Perturbative QCD for physics at colliders: high energies and soft emissions

Dimitri Colferai

In collaboration with: S. Catani, L.Cieri, F.Coradeschi, F. Deganutti

Florence theory group day

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Outline

- QFT@colliders in Florence (S.Catani, D.Colferai) is mainly focussed on phenomenology of QCD with perturbative techniques
- QCD is a very rich theory
  - It is the strongest interaction
  - Many aspects have still to be understood and tested
  - (Almost) every event at colliders involves QCD dynamics. Controlling QCD effects is crucial in order to study other interactions and to search for new physics
- My research activity in QCD:
  - Physics of semihard ($s \gg |t|$) processes
  - Factorization theorems and currents for soft gluon and quark emissions
- Transplanckian scattering and radiation in gravity

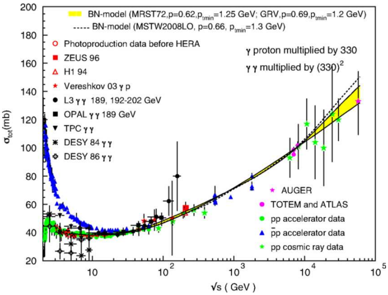
Semihard processes

Asymptotic energies $s \to \infty$
fixed t

Regge trajectory

$$A(s,t) \sim s^{1+\omega(t)}$$

$$\sigma(s,t) \sim s^{\omega(0)}$$



$$\sigma_{p\bar{p}} = 21.70 s^{0.0808} + 98.39 s^{-0.4525}$$

$$\sigma_{pp} = 21.70 s^{0.0808} + 56.08 s^{-0.4525}$$

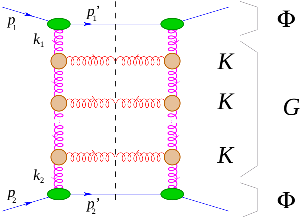
Pomeron

Resummation and factorization

Perturbatively: $(s \gg |t| \gg \Lambda_{qcd})$
$\log s$ enhancement in the PT series

$$A_8 = c_0 + c_1 \alpha_S \log \frac{s}{t} + c_2 \alpha_S^2 \log^2 \frac{s}{t} + \cdots \simeq \Gamma_A \frac{1}{t} \left(\frac{s}{t}\right)^{\alpha_S c(t)} \Gamma_B$$

Resummation → Gluon reggeization



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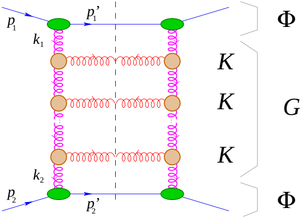


Factorization:

$$\sigma(s) = \int \mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2\; \Phi_A(\boldsymbol{k}_1)G(s,\boldsymbol{k}_1,\boldsymbol{k}_2)\Phi_B(\boldsymbol{k}_2)$$

$$\frac{\partial}{\partial \log s}G(s,\boldsymbol{k}_1,\boldsymbol{k}_2) = \int \mathrm{d}\boldsymbol{k}\; K(\boldsymbol{k}_1,\boldsymbol{k})G(s,\boldsymbol{k},\boldsymbol{k}_2)$$

$$K = \alpha_S K_0 + \alpha_S^2 K_1$$

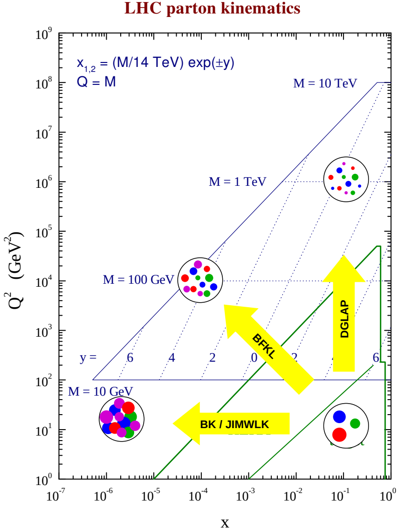
Brown blob = Lipatov vertex

Instability of high-energy expansion

$$\omega(t) = 2.77\alpha_S(t)[1-6.7\alpha_S(t)] \simeq \begin{cases} 0.5 & (LL) \\ -0.15 & (NLL) \end{cases} \quad \text{for } \alpha_S(t) \simeq 0.2$$

