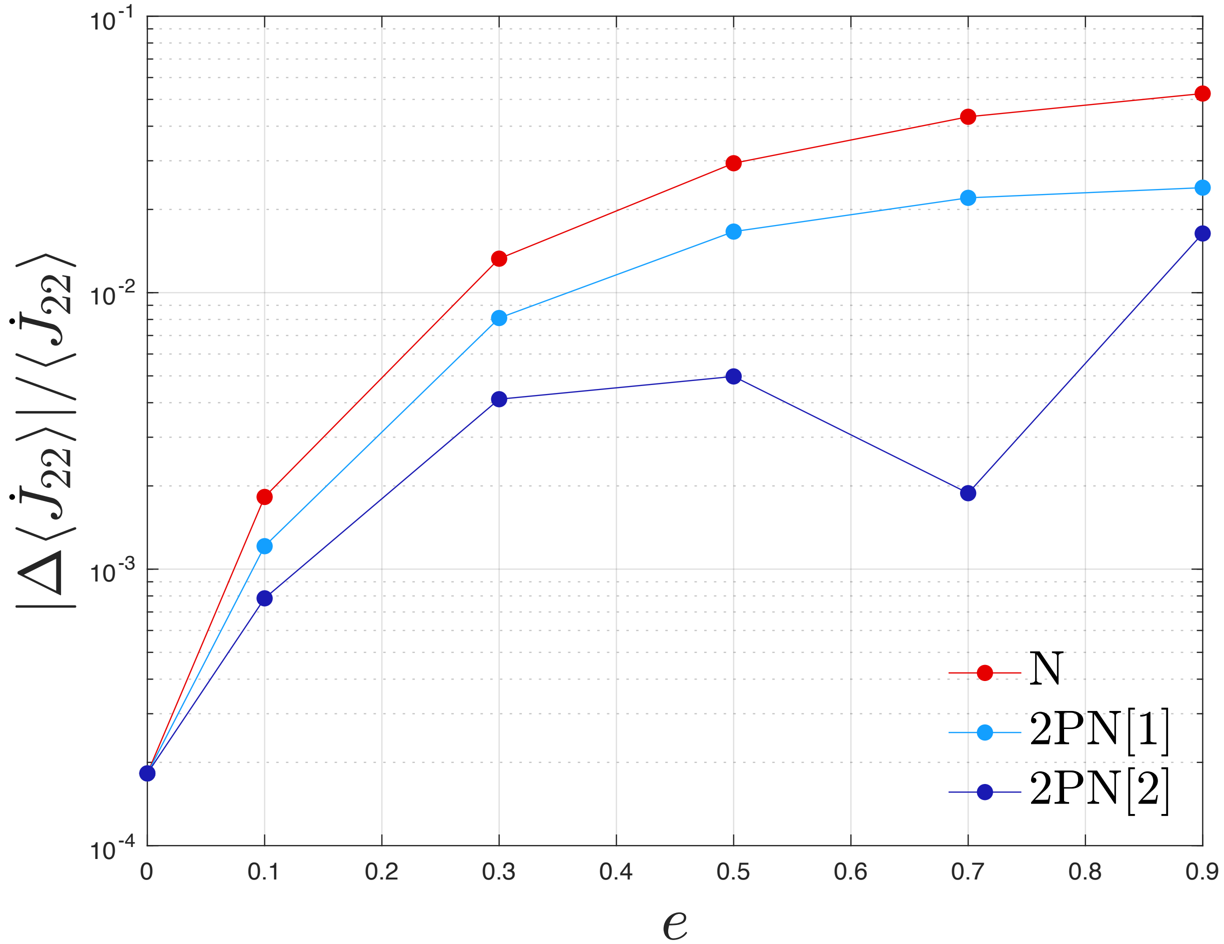
- N → $h_{\ell m} = h_{\ell m}^{N_{\rm qc}} \hat{h}_{\ell m}^{N_{\rm nc}} \hat{h}_{\ell m}^{\rm qc}$
- 2PN[1] → $h_{\ell m} = h_{\ell m}^{N_{\rm qc}} \hat{h}_{\ell m}^{N_{\rm nc}} \hat{h}_{\ell m}^{\rm qc} \; \hat{h}_{\ell m}^{\rm nc}[1]$
- 2PN[2] → $h_{\ell m} = h_{\ell m}^{N_{\rm qc}} \hat{h}_{\ell m}^{N_{\rm nc}} \hat{h}_{\ell m}^{\rm qc} \; \hat{h}_{\ell m}^{\rm nc}[2]$

