Conformal field theory at the edge of Quantum Hall droplets and Coulomb gas

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Based on [Oblak, Lapierre, Moosavi, JMS, Estienne arXiv:2301.01726] and [Work in progress]

Hamiltonian in a magnetic field + trapping potential

$$H = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2 + \frac{k}{2}(x^2 + y^2)$$

with

$$\mathbf{p} = -\mathrm{i}\hbar \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \qquad \mathbf{A} = \frac{B}{2} \begin{pmatrix} -y \\ x \end{pmatrix}$$

naturally leads to single particle wave functions ($\ell^2 = 1/B$)

$$\phi_m(z) = \frac{z^m}{\ell\sqrt{2\pi m!}} e^{-|z^2|/4\ell^2}$$

and ground state many-body wavefunction

$$\Psi(z_1, \ldots, z_N) = \det_{1 \le j,m \le N} (\phi_{m-1}(z_j))$$

Density profile for large N

Well-known free fermions model, everything determined from the two-point function:

$$\begin{aligned} K_N(z,w) &= \langle c^\dagger(z)c(w)\rangle \\ &= \sum_{m=0}^{N-1} \phi_m^*(z)\phi_m(w) \end{aligned}$$

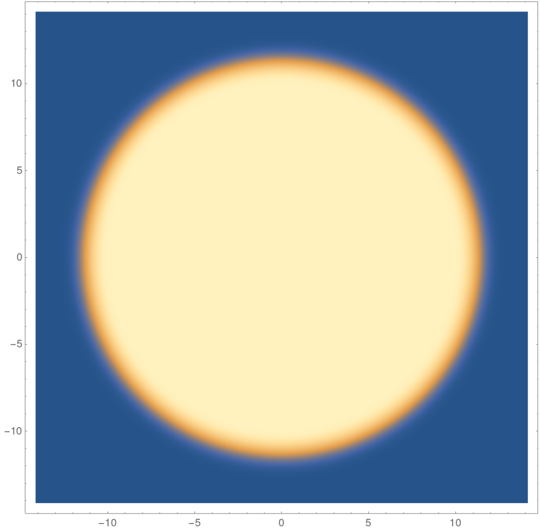
Density profile $K_N(z,z) = \rho_N(z)$. Constant density in the bulk.

Edge behavior:

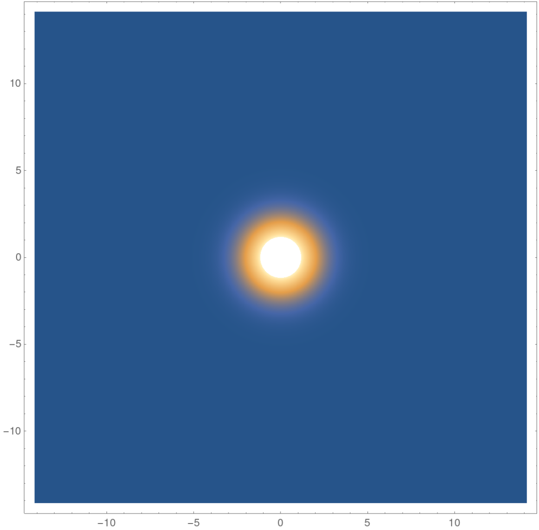
$$\rho_N(r,\varphi) \sim \frac{\mathrm{erfc}(d/\ell)}{4\pi\ell^2}$$

Droplet is a disc with radius $\ell\sqrt{2N}$, d is distance to the boundary.