- Origin of instability: presence of large collinear (double) logs $\alpha_S \log^2\left(\frac{\boldsymbol{k}_1^2}{\boldsymbol{k}_2^2}\right)$ in $G(s, \boldsymbol{k}_1, \boldsymbol{k}_2)$
- leading logs can be predicted by renormalization-group analysis and resummed to all orders

x-evolution VS Q-evolution



$$x \simeq \frac{Q^2}{s} \xrightarrow{s\to\infty} 0$$

Renormalization group improvement

$\log(s)$ resummation ($\sqrt{s} \gg \boldsymbol{k}_{1,2}$) at next-to-leading order:

$$G(s,\boldsymbol{k}_1,\boldsymbol{k}_2) = \int \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \left(\frac{s}{\boldsymbol{k}_1\boldsymbol{k}_2}\right)^{\omega} \int \frac{\mathrm{d}\gamma}{2\pi\mathrm{i}} \left(\frac{\boldsymbol{k}_1^2}{\boldsymbol{k}_2^2}\right)^{\gamma} \frac{1}{\omega - \chi(\alpha_\mathrm{S},\gamma)}$$

$$\chi(\alpha_\mathrm{S},\gamma) \simeq \alpha_\mathrm{S}\frac{1}{\gamma} + \alpha_\mathrm{S}^2\left(\frac{-1}{\gamma^3} + \frac{A}{\gamma^2}\right)$$

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$$\chi(\alpha_S, \gamma) \simeq \alpha_S \frac{1}{\gamma} + \alpha_S^2 \left(\frac{-1}{\gamma^3} + \frac{A}{\gamma^2}\right)$$

$\log(\boldsymbol{k}_1^2/\boldsymbol{k}_2^2)$ resummation ($\boldsymbol{k}_1 \gg \boldsymbol{k}_2$) at leading order (Altarelli-Parisi eqns):

$$\chi(\alpha_S, \gamma) = \alpha_S \frac{\omega P_{gg}(\omega)}{\gamma + \omega} = \alpha_S \frac{1 + \omega A}{\gamma + \omega}$$

$$\simeq \alpha_S \frac{1}{\gamma} + \omega \left(\frac{-1}{\gamma^2} + \frac{A}{\gamma}\right) + \mathcal{O}\left(\omega^2\right)$$

$$\left[\omega \to \frac{\alpha_S}{\gamma}\right] \qquad = \alpha_S \frac{1}{\gamma} + \alpha_S^2 \left(\frac{-1}{\gamma^3} + \frac{A}{\gamma^2}\right) + \mathcal{O}\left(\alpha_S^3\right)$$

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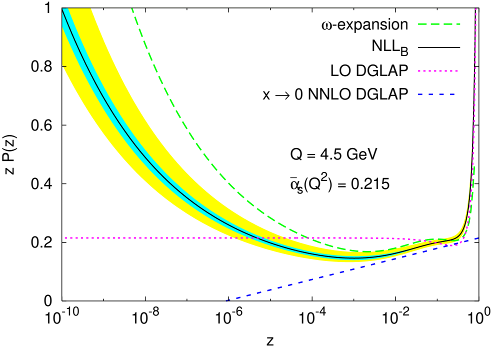
Trade γ-poles with ω-corrections *[Ciafaloni,DC,Salam,Stasto]*

Gluon splitting function

Splitting functions $P_{ab}(\alpha_S, z)$ are the fundamental objects to determine the partonic content of hadrons via Altarelli-Parisi equations *[Ciafaloni,DC,Salam,Stasto]*