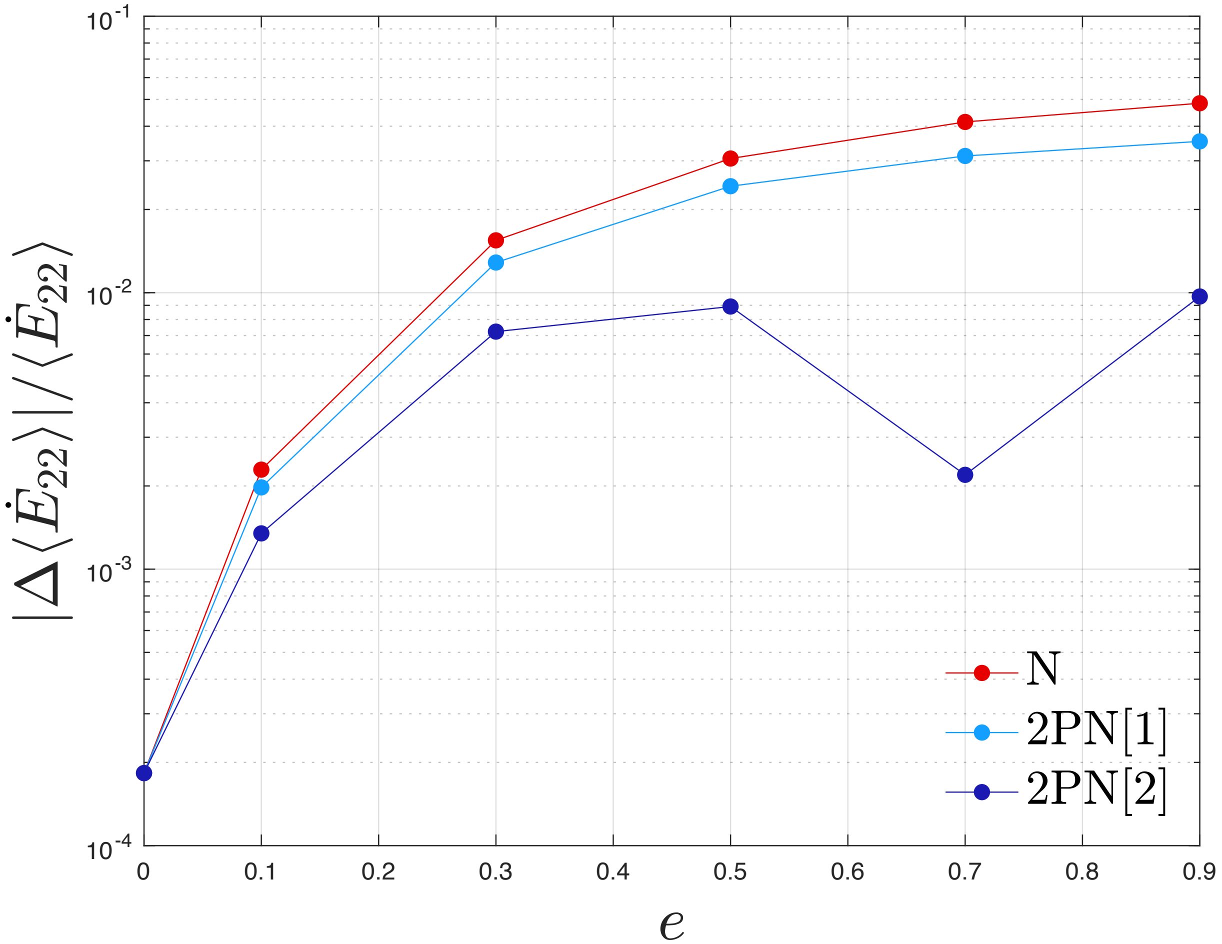
Orbit averaged fluxes:

$$\langle \dot{J}_{\ell m} \rangle = \frac{1}{T_r} \int_0^{T_r} \left[-\frac{1}{8\pi} m \Im (\dot{h}_{\ell m} h^*_{\ell m}) \right] dt$$

$$\langle \dot{E}_{\ell m} \rangle = \frac{1}{T_r} \int_0^{T_r} \left[\frac{1}{8\pi} |\dot{h}_{\ell m}|^2 \right] dt$$