Density profile for large N



$|K(x_0, w)|$ for some x_0 on the real line



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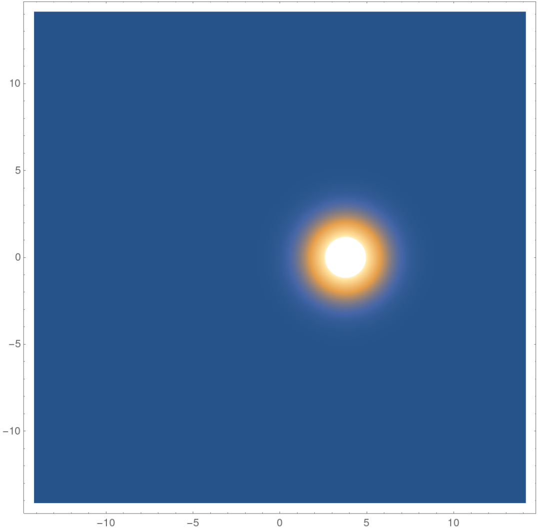

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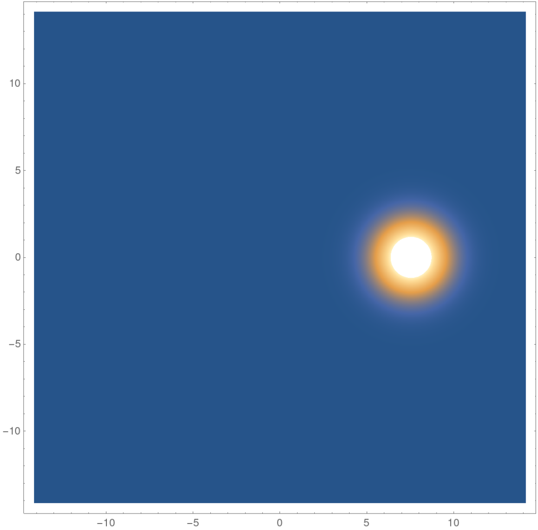

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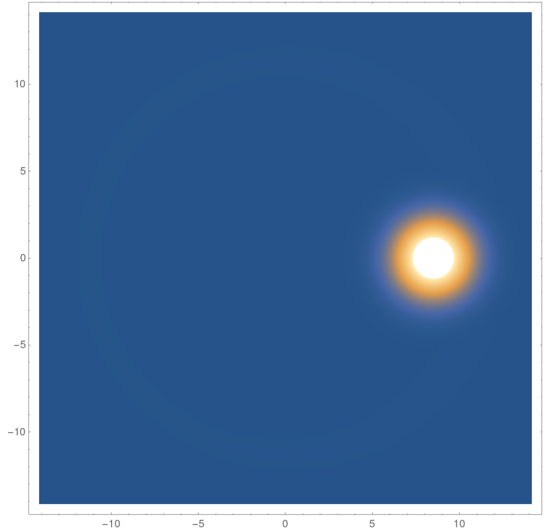

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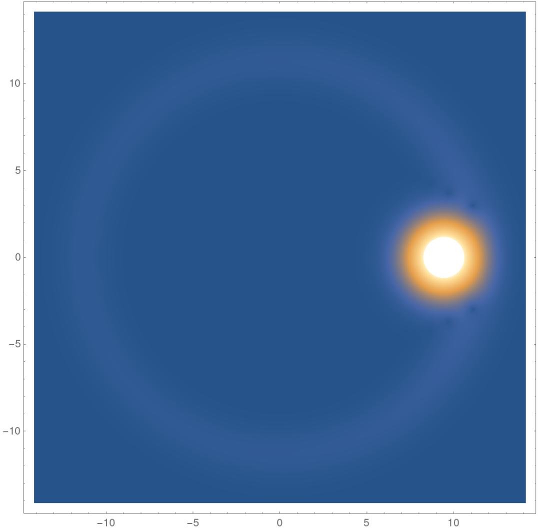

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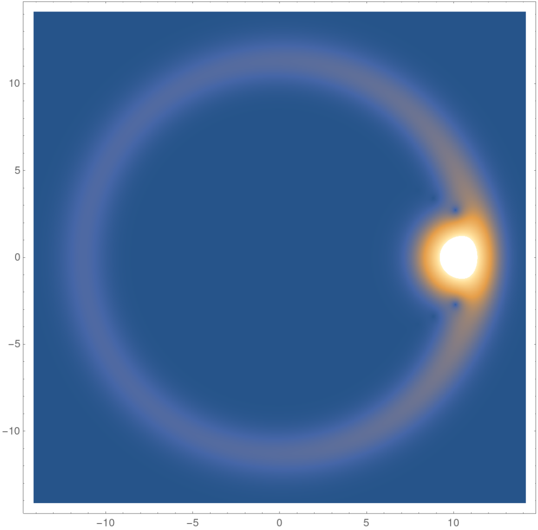

