Gravitomagnetic field and gravitational waves

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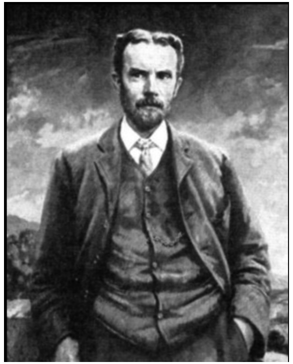
Plan of the Talk

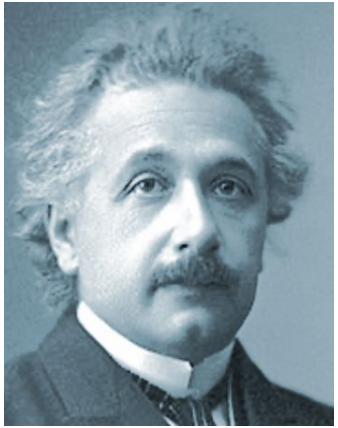
- Gravitomagnetism and its faces
  - Full Theory
  - Linear Theory
  - Curvature tensor
- Gravitational waves using the gravitoelectromagnetic formalism
- Gravitational waves and gravitomagnetic effects

The Origins of Gravitomagnetism

J.C. Maxwell



O. Heaviside



J. Lense

H. Thirring

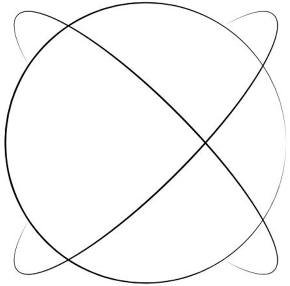
Inevitability of gravitomagnetism

If we want to put together Newtonian gravity and Lorentz invariance, the presence of a magnetic-like component of the gravitational field or, for short, a gravitomagnetic field, is mandatory.

In his letter to Thirring Einstein stated that these effects “remain far below any observable quantity”: both the speeds of the sources and that of the test particles must be compared to the speed of light. In weak GR these effects are very small

