Integrability tools for gauge theory and black holes physics

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Based on: arXiv:23**.*****, arXiv:2208.14031, arXiv:2112.11434, arXiv:1908.08030

with D. Fioravanti, R. Mahanta, M. Rossi, H. Shu

On quantum integrability

- Integrability can be considered as the study of **non-linear** phenomena in nature in a quantitative **exact, non perturbative** way.
- The hallmark of quantum integrability is the presence of **infinite (local) integrals of motion commuting** with each other

$$[\mathbf{I}_n, \mathbf{I}_m] = 0\,. \tag{1}$$

What objects are more general and abstract?

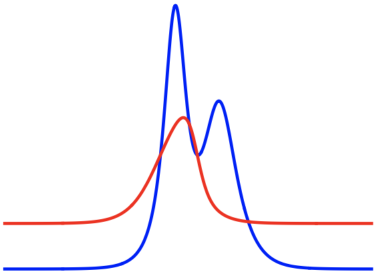


Figure: Solitons of mKdV (Wolfram Demo)

- The IM I_{2n-1} are also asymptotic expansion coefficients of the **Baxter's** Q, T **operators**

$$\ln \mathbf{Q}(\theta) \simeq -C_0 e^{\theta} - \sum_{n=1}^{\infty} e^{-n\theta} C_n \mathbf{I}_n \qquad \theta \to +\infty\,, \tag{2}$$

$$[\mathbf{Q}(\theta), \mathbf{T}(\theta)] = 0\,, \tag{3}$$

their eigenvalues being called Q, T functions.

- Q, T functions satisfy **functional relations**, like quantum wronskian QQ or Baxter's TQ (here for Sinh-Gordon IM)

$$Q(\theta - i\pi/2)Q(\theta + i\pi/2) - Q(\theta - i\pi a/2)Q(\theta + i\pi a/2) = 1\,, \tag{4}$$

$$T(\theta)Q(\theta) = Q(\theta + i\pi(a+1)/2) + Q(\theta - i\pi(a+1)/2)\,. \tag{5}$$

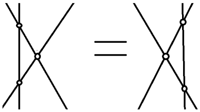
- Given suitable **asymptotic behaviours** these functional relations have **unique solutions**.

The ODE/IM correspondence

- The **ODE/IM correspondence** is an approach to integrability, in which the Q, T functions and all the (QQ, TQ...) **integrable structures** of some Integrable Model (IM) can be **derived from Ordinary Differental Equation** (ODE), even without knowing the corresponding IM.

$$\left[-\frac{d^2}{dx^2} + U\right]\psi = E\psi$$

ODE	IM
Spectral determinant	Q
Stokes multipliers	T_k
Central Connect. rels.	QQ relation
Lateral Connect. rels.	TQ, T-syst
ODE parameters	IM parameters
ODE ind. var. x	... cf. Rossi's talk



ODE/IM allows to apply integrability it to very different physical theories!

Three different physical theories, same mathematics!

$$-\frac{d^2}{dy^2}\psi(y) + \left[2e^{2\theta}\cosh y + P^2\right]\psi(y) = 0 \qquad (6)$$

INTEGRABILITY (sd-Liouville)

$$\Downarrow \qquad (7)$$

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda^2\cosh y + u]\psi(y) = 0 \qquad (8)$$

N=2 GAUGE TH. ($SU(2)$ $N_f = 0$)

$$\Downarrow \qquad (9)$$

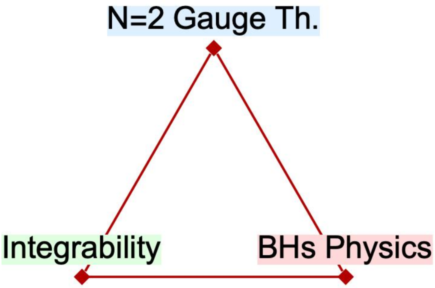
$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1 + \frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi(r) = 0 \qquad (10)$$

BLACK HOLES PERT. (D3 brane)

- We have shown this **triality** generally for Mathieu , Doubly Confluent Heun , Confluent Heun ODEs .