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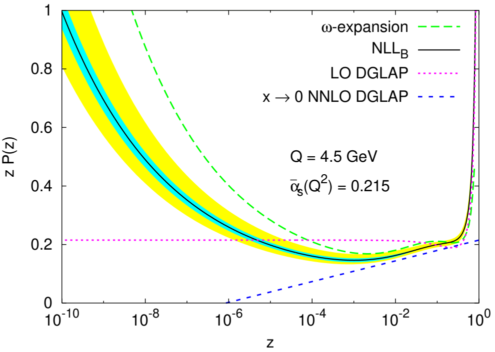
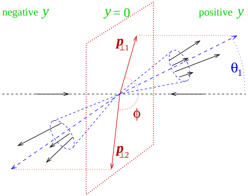


Can be extended to quark sector (matrix formulation)

Jet production at LHC

At LHC large $Y = y_{J1} - y_{J2} \simeq \log(s/\boldsymbol{k}_J^2) \sim 4 \div 9$



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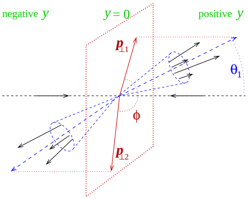

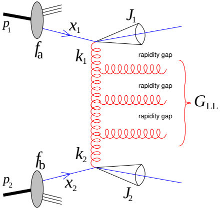

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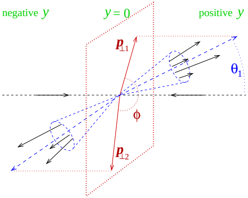
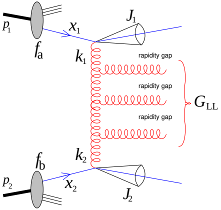
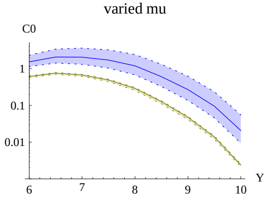
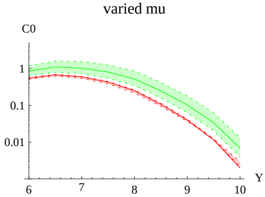




Measuring such processes requires dedicated runs at zero pileup
$Y \sim 12$ accessible with very forward detectors



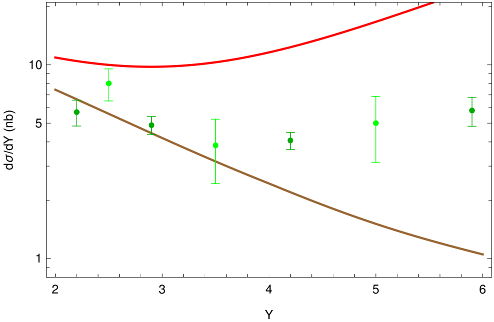


LL
full NLL
NL IF, LL GGF
RGI

[DC,Schwensen,Szymanowski,Wallon]

$\gamma^*\gamma^*$ cross section

Cross section of virtual photons at lepton colliders *[DC,Li,Stasto]*



- Opal and L3 data
- LO + LL
- LO + NLL
- LO + RGI

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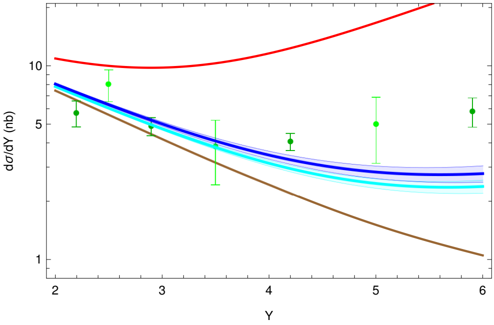

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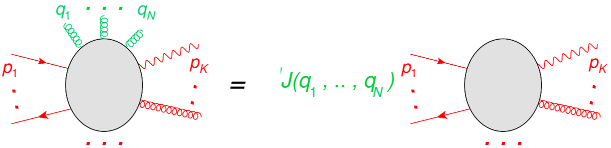
Soft theorems

- Behaviour of scattering amplitude $\mathcal{M}(\{p_k\},\{q_i\})$ when some external gluons $q_i = \xi\bar{q}_i$ become soft ($\xi \to 0$)