PROPOSAL FOR A SATELLITF TEST OF THE CORIOLIS PREDICTION OF GENERAL RELATIVITY*

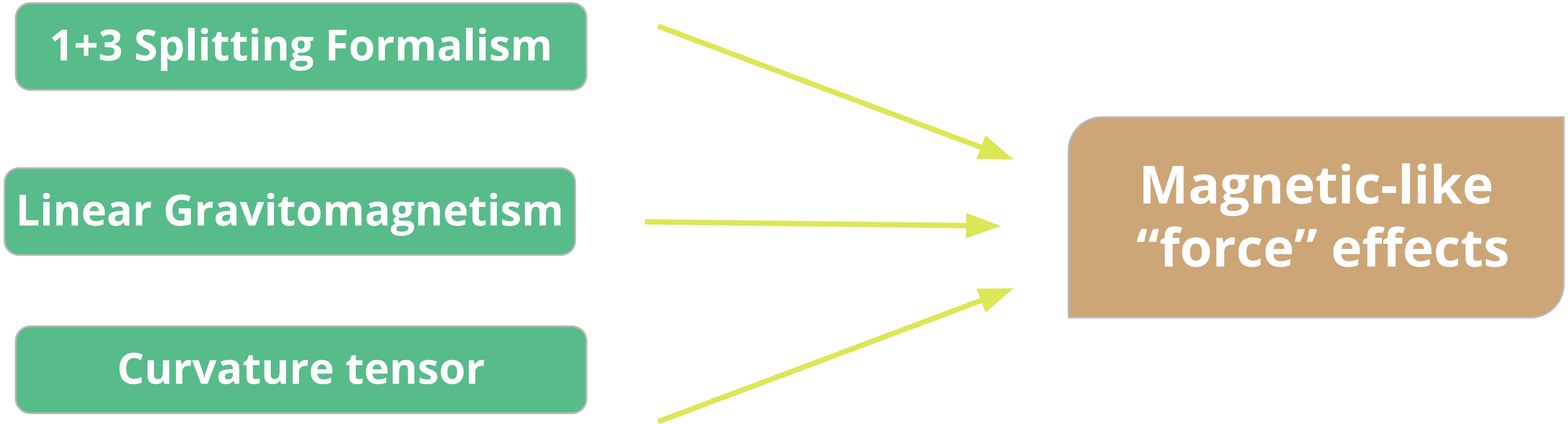
George E. Pugh
12 November 1959



A tale of analogies

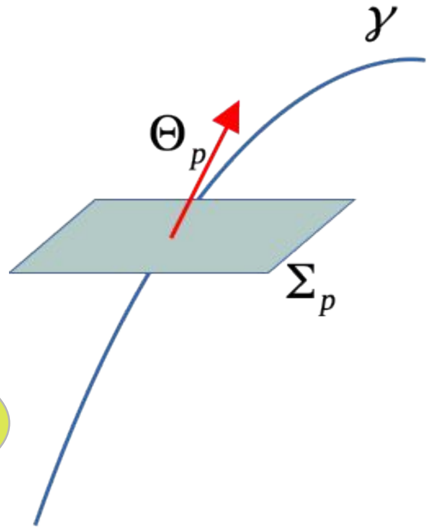
And I cherish more than anything else the Analogies, my most trustworthy masters. They know all the secrets of Nature, and they ought to be least neglected in Geometry
(J. Kepler)

The many faces of gravitomagnetism



The many faces of gravitomagnetism

1+3 Splitting Formalism



Splitting spacetime into space plus time is fundamental in GR to get a better understanding on the basis of our 3-dimensional experience of what happens in the 4- dimensional geometry.

PROJECTIONS

4-dimensional quantities → **space-like+time-like quantities**

At each point p in spacetime, the tangent space Tp can be split into the direct sum of two subspaces: the local time direction Θp, and Σp, the local space platform: a 3-dimensional subspace which is orthogonal to Θp

$$T_p = \Theta_p \oplus \Sigma_p.$$

The many faces of gravitomagnetism

1+3 Splitting Formalism

To describe the test particle dynamics we need to project the geodesic equations onto the tangent space:

acceleration

$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} U^\beta U^\gamma = 0 \quad \longrightarrow \quad \frac{\hat{D}\widetilde{p}_i}{dT} = m\widetilde{G}_i \qquad \widetilde{G}_i = -c^2 \widetilde{C}_i + c\widetilde{\Omega}_{ij}\widetilde{v}^j$$

rotation

A gravitoelectromagnetic analogy emerges in full GR when we are dealing with a non time-orthogonal metric

rotation

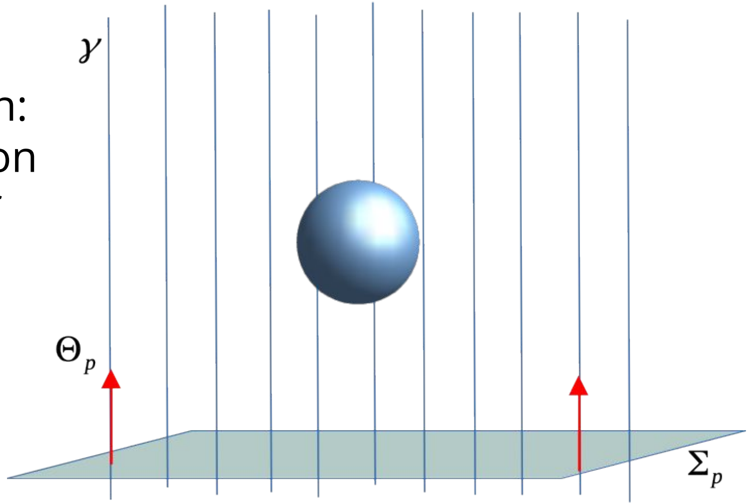
$$ds^2 = g_{00}c^2dt^2 + 2g_{0i}c\,dt\,dx^i + g_{ij}dx^i dx^j,$$

$$\frac{\hat{D}\widetilde{p}_i}{dT} = m\widetilde{E}^G_i + m\gamma_0 \left(\frac{\tilde{\boldsymbol{v}}}{c} \times \tilde{\boldsymbol{B}}_G\right)_i$$

The many faces of gravitomagnetism

Linear Gravitomagnetism

Weak-field and slow-motion approximation: the gravitational field is a small perturbation $h_{\mu\nu}$ of flat spacetime $\eta_{\mu\nu}$ and the speed (of both sources and test particles) are small w.r.t. the speed of light.



