Integrability and exact solution of the multi-species non-equilibrium stirring process

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Based on a joint work with Rouven Frassek, Cristian Giardinà
Soon on the ArXiv

10th workshop on conformal field theory and integrable models

Stochastic interacting particle systems

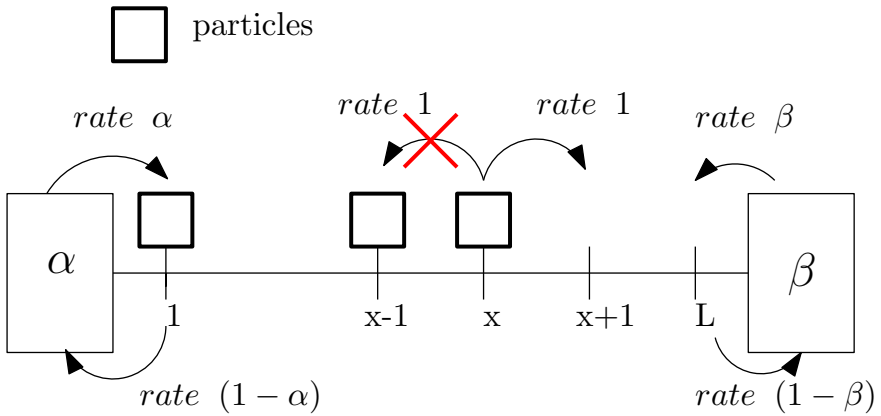
- **Stochastic particle systems** = continuous time Markov processes on a lattice
- **States**: $|n(t)\rangle = (n^1(t), \ldots, n^L(t)) \in \Omega$
- **Transitions**: Exponential clock. Hamiltonian H

$$h(n, n') \geq 0 \qquad \text{rate from } n \text{ to } n'$$

$$\frac{d}{dt}|n(t)\rangle = H|n(t)\rangle$$

- **Description**: $H \iff$ Markov process
- **Simplest example**: independent random walk

Non-equilibrium symmetric simple exclusion process



Original process: exchange particle with the reservoirs

$n^x \in \{1, 2\}$ where 2= empty

$$H = H_{left} + \sum_{x=1}^{L-1} H_{x,x+1} + H_{right} \quad \text{on} \quad \Omega := \bigotimes_{x=1}^{L} \mathbb{C}^2$$

- **Hamiltonian density and boundary Hamiltonian**

  $H_{x,x+1} = P - I$ $\quad$ H_{left}, H_{right} exchange particles with reservoirs

- **open integrable XXX Heisenberg spin chain** (QISM)

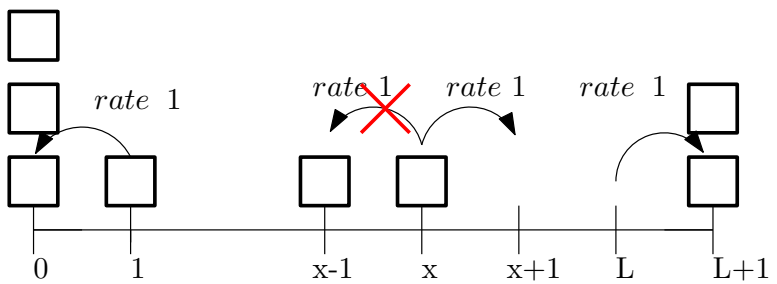
Stochastic duality

Definition

Consider a Markov process H acting on Ω and a second Markov process $\widetilde{H}$ acting on $\mathbb{N}_0 \otimes \Omega \otimes \mathbb{N}_0$ (extra-sites 0 and $L+1$). They are dual if there exists a matrix D (duality matrix) such that

$$H^T D = D\widetilde{H}$$

$\widetilde{H}$ has triangular boundaries. Eventually voids dual chain.



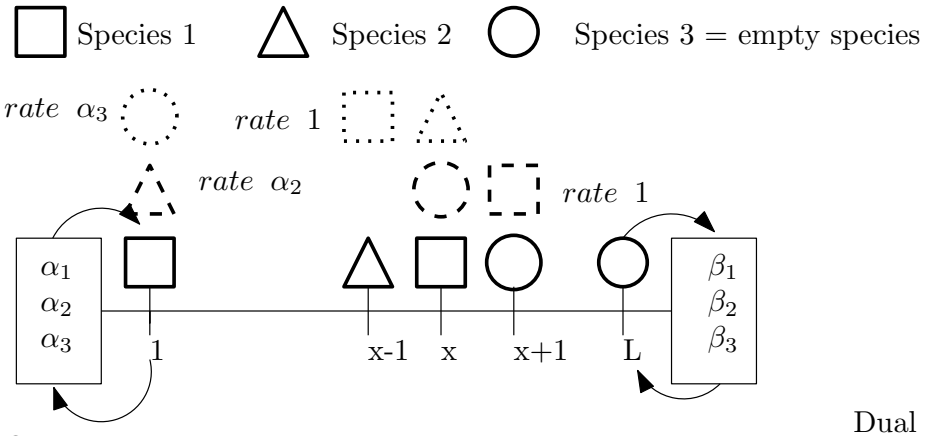
Dual process: absorbing extra-sites 0, L+1 in place of the reservoirs.
All particles are eventually absorbed in 0 or in L+1.

Absorption probabilities! (Hard to find)

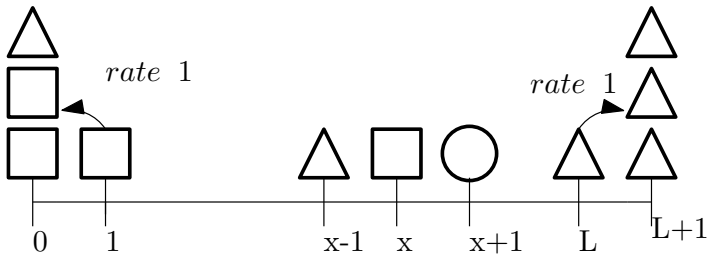
Duality meets integrability

- Some stochastic particles systems are integrable
- Prove duality (if it exists)
- Bring the chain to a simpler dual one
- Apply MPA, QISM to the dual chain; find absorption probabilities
- Go back to the original process

Multi-species non-equilibrium stirring process



Original process: stirring dynamics
Contact with external reservoirs



Dual dynamics: replace reservoirs with absorbing non-empty particles.
Eventually voids the chain

$n^x \in \{1, 2, \ldots, N\}$ where N= empty.

- **Configuration space**: $\Omega = \bigotimes_{x=1}^{L} \mathbb{C}^N$
- **Hamiltonian density and boundary Hamiltonian**

$$H_{x,x+1} = P - I \qquad H_{left}, H_{right} \text{ exchange types with reservoirs}$$

- **Higher rank open XXX Heisenberg spin chain**

Our results

1. **Prove duality**: Lie algebra $gl(N)$ symmetries
2. **Find exact formula for the steady state**:

$$H|\Psi\rangle = 0$$

   MPA commutators become easier $\Rightarrow$ absorption probabilities
3. **Map non-equilibrium onto equilibrium**: perturbation theory

$$|\Lambda\rangle = W|\Lambda^0\rangle \implies \widetilde{H}W = WH_0$$

   Equilibrium eigenvectors from Bethe ansatz.
   W function of conserved charges

Thank you for the attention

For details come to see my poster!