Outline

1. Introduction to ODE/IM
2. Integrability for gauge theory
   - A new gauge / integrability duality
   - ODE / IM vs AGT duality
3. Integrability for black holes
   - QNMs as Bethe roots
   - Computing QNMs from TBA
   - Other applications
   - Connection to astrophysics
4. Conclusions



Introduction to ODE/IM

ODE/IM correspondence examples I

- The first found was the anharmonic oscillator ODE [Dorey-Tateo,Bazhanov-Lukyanov-Zamolodchikov]

$$-\frac{d^2}{dx^2}\psi(x)+\left(x^{2\alpha}+\frac{l(l+1)}{x^2}-E\right)\psi(x)=0\,. \tag{11}$$

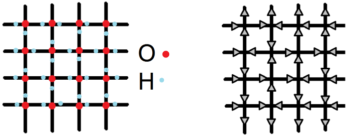


Figure: Six vertex model (Wikicommons/Moali)

- The vacuum Q function is defined as

$$Q_+ = W[\chi_0,\psi_0] \qquad \text{with} \qquad \begin{cases}\chi_0 \to 0 & x\to\infty \\ \psi_0 \to 0 & x \to 0\end{cases}\,. \tag{12}$$

Similarly the T function and find realization of TQ system of **Scaling 6-vertex IM!**

- Holding for values of α in correspondence with **KdV / Minimal Models** CFT $c<1$:

$$\alpha \geq -1 \qquad \dots \qquad \textbf{what for other } \alpha \textbf{ ?} \tag{13}$$

ODE/IM correspondence examples II

- We can reach the regime $\alpha < -1$, by a formal change of variable, leading to the ODE for the to **Liouville model** , that is CFT central charge $c = 1 + (b + 1/b)^2 \geq 1$

$$-\frac{d^2}{dy^2}\psi(y) + \left[e^{2\theta}(e^{y/b} + e^{-by}) + P^2\right]\psi(y) = 0\,, \tag{14}$$

which reduces to (modified) Mathieu equation for $b = 1$.

- Liouville model is dual to 4D $\mathcal{N} = 2$ susy by **AGT duality**, but we will show through ODE/IM another connection to it!

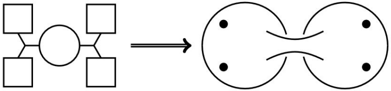


Figure: AGT duality quiver (NLab)

ODE/IM correspondence examples III

- **Perturbed Hairpin IM** for $n = 1, 2$ is Doubly Confluent Heun (DCHE)

$$-\frac{d^2}{dx^2}\psi(x) + \left[2kqe^x + k^2(e^{2x} + e^{-nx}) + P^2\right]\psi(x) = 0\,. \tag{15}$$

- **Paperclip IM** for $n = 2$ is the Confluent Heun equation (CHE)

$$-\frac{d^2}{dx^2}\psi(x) + \left[k^2\left(e^x + 1\right)^n - \frac{p^2 e^x}{e^x + 1} - \frac{\left(q^2 - \frac{1}{4}\right)e^x}{\left(e^x + 1\right)^2}\right]\psi(x) = 0\,. \tag{16}$$

- They are both related **2D IQFTs** with **non-conformal boundary interaction**.

Fig. 2. The paperclip formed by a junction of two hairpins.

Figure: Lukyanov-Vitchev-Zamolodchikov

ODE/IM correspondence applications

- Over 20+ years, ODE/IM found fruitful applications applications **beyond integrability** in
  - Foundations of quantum mechanics
  - Statistical mechanics
  - Condensed matter physics
  - Supersymmetric gauge theories
  - Pure mathematics

A new very promising application is to **black holes physics**.

On the Origin of ODE/IM

- For a long time, ODE/IM has been regarded as either **mysterious correspondence** or just a mathematical coincidence.
- However, new work - cf. Rossi's talk - has shed light on its **origin**, by "reversing it":

$QQ/TQ/$Bethe Ansatz

$\Downarrow$

Marchenko-like int. eq.

$\Downarrow$

Schrödinger ODEs

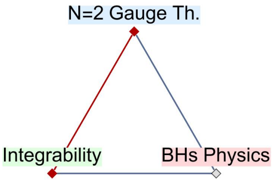
- This appears general and in principle would provide ODE/IM for also **non-QFTs**, noticebly **spin chains**.