$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

LINEARIZATION

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \Longrightarrow \quad \Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Analogy: Maxwell's Equations

The many faces of gravitomagnetism

Linear Gravitomagnetism

Same equations, same solutions:

$$\bar{h}_{\mu\nu} = \frac{4G}{c^4}\int_V \frac{T_{\mu\nu}(ct - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'$$

$$\Phi = G\int_V \frac{\rho(ct - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x' \ ,$$

gravitoelectric e.g. Newtonian potential

$$A_i = \frac{2G}{c}\int_V \frac{j^i(ct - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'$$

gravitomagnetic potential

$$\mathrm{d}s^2 = -c^2\left(1 - 2\frac{\Phi}{c^2}\right)\mathrm{d}t^2 - \frac{4}{c}A_i\mathrm{d}x^i\mathrm{d}t + \left(1 + 2\frac{\Phi}{c^2}\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

Solar System Tests, e.g. GPB, LAGEOS/LARES

The many faces of gravitomagnetism

Linear Gravitomagnetism

Non Maxwellian term!

Motion of test particles

LINEARIZATION

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0,$$

$$\frac{dv^i}{dt} = -E^i - \frac{1}{c}2(\boldsymbol{v}\times\mathbf{B})_i - 3\frac{v^i}{c}\frac{\partial\Phi}{c\partial t}$$

with the definitions of the fields:

$$\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}, \quad \mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{2}{c}\frac{\partial\mathbf{A}}{\partial t}$$

If the metric is not time-depending the motion of test particles is described in analogy with electromagnetism

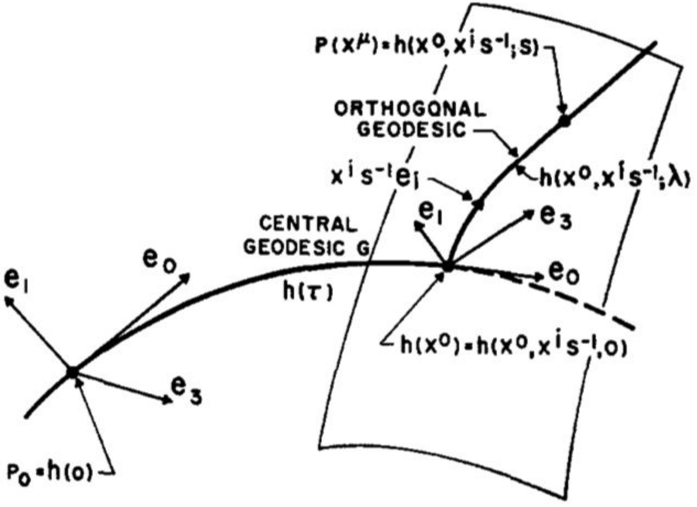
$$\frac{dv^i}{dt} = -E^i - \frac{1}{c}2(\boldsymbol{v}\times\mathbf{B})_i$$

The many faces of gravitomagnetism

Curvature tensor

In the tangent space along the observer's world-line we consider an orthonormal tetrad. The time coordinate is the proper time, while the space coordinates are defined using space-like geodesics.

Fermi Coordinates



from Misner, Manasse (1963)

The expression of the spacetime element in Fermi coordinates depends both on the properties of the reference frame (the acceleration and rotation of the congruence) and on the spacetime curvature, through the Riemann curvature tensor.

The many faces of gravitomagnetism

Curvature tensor

$R_{\alpha\beta\gamma\delta}(T)$ ıe projection of the Riemann curvature tensor on the orthonormal tetrad of the reference observer, parameterized by the proper time

Inertial effects

$$ds^2 = -\left[\left(1+\frac{\mathbf{a}\cdot\mathbf{X}}{c^2}\right)^2 - \frac{1}{c^2}(\boldsymbol{\Omega}\wedge\mathbf{X})^2 + R_{0i0j}X^iX^j\right]c^2dT^2 + \left[\frac{1}{c}(\boldsymbol{\Omega}\wedge\mathbf{X})_i - \frac{4}{3}R_{0jik}X^jX^k\right]cdTdX^i +$$

$$+\left(\delta_{ij} - \frac{1}{3}R_{ikjl}X^kX^l\right)dX^idX^j.$$

curvature effects

Gravitoelectromagnetic formalism $\Phi(T,\mathbf{X}) = \Phi^I(\mathbf{X}) + \Phi^C(T,\mathbf{X}), \quad , \quad \mathbf{A}(T,\mathbf{X}) = \mathbf{A}^I(\mathbf{X}) + \mathbf{A}^C(T,\mathbf{X}),$

$$\Phi^I(\mathbf{X}) = -\mathbf{a}\cdot\mathbf{X} - \frac{1}{2}\frac{(\mathbf{a}\cdot\mathbf{X})^2}{c^2} + \frac{1}{2}\left[|\boldsymbol{\Omega}|^2|\mathbf{X}|^2 - (\boldsymbol{\Omega}\cdot\mathbf{X})^2\right]$$

$$A_i^I(\mathbf{X}) = -\left(\frac{\boldsymbol{\Omega}c}{2}\wedge\mathbf{X}\right)_i$$

$$\Phi^C(T,\mathbf{X}) = -\frac{1}{2}R_{0i0j}(T)X^iX^j$$

$$A_i^C(T,\mathbf{X}) = \frac{1}{3}R_{0jik}(T)X^jX^k$$

GWs curvature

The many faces of gravitomagnetism

Curvature tensor

All quantities involved are relative to the reference observer at the origin of the frame!