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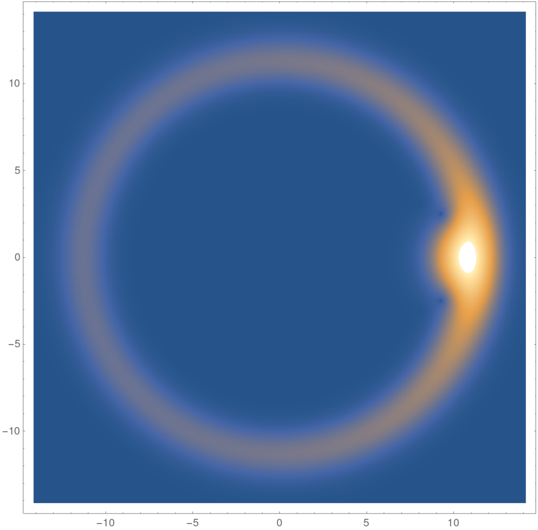

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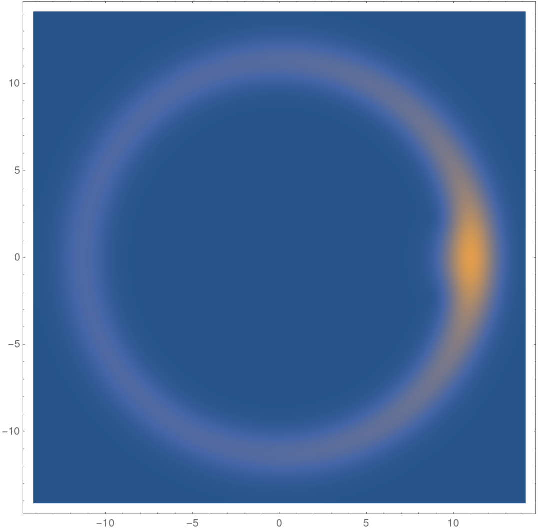

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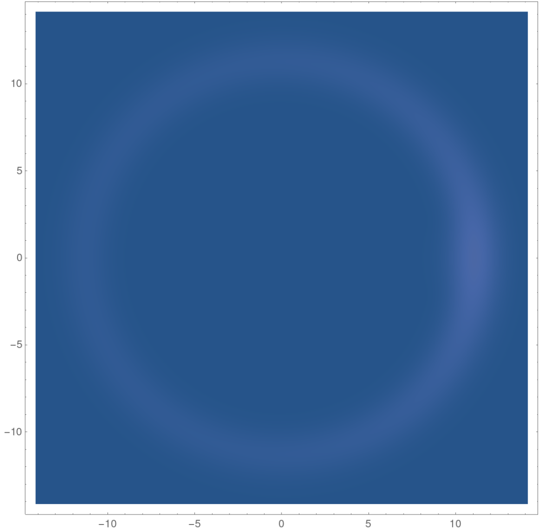


Power law decay for large N and $\theta \in (0, 2\pi)$ [Jancovici 1982]:

$$|K_N(\ell\sqrt{2N}, \ell\sqrt{2N}e^{\mathrm{i}\theta})| \sim \frac{\text{constant}}{\ell^2\sqrt{N}\sin\frac{\theta}{2}}$$

Ideas of droplet deformations: symmetries [Cappelli, Trugenberger & Zemba, 1993], exact ellipse calculation [Di Francesco, Gaudin, Itzykson, Lesage 1994], electrostatics and conformal maps [Choquard, Piller & Rentsch 1986], generalizations [Zabrodin & Wiegmann 2006], free field, proofs [Leblé & Serfaty 2018], [Ameur & Cronvall 2022], euclidean conformal field theory [Moore & Read 1991].