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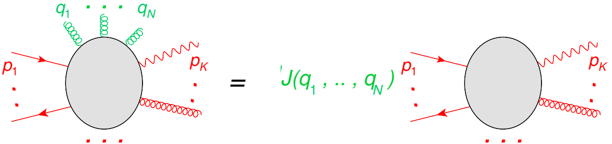
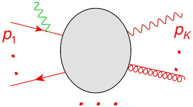


- Leading $1/\xi$: only gluon insertions on the external hard legs

QED: $$J(q) = J^{\mu}(q)\varepsilon_{\mu}(q) = \sum_{k=1}^{K} \frac{e_k p_k^{\mu}}{p_k \cdot q} \varepsilon_{\mu}(q)$$



QCD: $$J^a(q) = J^{a,\mu}(q)\varepsilon_{\mu}(q) = \sum_{k=1}^{K} \frac{g T_k^a p_k^{\mu}}{p_k \cdot q} \varepsilon_{\mu}(q)$$

$$\begin{cases} (T_q^a)_{bc} = t_{bc}^a \\ (T_{\bar{q}}^a)_{bc} = -t_{cb}^a \\ (T_g^a)_{bc} = \mathrm{i} f^{abc} \end{cases}$$

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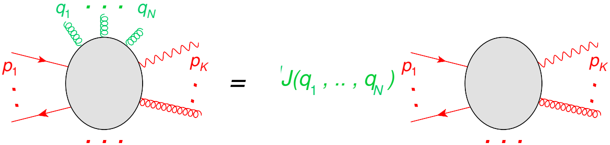

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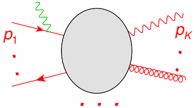

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$$\sum_{k=1}^{K} e_k = 0\,, \qquad \sum_{k=1}^{K} T_k^a \stackrel{\text{cs}}{=} 0 \qquad \Longrightarrow \qquad q_\mu J^{a,\mu} \stackrel{\text{cs}}{=} 0$$

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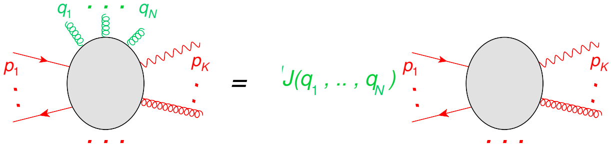

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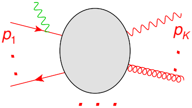

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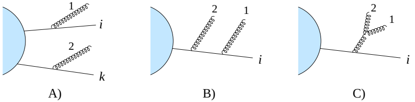
GRAV: $$J(q) = J^{\mu\nu}(q)\varepsilon_{\mu\nu}(q) = \sum_{k=1}^{K} \frac{\kappa p_k^{\nu} p_k^{\mu}}{p_k \cdot q}\varepsilon_{\mu\nu}(q)$$

2 soft gluon emission



[Catani, Grazzini] "Independent" + max. non-abelian correlation

$$J^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) = J^{a_1}_{\mu_1}(q_1) * J^{a_2}_{\mu_2}(q_2) + \Gamma^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2)$$

$A * B \equiv \frac{1}{2}(AB + BA)$

$$\Gamma^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) = \mathrm{i} f^{a_1 a_2 \, b} \sum_{k \in \text{hard}} T^b_k \, \gamma^{\mu_1 \mu_2}_k(q_1, q_2)$$

$$\gamma^{\mu_1 \mu_2}_k(q_1, q_2) = \frac{1}{p_k \cdot (q_1 + q_2)} \left\{ \frac{p_k^{\mu_1} p_k^{\mu_2}}{2 \, p_k \cdot q_1} + \frac{1}{q_1 \cdot q_2} \left(p_k^{\mu_1} q_1^{\mu_2} + \frac{1}{2} g^{\mu_1 \mu_2} p_k \cdot q_2 \right) \right\} - \left(1 \leftrightarrow 2 \right)$$