Neglecting inertial effects:

$$ds^2 = -c^2\left(1-2\frac{\Phi}{c^2}\right)dT^2 - \frac{4}{c}A_i dX^i dT + \left(\delta_{ij}+2\frac{\Psi_{ij}}{c^2}\right)dX^i dX^j$$

$$\Phi(T,X^i) = -\frac{c^2}{2}R_{0i0j}(T)X^iX^j, \qquad A_i(T,X^i) = \frac{c^2}{3}R_{0jik}(T)X^jX^k, \qquad \Psi_{ij}(T,X^i) = -\frac{c^2}{6}R_{ikjl}(T)X^kX^l,$$

with the definition of the fields:

$$\mathbf{E} = -\nabla\Phi - \frac{2}{c}\frac{\partial \mathbf{A}}{\partial T}, \qquad \mathbf{B} = \nabla \wedge \mathbf{A},$$

Geodesic Equations:

$$\frac{d^2X^i}{dT^2} = -E^i - 2\left(\frac{\mathbf{V}}{c}\times\mathbf{B}\right)^i - 2\frac{V^j}{c}\frac{\partial\Psi_{ij}}{c\partial T} - \frac{V^i}{c}\frac{\partial\Phi}{c\partial T}.$$

Non Maxwellian terms!

Only if we confine ourselves **to linear displacements** from the reference world-line we get a Lorentz-like force equation

Summarizing

- Gravitational dynamics can be described in analogy with electromagnetism: in all cases, the dynamics of free particles is formally described by Lorentz-like equations when suitable hypotheses are taken into account.

- The similarity with Maxwell's theory, then, allows to explain and investigate gravitational effects in terms of known electromagnetic ones, according to the reasonable principle for which similar equations lead to similar solutions.

Gravitation ≠ Electromagnetism

Gravitational Waves

A plane gravitational wave

For a gravitational wave propagating along the x direction, in TT gauge: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$h_{xx} = 1, \quad h_{yy} = 1 - A^{+} \sin(\omega t - kx), \quad h_{zz} = 1 + A^{+} \sin(\omega t - kx), \quad h_{zy} = -A^{\times} \cos(\omega t - kx)$$

If we apply the gravitoelectromagnetic formalism, we obtain the following components

$$E_X = 0, \quad E_Y = -\frac{\omega^2}{2}\left[A^{+}\sin(\omega T)\,Y + A^{\times}\cos(\omega T)\,Z\right], \quad E_Z = -\frac{\omega^2}{2}\left[A^{\times}\cos(\omega T)\,Y - A^{+}\sin(\omega T)\,Z\right]$$

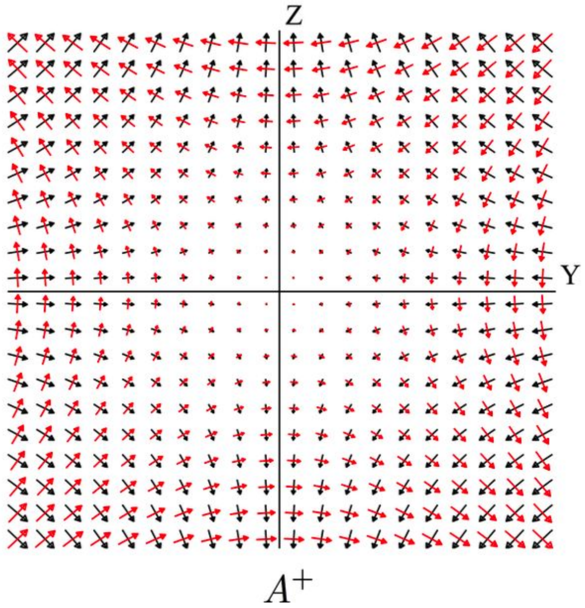
$$B_X = 0, \quad B_Y = -\frac{\omega^2}{2}\left[-A^{\times}\cos(\omega T)\,Y + A^{+}\sin(\omega T)\,Z\right], \quad B_Z = -\frac{\omega^2}{2}\left[A^{+}\sin(\omega T)\,Y + A^{\times}\cos(\omega T)\,Z\right].$$