Figure: The Thinker (Rodin)

Integrability for gauge theory

A new ($\mathcal{N}=2$) gauge / integrability duality

- Using ODE/IM correspondence, we **connected** the **basic integrability Q, T (and Y) functions** to the **gauge exact quantum periods** a, a_D



$$Q(\theta) = \exp\left[\frac{1}{\hbar} a_D\right] \qquad T(\theta) = 2\cos\left[\frac{2\pi}{\hbar} a\right], \tag{17}$$

(here for $c = 25$ Liouville and $SU(2)$ $N_f = 0$).

- a, a_D build up the 4D $\mathcal{N}=2$ **gauge prepotential**.
- This basic identification allowed us to find several **new results for both sides**. For instance, QQ and TQ functional relations get translated in relations for gauge periods

$$QQ \qquad \& \qquad TQ \tag{18}$$

$$\Downarrow$$

$$a_D(\theta + i\pi/2, a) = a_D(\theta, a) + \hbar \ln\left\{-\cos\left[\frac{2\pi}{\hbar}a\right] - \sqrt{\cos^2\left[\frac{2\pi}{\hbar}a\right] - 1 - \exp\left[-\frac{2}{\hbar}a_D(\theta, a)\right]}\right\}.$$

- Until now, we showed this new **2D integrability / 4D gauge duality** for the following models and ODEs

	ODE	IM	$\mathcal{N}=2$ susy
(a)	Mathieu	Liouville	$SU(2)\ N_f=0$
(b)	Doub. Confluent Heun	Perturbed Hairpin (+)	$SU(2)\ N_f=1,2$
(c)	Confluent Heun	Paperclip (+)	$SU(2)\ N_f=3$
(d)	some 3rd degree ODE ...	A_3 Toda	$SU(3)\ N_f=0$
	...	...	...

in the following papers

(a)	arXiv:1908.08030	DG, Fioravanti
(b)	arXiv:2112.11434	DG, Fioravanti
(b^+)	arXiv:2208.14031	DG, Fioravanti, Shu
(c)	arXiv:23**.*****	DG, Fioravanti, Mahanta, Rossi
(d)	arXiv:1909.11100	Fioravanti, Poghossian, Poghosyan

ODE/IM vs AGT duality

- Through linear connection relations among solutions ψ_k one easily derives QQ-systems like

$$Q(\theta + i\pi/2, a)Q(\theta - i\pi/2, a) - Q(\theta, a)^2 = 1\,. \tag{19}$$

- From purely ODE/IM and gauge theory we proved the QQ **system** is solved by formulae like

$$Q(\theta, a) = \frac{\sinh\left[\frac{1}{\hbar}A_D(\theta, a)\right]}{\sin\left[\frac{2\pi}{\hbar}a\right]}\,, \quad \text{with} \quad A_D\,(\theta + i\pi/2, a) = A_D(\theta, a) - 2\pi i a\,, \tag{20}$$

where a and A_D are $\mathcal{N} = 2$ gauge periods computable from **NS prepotential instanton expansion**

$$\begin{cases} A_D & = \partial\mathcal{F}/\partial a \\ a & = \partial\mathcal{F}/\partial\Lambda_0 \end{cases} \qquad \mathcal{F} = \mathcal{F}^{pert} + \sum_{n=1}^{\infty} \mathcal{F}_n^{inst}\Lambda_0^{4n}\,. \tag{21}$$

An **alternative derivation** is possible through **AGT duality**

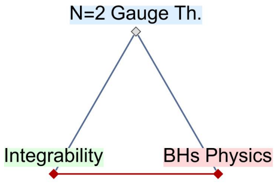
Integrability for black holes

Quasinormal modes of black holes as Bethe roots

- Through the ODE/IM formula $Q = W[\psi_+, \psi_-]$ we proved that the **Bethe root** (zero) condition on the function

$$Q(\theta_n) = 0\,, \tag{22}$$

  is equivalent to the mathematically rigorous definition of **quasinormal modes (QNMs)**!
- QNMs are the characteristic complex (damped) frequencies of **gravitational waves** in the final - **ringdown** - phase of black holes (BHs) merging.