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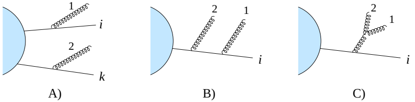

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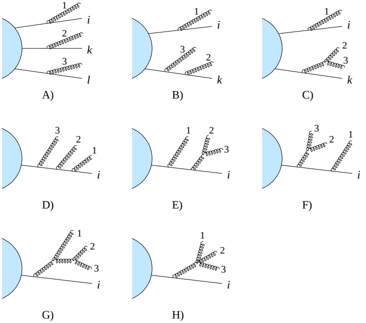
- conservation of current: $q_1^{\mu_1} J^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) \, \varepsilon^{\mu_2}(q_2) \overset{\text{cs}}{=} 0$
- Abelian case ($f^{abc} = 0$): only independent emission
- QCD: $J(1) * J(2) \implies$ colour-correlations with hard partons (furthermore currents do not commute)

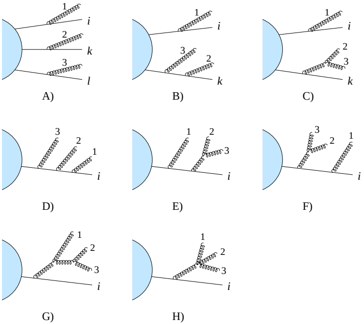
3 soft gluon emission



[Catani, DC, Torrini]

- Eikonal vertices on hard lines
- Exact vertices elsewhere
- Exact propagators of soft gluons (light-cone / covariant) gauges

3 soft gluon emission



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- Eikonal vertices on hard lines
- Exact vertices elsewhere
- Exact propagators of soft gluons (light-cone / covariant) gauges
- Current is conserved on colour-singlet states
- Irreducible term has colour structure $T^b \sum_s f^{a_1 a_2 s} f^{s a_3 b}$

$$J(1,2,3) = J(1) * J(2) * J(3) + \Big[\sum_{\text{cyc.}123} J(1) * \Gamma(2,3) \Big] + \Gamma(1,2,3)$$

$$\Gamma(1,2,3) = \sum_{k \in \text{hard}} T_k^b \sum_{\text{cyc.}123} f^{a_1 a_2,\, a_3 b}\, \gamma_k^{\mu_1 \mu_2 \mu_3}(q_1, q_2; q_3)$$

Properties and applications

- Computed also soft current for soft boson-fermion-antifermion emission *[Catani,Cieri,DC,Coradeschi]*
- Non-trivial organization in group-theoretical (colour) structures
- Combinatorics can be organized with non-commuting exponentiations (Magnus expansion)

The square of soft-currents are very important:

- Soft particles are easily produced in most reactions
- Key ingredient in MonteCarlo event generators for numeric treatment of arbitrary number of soft emissions
- Analytic tool for the cancellation of IR singularities: phase-space integration of soft particles is IR singular; combining with (IR singular) virtual corrections provide finite cross sections

Effective vertices in gravity

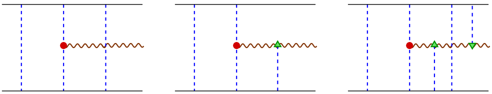
Both techniques (high-energy scattering and soft emissions) can be exported in gravity: effective vertex for graviton production at transplanckian energies is essentially the "double-copy" of the (colour-stripped) Lipatov vertex in QCD:

$$J_L^{\mu\nu}(q) = J_L^{\mu}(q) J_L^{\nu}(q) - J_W^{\mu}(q) J_W^{\nu}(q)$$
$$J_w^{\mu\nu}(q) = J_W^{\mu}(q) J_W^{\nu}(q)$$

[Ciafaloni,DC,Veneziano] found an effective vertex which interpolates between Lipatov's in the high-energy, central rapidity, region and the Weinberg vertex in the soft, arbitrary rapidity, region.
By resumming the leading diagrams to all order, we can find the spectrum of graviton radiation in transplanckian scattering