$$\mathbf{E} \cdot \mathbf{B} = 0 \qquad |\mathbf{E}|^2 - |\mathbf{B}|^2 = 0$$

E and **B** are transverse, i.e. orthogonal to the propagation direction

A plane gravitational wave

The gravitoelectric **E** (black) and gravitomagnetic field **B** (red) for a wave with A+ polarization: notice that the two fields everywhere are orthogonal.



The simplest GW-antenna is made of two free masses, which are at rest before the passage of the wave: their *relative* motion is determined by

$$\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}T^2} = -\mathbf{E} \longrightarrow \quad X(T) = 0, \quad Y(T) = L\left[1 - \frac{A^+}{2}\sin(\omega T)\right], \quad Z(T) = 0.$$

The masses are along the Y direction, and their distance before the passage of the wave is L

A plane gravitational wave at second order

If we look for the solutions accurate up to second order in the displacements from the reference world line, care must be paid in using the gravitoelectromagnetic formalism.

For test masses at rest before the passage of the wave, motion is determined by the gravitoelectric field only.

$$\mathbf{E} = -\nabla\Phi - \frac{2}{c}\frac{\partial \mathbf{A}}{\partial T},$$

$$A_i(T, X^i) = \frac{c^2}{3} R_{0jik}(T) X^j X^k,$$ quadratic

$$\Phi(T, X^i) = -\frac{c^2}{2} R_{0i0j}(T) X^i X^j,$$ linear

quadratic $$\Phi = -\frac{c^2}{2} R_{0i0j}(T) X^i X^j - \frac{c^2}{6} R_{0i0j,m}(T) X^i X^j X^m$$

A plane gravitational wave at second order

quadratic

$$\mathbf{E} = \mathbf{E}^{(0)} + \mathbf{E}^{(1)}$$

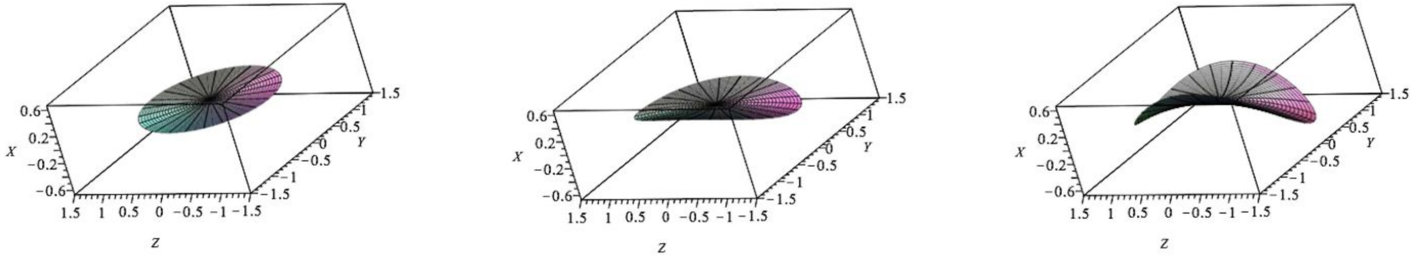
linear

$$\frac{|\mathbf{E}^{(1)}|}{|\mathbf{E}^{(0)}|} \simeq \frac{\omega L}{c} \simeq \frac{L}{\lambda}.$$

$$E_X^{(1)} = -\frac{1}{4}\frac{\omega^3}{c}\left[A^+ \cos(\omega T)\left(Y^2 - Z^2\right) - 2A^\times \sin(\omega T)\, ZY\right]$$

$$E_Y^{(1)} = \frac{1}{2}\frac{\omega^3}{c}\left[A^+ \cos(\omega T)\, YX - A^\times \sin(\omega T)\, ZX\right],$$

$$E_Z^{(1)} = -\frac{1}{2}\frac{\omega^3}{c}\left[A^\times \sin(\omega T)\, YX + A^+ \cos(\omega T)\, ZX\right].$$