Coincidence

$$|\Psi(z_1,\ldots,z_N)|^2 = \prod_{j<k} |z_j - z_k|^\Gamma e^{-\sum_{j=1}^N |z_j|^2}$$

where $\Gamma = 2$. Other values of Γ: Laughlin state. Gas of N classical particles with logarithmic interactions at some inverse temperature

$$E(z_1,\ldots,z_N) = -\sum_{j<k} \log|z_j - z_k| + \sum_{j=1}^N W(z_j)$$

Quadratic W corresponds to quadratic V [Di Francesco, Gaudin, Itzykson, Lesage 1994] [Forrester & Jancovici 1995], but no general correspondence.

This talk

Similarities and differences between the Coulomb log gas problem, and the quantum Hall problem?

Deformed quantum Hall

Edge deformed Hall droplets

Radially symmetric potentials $V_0(r^2)$ yield circular droplets.

Area preserving deformations $(r^2, \varphi) \mapsto (\frac{r^2}{f'(\varphi)}, f(\varphi))$

Motivates the study of potentials of the form $V(r, \varphi) = V_0\left(\frac{r^2}{f'(\varphi)}\right)$

Example of a diffeo

$$e^{\mathrm{i}kf(\varphi)} = \frac{\cosh\lambda\, e^{\mathrm{i}k\varphi} + \sinh\lambda}{\sinh\lambda\, e^{\mathrm{i}k\varphi} + \cosh\lambda}$$

$k = 2$ maps the circle to an ellipse, $k = 3$ maps the circle to a "flower" with 3 petals, etc.

Strategy

The single particle wave functions are not explicit anymore, but nevertheless perform similar computations as before:

- For large m, solve asymptotically the Schrödinger equation using a WKB method projected to LLL.

$$\phi_m(r,\varphi) \sim \frac{e^{\mathrm{i}\Theta_m(r,\varphi)}}{\ell\sqrt{2\pi\sigma(\varphi)}} \frac{e^{-d^2/(2\ell^2)}}{(2\pi m)^{1/4}}$$

  where d is distance to the equipotential.

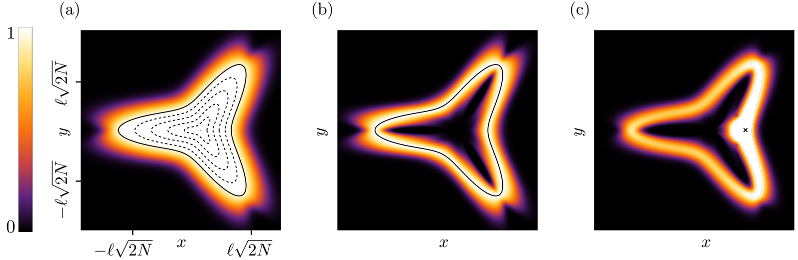
- Plug this in $K_N(z,w) = \sum_{k=0}^{N-1} \phi_m^*(z)\phi(w)$

Edge correlator

Near the edge

$$K_N(z_1, z_2) \sim \frac{e^{\mathrm{i}\cdots}}{\ell^2\sqrt{N}} \frac{e^{-(d_1^2+d_2^2)/2\ell^2}}{\sin\left(\frac{f(\varphi_1)-f(\varphi_2)}{2}\right)}$$

From left to right (flower 3): density, current, edge correlator.



Deformed Coulomb gas with Boltzmann weights $e^{-\beta E}$ with

$$E(z_1, \ldots, z_N) = -\sum_{j<k} \log |z_j - z_k| + \sum_{j=1}^{N} W(z_j)$$

Droplet A for the classical particles, set by the potential W.

The coarsed-grained potential felt by a test particle at position z is

$$U(z) = -\int_A \rho(w) \log |z-w|^2 d^2w$$

for large N. In the bulk, it satisfies

$$W(z) + U(z) = \lambda \quad , \quad z \in A$$

Outside of the droplet, $U(z)$ is harmonic.
Byproduct: density profile satisfies $\rho(z) = \frac{\Delta W}{4\pi}$ in the bulk.

Edge correlations

Dirichlet (or electrostatics or boundary CFT) problem naturally pop up in the Coulomb gas setup.