Resummed amplitude and coherent state

Transplanckian scattering: $\sqrt{s} \gg M_P$



$\boldsymbol{x}$ = graviton transv. position
$\boldsymbol{b}$ = opposite particle " "

interaction = $G\omega\sqrt{s}\Delta(\boldsymbol{b}-\boldsymbol{x})$

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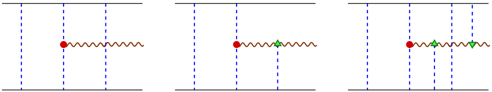
Transplanckian scattering: $\sqrt{s} \gg M_P$



$\boldsymbol{x}$ = graviton transv. position

$\boldsymbol{b}$ = opposite particle " "

interaction = $G\omega\sqrt{s}\Delta(\boldsymbol{b}-\boldsymbol{x})$

$$\mathcal{M}_{\text{tot}}(\boldsymbol{b},\boldsymbol{k}) = \mathrm{e}^{\mathrm{i}2\alpha\Delta(\boldsymbol{b})}\mathfrak{M}_\lambda(\boldsymbol{b},\boldsymbol{k}) \qquad (\boldsymbol{k}=\omega\boldsymbol{\theta}\,, \quad \hbar = 1)$$

$$\mathfrak{M}_\lambda(\boldsymbol{b},\boldsymbol{k}) = \frac{\mathrm{e}^{\mathrm{i}\lambda\phi_{\boldsymbol{\theta}}}}{(2\pi)^2}\frac{\sqrt{\alpha}}{\mathrm{i}\omega}\int\frac{\mathrm{d}^2\boldsymbol{x}}{\boldsymbol{x}^2}\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{\mathrm{e}^{\mathrm{i}\lambda\phi_{\boldsymbol{x}}}}\left\{\mathrm{e}^{\mathrm{i}2\alpha\left[\Delta(\boldsymbol{b}-\frac{\omega}{E}\boldsymbol{x})-\Delta(\boldsymbol{b})\right]} - \mathrm{e}^{-\mathrm{i}2\omega R[\Delta(\boldsymbol{b})-\right.$$

Resummed amplitude and coherent state

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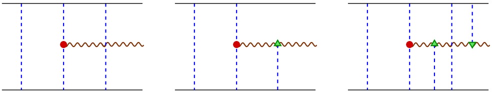

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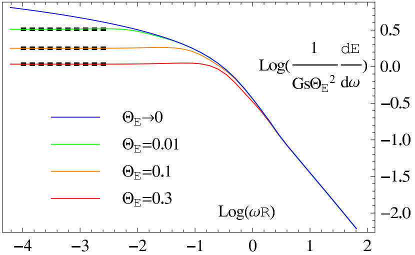
$$|\text{gravitons}\rangle = e^{i2\delta_0(b)}\exp\left\{i\sum_{\lambda=\pm}\int\frac{d^3k}{2\omega_k}\mathfrak{M}^{(\lambda)}(k)a^{(\lambda)\dagger}(k) + \text{h.c.}\right\}|0\rangle$$

$$= S|0\rangle$$

virtual contributions à la Weinberg

Spectrum

[Ciafaloni,DC,Veneziano]



- $\sim \log(1/\omega R)$ for $b^{-1} < \omega < R^{-1}$; $\sim 1/\omega$ for $\omega R \gg 1$
- Almost universal (b-independent) shape, Characteristic frequency $\langle\omega\rangle \sim 1/R$ that decreases as $\sqrt{s}$ increases, like Hawking radiation

$$\left.\frac{\mathrm{d}E^{\mathrm{GW}}}{\mathrm{d}\omega}\right|_{\substack{\text{ener}\\ \text{cons}}} \simeq \left.\frac{\mathrm{d}E^{\mathrm{GW}}}{\mathrm{d}\omega}\right|_0 \times \mathrm{e}^{-\hbar\omega/\tau} \qquad \left(\tau \overset{b\to b_c}{\sim} \frac{\hbar}{R} \simeq 0.1\, k_B\, T_{\text{Hawking}}\right)$$