A plane gravitational wave at second order

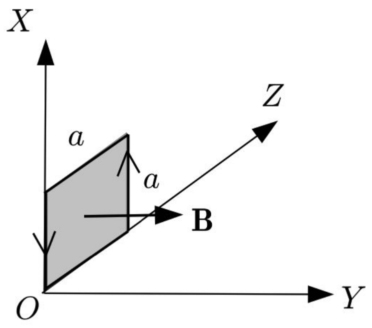
A physical interpretation

By their very definition, the gravitoelectromagnetic fields satisfy the homogenous equations

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{2}{c}\frac{\partial \mathbf{B}}{\partial T} = 0,$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0.$$

A test mass m initially at rest undergoes a **gravitomagnetic force** $\frac{2m}{c}\frac{\partial \mathbf{A}}{\partial T}$

The displacements along the propagation direction can be explained in terms of gravitational induction: the gravitomagnetic field is orthogonal to the propagation direction and induces a gravitoelectric field parallel to the propagation direction.



GWs in the Gravitoelectromagnetic formalism

$$m\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}T^2} = -m\mathbf{E} - 2m\frac{\mathbf{V}}{c}\times\mathbf{B}$$

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}T} = \frac{1}{c}\mathbf{B}\times\mathbf{S}$$

Gravitoelectric effects (interferometers)

Gravitomagnetic effects (new!)

$$\Phi(T,\mathbf{X}) = \Phi^I(\mathbf{X}) + \Phi^C(T,\mathbf{X}), \quad , \quad \mathbf{A}(T,\mathbf{X}) = \mathbf{A}^I(\mathbf{X}) + \mathbf{A}^C(T,\mathbf{X}),$$

inertial

$$\Phi^I(\mathbf{X}) = -\mathbf{a}\cdot\mathbf{X} - \frac{1}{2}\frac{(\mathbf{a}\cdot\mathbf{X})^2}{c^2} + \frac{1}{2}\left[|\boldsymbol{\Omega}|^2|\mathbf{X}|^2 - (\boldsymbol{\Omega}\cdot\mathbf{X})^2\right]$$

$$A_i^I(\mathbf{X}) = -\left(\frac{\boldsymbol{\Omega}c}{2}\wedge\mathbf{X}\right)_i$$

curvature (GWs)

$$\Phi^C(T,\mathbf{X}) = -\frac{1}{2}R_{0i0j}(T)X^iX^j$$

$$A_i^C(T,\mathbf{X}) = \frac{1}{3}R_{0jik}(T)X^jX^k$$

Gravitomagnetic Resonance

In a rotating Fermi frame with a given rotation rate Ω, the total gravitomagnetic field is

$$\mathbf{B}' = \mathbf{B}^I + \mathbf{B}^C, \text{ where } \mathbf{B}^I = -\mathbf{\Omega} c$$

If **Ω** is along the propagation direction of the Gw the spin evolution equation is

$$\frac{d\mathbf{S}}{dT} = \frac{1}{c}\left[\mathbf{B}^C(T) + \mathbf{B}^I\right] \times \mathbf{S} \quad \Rightarrow \quad \left(\frac{d\mathbf{S}}{dT}\right)_{\mathrm{rot}} = \left[\Delta\omega \mathbf{u}_{X'} + \frac{1}{c}\mathbf{B}^C\right] \times \mathbf{S} = \frac{1}{c}\mathbf{B}_{eff} \times \mathbf{S}$$

$$\omega - \frac{1}{c}B^I = \omega - \Omega = \Delta\omega$$

In the rotating frame spin precesses around the B_{eff} field.

In resonance condition (<u>the rotation rate Ω is equal to the wave frequency ω</u>) the passage of the wave completely flips the spin.

GRAVITOMAGNETIC RESONANCE

Gravitomagnetic Resonance

Probability transition for a 2-level system

$$P_{g\to e}(T) = \frac{(\omega^*)^2}{(\omega^*)^2 + \Delta\omega^2} \sin^2\left(\sqrt{(\omega^*)^2 + \Delta\omega^2}\frac{T}{2}\right)$$