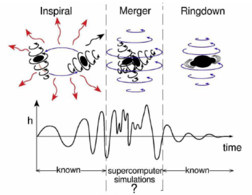


Figure: BH merging (K. Thorne)

Derivation: mathematical definition of QNMs vs. ODE/IM correspondence

- By standard techniques (PDE $\to$ Laplace tr. $\to$ non-hom. ODE $\to$hom. ODE) we can express the perturbation Φ as an expansion over the **quasinormal modes (QNMs)** frequencies ω_n

$$\Phi(t,y) = \sum_n e^{i\omega_n t} \text{Res} \left(\frac{1}{W(s)} \right) \Bigg|_{\omega_n} \int_{-\infty}^{\infty} \psi_-(\omega_n, y_<) \psi_+(\omega_n, y_>) \mathcal{I}(\omega_n, y')\, dy' + \text{ (other contributions)}\,. \tag{23}$$

- QNMs ω_n are so found to be **zeros of wronskian of the fundamental regular solutions** at $y \to \pm\infty$ of the ODE [Nollert:1999], which also defines Q function as

$$W[\psi_+, \psi_-](\omega_n) = 0\,, \qquad \text{with} \quad \psi_\pm(y) \to 0 \quad y \to \pm\infty\,. \tag{24}$$

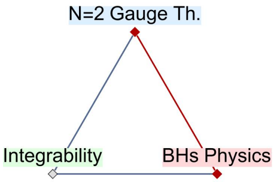
On the new ($\mathcal{N}=2$) gauge / gravity duality

- Since proved a **relation between the Q function and the gauge periods**

$$Q(\theta, \mathbf{m}, a) = \mathfrak{F}(\theta, \mathbf{m}, a) \frac{\sinh \frac{1}{\hbar} A_D(\theta, \mathbf{m}, a)}{\sin \frac{2\pi}{\hbar} a}, \qquad (25)$$

from our formalism **it follows immediately that QNMs are given by quantization conditions on the gauge periods**

$$Q(\theta, \mathbf{m}, a) = 0 \quad \Longrightarrow \quad A_D(\theta, \mathbf{m}, a) = 2i\pi\hbar\, n \quad \text{for } n \in \mathbb{Z}. \qquad (26)$$



- (26) was previously found heuristically and allowed:
  - a **novel analytic characterization of QNMs**;
  - to **find new results for the BHs theory** (by many other authors);
  - an **unexpected application of Supersymmetry**.

Computing QNMs from Thermodynamic Bethe Ansatz

- The Q (or $Y = Q^2$) functions satisfy functional equations which can be inverted into the **Thermodynamic Bethe Ansatz (TBA)** for $\varepsilon(\theta) = -2\ln Q(\theta)$ (here for $SU(2)$ $N_f = 0$):

$$\varepsilon(\theta) = \frac{16\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2} e^{\theta} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\{-\varepsilon(\theta')\}\right]}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi}, \qquad (27)$$

  with $\varepsilon(\theta, P) \simeq 8P\theta$, $P > 0$ as $\theta \to -\infty$.

- The **QNMs condition** in gravity variables **reads also as**

$$\varepsilon(\theta_n - i\pi/2) = -i\pi(2n+1)\,. \qquad (28)$$

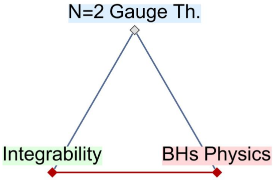
n	l	TBA	Leaver
0	0	1.36912 − 0.504048i	1.36972 − 0.504311i
0	1	2.09118 − 0.501788i	2.09176 − 0.501811i
0	2	2.8057 − 0.501009i	2.80629 − 0.501000i
0	3	3.51723 − 0.500649i	3.51783 − 0.500634i
0	4	4.22728 − 0.500453i	4.22790 − 0.500438i

Table: Comparison of QNMs of the D3 brane from TBA (27) (through (28) with $n = 0$), Leaver method (with $L = 1$).

Through TBA we have a **new way to numerically compute QNMs**.

Other applications of integrability

- We notice that **much of the BH theory seems to go in parallel to the ODE/IM correspondence** construction and its 2D integrable field theory interpretation, beyond the determination of QNMs!
  - For instance, also the **greybody factor** that parametrizes the **Hawking radiation** seems to be **ratio of Qs**
  - Intepretation and use of T, TQ relation...