T_r → radial period

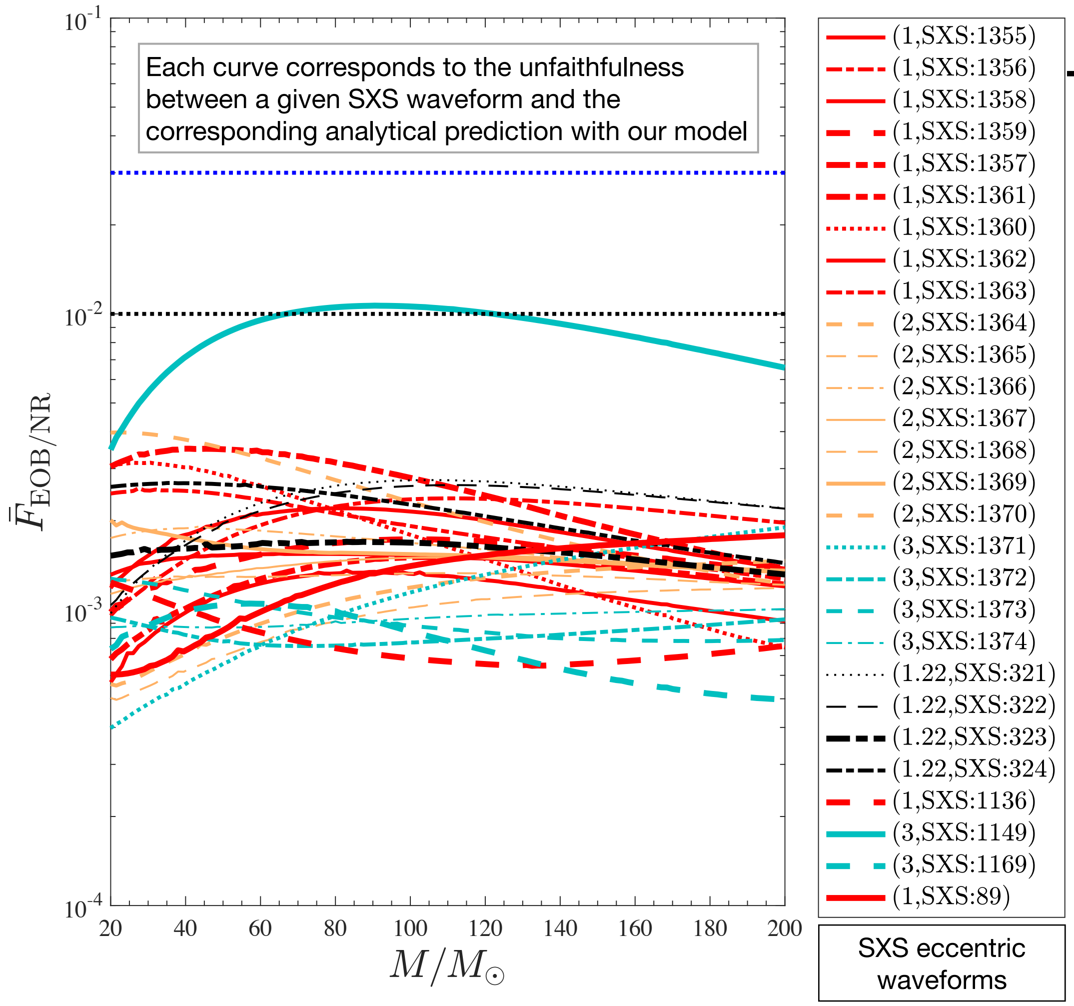




Waveform results [1]-[2]: comparable masses

Comparisons with the waveforms of the Simulating eXtreme Spacetime (SXS) catalog

[Placidi et al. 2021]: **EOB/NR unfaithfulness analysis** for the model with $\hat{h}^{\rm nc}_{\ell m}[1]$



$$\bar{F}(M) \equiv 1 - F = 1 - \max_{t_0,\phi_0} \frac{\langle h_1, h_2\rangle}{||h_1||\,||h_2||}$$

→estimates the mismatch between two given waveforms

$$||h|| \equiv \sqrt{\langle h, h\rangle}$$

$$\langle h_1, h_2\rangle \equiv 4\Re \int_{f^{\rm NR}_{\rm min}(M)}^{\infty} \tilde{h}_1(f)\tilde{h}_2^*(f)/S_n(f)\, df$$

⇨

$$\bar{F}^{\rm max}_{\rm EOB/NR} < 0.7\%$$

(modulo an outlier simulation for which is ~1%)

[Albanesi et al. 06/2022]: still no unfaithfulness analysis for the $\hat{h}^{\rm nc}_{\ell m}[2]$ model but we checked that the additional improvement over $\hat{h}^{\rm nc}_{\ell m}[1]$ seen in the test-mass limit carries over to the comparable mass case, where the **amplitude corrections** at the radial turning points are **even more relevant**

EOB approach for more general dynamics

Aligned/antialigned spins:

$r_c \equiv$ centrifugal radius

- $H_{\rm eff} \to H_{\rm eff}\big|_{r\to r_c}$ + spin-orbit terms
- $A(r) \to A(r_c)\dfrac{1+2M/r_c}{1+2M/r}, \quad D(r) \to D(r_c)\dfrac{r^2}{r_c^2}$
- $\rho_{\ell m} \to \rho_{\ell m}^{\rm orb}\,\rho_{\ell m}^{\rm spin}$

Tidal deformations (Neutron stars):

- $A(r) \to A(r) + A_{\rm tidal}^{\rm 5PN}(r, k_\lambda)$

Precession

- Euler rotating aligned-spin (non-precessing) waveforms from a precessing frame to an inertial frame

Eccentricity e and semilatus rectum p

There is no gauge invariant definition, we define them in analogy with Newtonian mechanics as:

$$r(\varphi) = \frac{p}{1 - e\cos\varphi}, \quad \rightarrow \quad r_p = \frac{p}{1+e}, \quad r_a = \frac{p}{1-e}$$

$$\rightarrow \quad e = \frac{r_a - r_p}{r_a + r_p}, \quad p = \frac{2 r_a r_p}{r_a + r_p}$$

where the numerical values of periastron and apastron (r_p, r_a) follows from Hamilton's equations of motion in terms of the EOB Hamiltonian

Notice: this definition is valid as long as bound orbits are considered