GRAVITOMAGNETIC RESONANCE

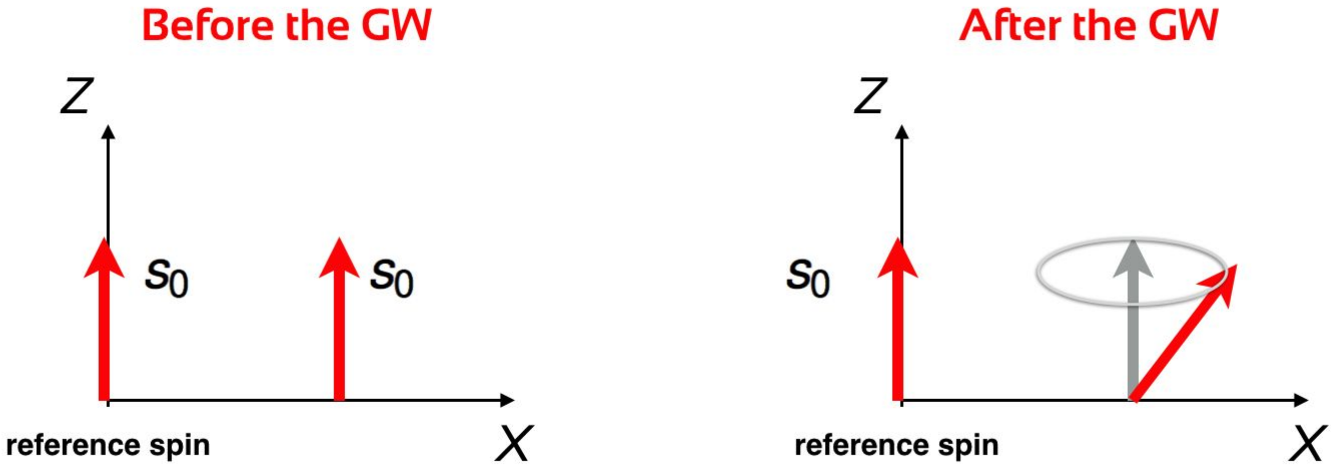
When Δω =0 the probability of transition is equal to 1 independently of the strength of the gravito-magnetic field

$$T = \frac{2n+1}{(\omega^*)}\pi \qquad \frac{1}{c}B^C = \omega^*$$

The time required is inversely proportional to the gravitomagnetic field of the wave

High Frequency GWs are ideal candidates

Gravitomagnetic Resonance



The passage of Gws can modify the magnetization of a sample (detection)

Gravitomagnetic Resonance

A tentative scheme of work

Resonance is obtained by combining the GW field with with a rotation field with the same frequency: it is not possible to obtain arbitrarily high frequencies

Exploit Larmor Theorem

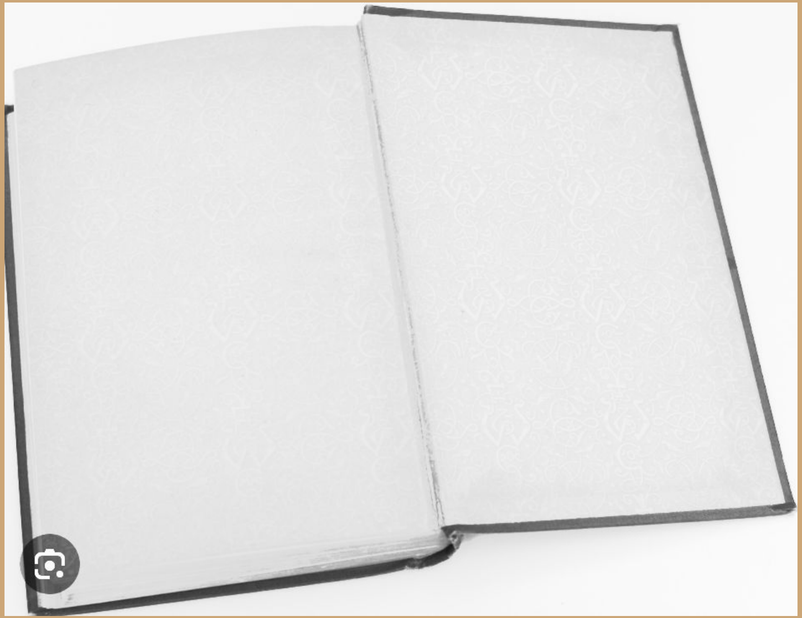
Equivalence between a system of electric charges in a magnetic field, and the same system rotating with the Larmor frequency

The passage of the wave modifies the spin of a sample.