Details: Alternative QNMs quantizations condition for D3 brane

- From the relation between T and a it follows a proof of the **alternative quantization condition for quasinormal modes** (only for the $N_f = 0$ $SU(2)$ theory, that is the D3 brane)

$$a(\hbar, \Lambda_0) = n/2 \,, \qquad n \in \mathbb{Z} \,. \tag{29}$$

- Indeed, the TQ relation $T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)$ means $Q(\theta_n - i\pi/2) + Q(\theta_n + i\pi/2) = 0$. This and the QQ relation $Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q^2(\theta)$ actually fixes $Q(\theta_n + i\pi/2)Q(\theta_n - i\pi/2) = 1$ and then

$$Q(\theta_n \pm i\pi/2) = \pm i \tag{30}$$

are fixed, too. Again the QQ relation around θ_n forces $Q(\theta + i\pi/2) = i \pm Q(\theta) + \dots$ and $Q(\theta - i\pi/2) = -i \pm Q(\theta) + \dots$ up to smaller corrections (dots). Therefore, TQ relation imposes

$$T(\theta_n) = 2\cos 2\pi a(\theta_n) = \pm 2 \,. \tag{31}$$

No jokes : Schwarshild black hole !

- ODE (16) for **Paperclip IM generalizes** to that for $N_f = 3$ $SU(2)$ **gauge th.**

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \Big\{ e^{2y}\Lambda_3 \left(4(m_1 - m_2)^2\right) + 4e^{y}\sqrt{\Lambda_3}\left(-2\hbar^2 + 8m_1 m_2 + \Lambda_3 m_3 - 8u\right) + \left(\Lambda_3^2 - 24\Lambda_3 m_3 + 64u\right) + 4e^{-y}\sqrt{\Lambda_3}\left(8m_3 - \Lambda_3\right) + 4\Lambda_3 e^{-2y} \Big\} \frac{1}{16\left(\sqrt{\Lambda_3}e^{y} - 2\right)^2}\psi(y) = 0\,. \tag{32}$$

- Also through the following map

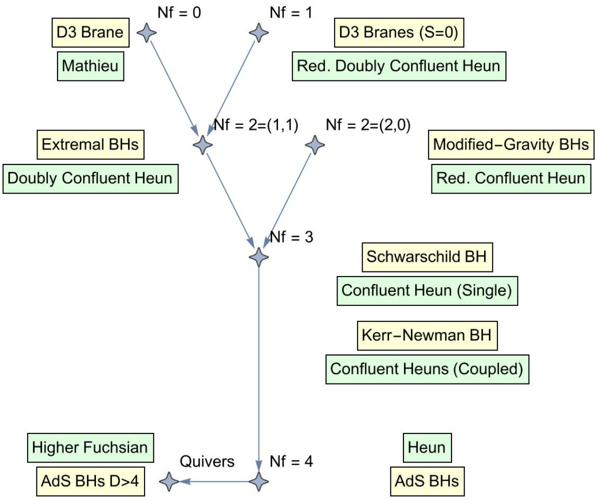
$$r = \frac{4M}{\Lambda_3}e^{-y}\,, \qquad \hbar = 1\,, \qquad \Lambda_3 = -16iM\omega\,, \qquad u = l(l+1) - 8M^2\omega^2 + \frac{1}{4}\,, \qquad m_1 = s - 2iM\omega\,, \qquad m_2 = -s - 2iM\omega\,, \qquad m_3 = -2iM\omega\,, \tag{33}$$

becomes the master equation for **linear perturbations of the Schwarschild BH !**

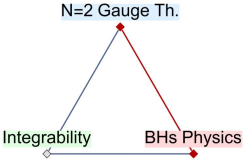
$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - f(r)\left[\frac{l(l+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right]\right]\phi(r) = 0\,, \quad \text{with } f(r) = \left(1 - \frac{2M}{r}\right) \tag{34}$$

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

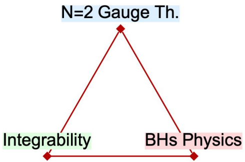
General picture of new gauge-gravity-ODE/IM triality



- $SU(2)$ $\mathcal{N} = 2$ (NS) gauge theories and BHs models.