On can pushing this logic further, introducing a density of particles per unit line $\sigma(z)$ near the boundary.

[Alastuey & Jancovici 1984; Choquard, Piller & Rentsch 1986; Jancovici 1995]

$$\langle \sigma(z)\sigma(w) \rangle = \frac{1}{2\pi^2\beta} \frac{|G'(z)G'(w)|}{|G(z)\overline{G(w)} - 1|^2}$$

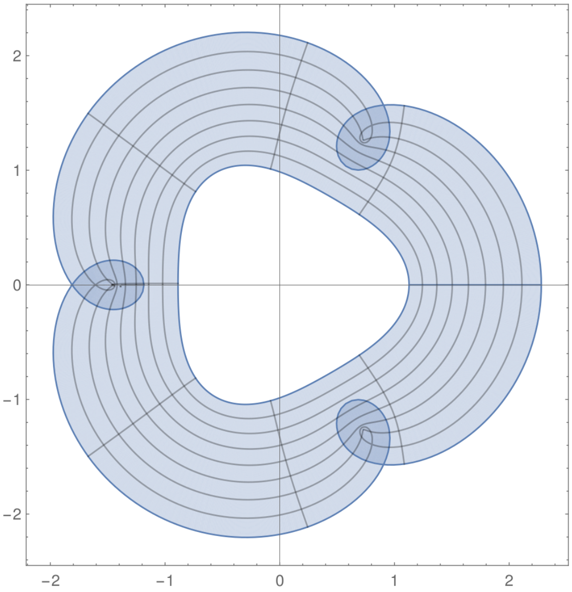
where G is the conformal map from the exterior of the droplet (+point at ∞) to the exterior of the unit disk (+ point at ∞).

Conformal maps

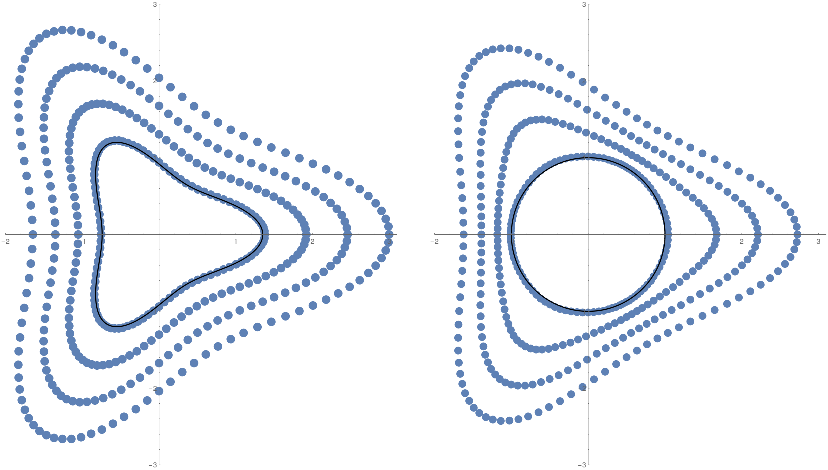
Same form as the edge correlator for integer quantum hall, with the identification $G^{-1}(e^{\mathrm{i}\theta}) = \frac{e^{\mathrm{i}\, f(\theta)}}{\sqrt{f'(\theta)}}$.

Seemingly "just" need to extend $e^{\mathrm{i}\theta}$ to z with $|z| \geq 1$ and the two edge results have the same form.

However this is typically not possible.



This is not an (inverse) conformal map!



Correct (numerical) conformal map G: analytic in the whole exterior of the droplet, bijective, analytic inverse.

Conclusion

- Many similarities between the two systems at the free fermions point: density, gaussian decay.
- But edge correlator seem generically different.
- Quantum droplet shape measurable through microwave absorption in the quantum setup.
- Other signatures by studying entanglement, full counting statistics, e.g. [Estienne & JMS 2019; Estienne, JMS & Witczak-Krempa 2022], density profile at the edge [Can, Forrester, Tellez & Wiegmann 2013; Cardoso, JMS & Abanov 2021], etc.

Thank you!