Obtain a differential magnetization of samples at different locations

Measurement of a variation of the magnetic field of the samples

Conclusions & Perspectives



Conclusions

- A survey of different gravitoelectromagnetic analogies
- Pros and cons of the analogies
- Describing GWs in Fermi coordinates shows their gravitomagnetic components
- Spinning particles could be used as a probe to measure gravitomagnetic effects due to GWs

More stuff in these publications

- Ruggiero, Matteo Luca, and Antonello Ortolan. "Gravitomagnetic resonance in the field of a gravitational wave." *Physical Review D* 102.10 (2020): 101501.
- Ruggiero, Matteo Luca, and Antonello Ortolan. "Gravito-electromagnetic approach for the space-time of a plane gravitational wave." *Journal of Physics Communications* 4.5 (2020): 055013.
- Ruggiero, Matteo Luca. "Gravitational waves physics using Fermi coordinates: a new teaching perspective." *American Journal of Physics* 89, 639 (2021)
- Iorio, Lorenzo and Ruggiero, Matteo Luca. "Perturbations of the orbital elements due to the magnetic-like part of the field of a plane gravitational wave" *Int. J. Mod. Phys. D, 30, 21500088, (2021)*

A plane gravitational wave at second order

Variation of the distance between two test masses

$$D(t) = D_0 + \frac{2A^{\times} Z_0 Y_0 \left(1 - \cos\left(\omega T\right)\right) + A^{+}\left(Z_0^2 - Y_0^2\right)\sin\left(\omega T\right)}{2D_0}$$

"linear"

$$+ \frac{\omega X_0}{c} \frac{A^{+}\left(1 - \cos\left(\omega T\right)\right)\left(Z_0^2 - Y_0^2\right) - 2A^{\times}\sin\left(\omega T\right) Z_0 Y_0}{4D_0}.$$

"quadratic"

The correction term is present only if the masses are not in the plane orthogonal to the propagation direction

Generalization

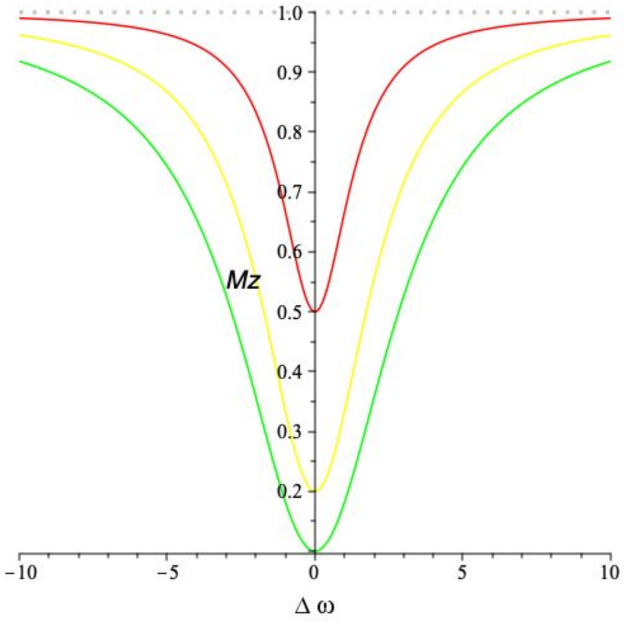
$$\mathbf{E}^{(n)} = -\boldsymbol{\nabla}\Phi^{(n)} - \frac{2}{c}\frac{\partial \mathbf{A}^{(n-1)}}{\partial T},$$

$$\Phi^{(n)} = -c^2 \frac{n+3}{(n+3)!} R_{0i0j,m_1\ldots m_n} X^i X^j X^{m_1} \ldots X^{m_n}.$$

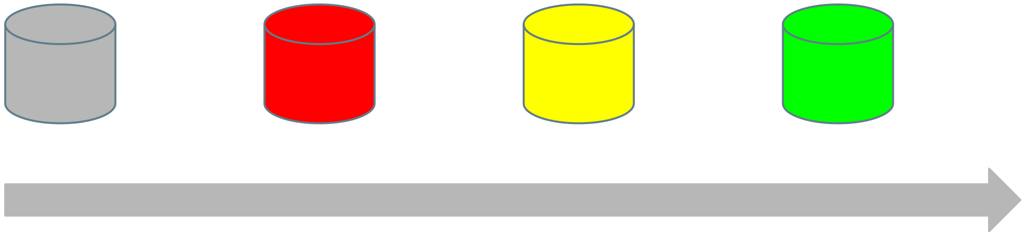
$$\mathbf{A}^{(n)} = c^2 \frac{n+2}{(n+3)!} R_{0lij,m_1\ldots m_n} X^l X^j X^{m_1} \ldots X^{m_n}.$$

Gravitomagnetic Resonance

In a magnetic resonance experiment we do not observe the one single spin, but a great number of particles: in our case, what we can measure is the variation of the magnetization of samples at different locations.



$$B^C \propto L$$



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