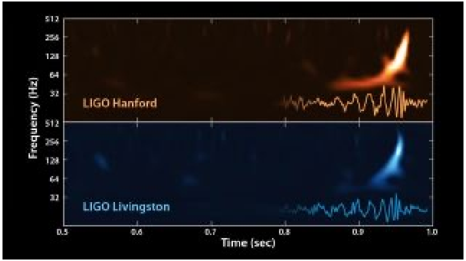


- We extended the triality with ODE/IM until $N_f = 3$ (single).



Importance for modified gravity and phenomenology

- Even if QNMs in GR are well understood, it is still an open problem to compute them in **modified gravity**, so developing **new analytic characterizations and numerical methods** is very important.
- Integrability could become a new powerful mathematical in **gravitational phenomenology** and search of new physics.



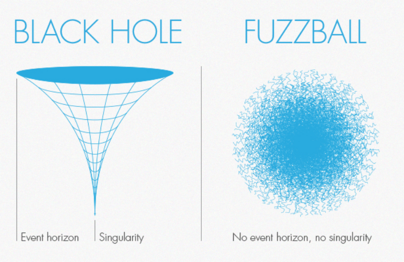
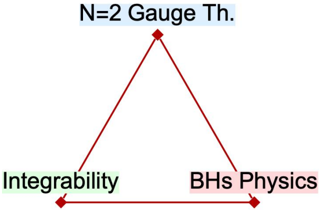


Figure: GW150914 (LIGO); Olena Shmahalo/Quanta Magazine

Conclusions

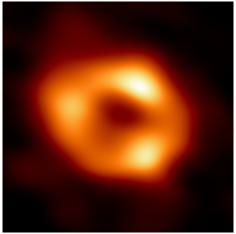
Summary

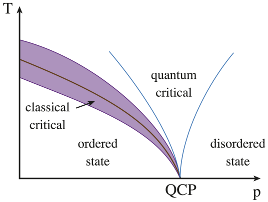
- We have shown how **integrable models** when studied in **ODE/IM correspondence** approach can find:
  - natural connection to (deformed) $\mathcal{N} = 2$ **SUSY gauge theory**
  - as well as to **black hole perturbation theory**
  - and shed light on the **relation** recently found **between the two**.
- This **new triality** besides being interesting in itself allows to:
  - **find new results** an all three sides of the correspondence
  - at the **non-perturbative exact level**.
- Especially for QNMs of BHs we give:
  - **new analytic characterizations** from Q, Y, T functions
  - **new method of computation** as Thermodyn. Bethe Ansatz.

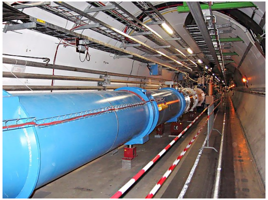


Future developments

- $\mathcal{N} = 2$ gauge theories and BHs constitute **novel fields of application of integrability** many future developments could be pursued:
  - **model generalization**;
  - **theory depth**;
  - **connection to other fields** (like holography).







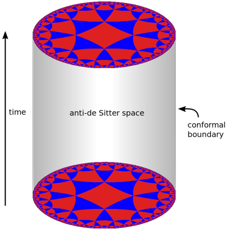
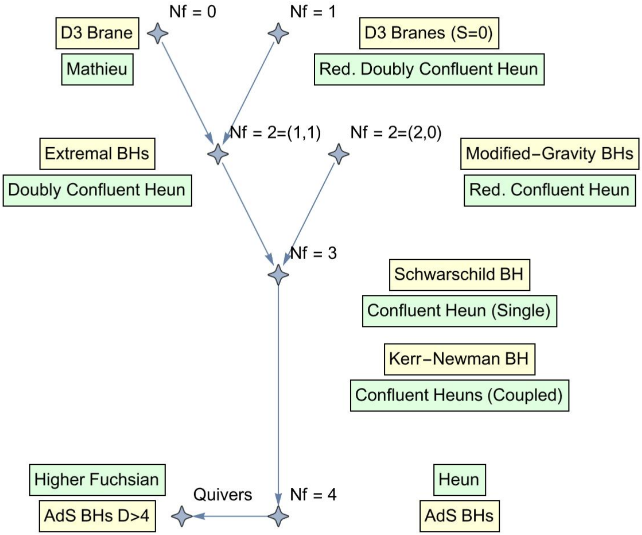


Figure: (Wikicommons/Herzog,Dunkel)



Thank you

for your attention!