All Regular 4×4 solutions of the Yang–Baxter Equation

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Yang–Baxter Equation

We study solutions to the Yang–Baxter equation

$$R_{12}(u,v)R_{23}(v,w)R_{13}(u,w) = R_{13}(u,w)R_{23}(v,w)R_{12}(u,v),$$

where $R(u,v) : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$.

YBE ensures the existence of an integrable spin chain Hamiltonian

$$\mathbb{Q}_2(u) = \sum_{i=1}^{L} \mathcal{H}_{i,i+1}(u), \qquad \mathcal{H}_{i,i+1}(u) = P\partial_u R_{i,i+1}(u,v)\Big|_{v\to u}$$

and higher commuting charges $\mathbb{Q}_3(u), \mathbb{Q}_4(u), \dots$.

Regularity $R(u,u) = P$ is a natural assumption which ensures that $\mathbb{Q}_1 = U$.

Sutherland Equations and Boost Operator

Goal: For $n = 2$, classify all the regular solutions of YBE.

The approach is based on two main ingredients.

- Sutherland equations

$$[R_{13}R_{23}, \mathcal{H}_{12}(u)] = \dot{R}_{13}R_{23} - R_{13}\dot{R}_{23},$$
$$[R_{13}R_{12}, \mathcal{H}_{23}(v)] = R_{13}R'_{12} - R'_{13}R_{12}.$$

- Boost operator $\mathcal{B} : \mathbb{Q}_r \to \mathbb{Q}_{r+1}$

$$\mathbb{Q}_3(u) = \partial_u \mathbb{Q}_2(u) - \sum_{j=1}^{L} [\mathcal{H}_{j,j+1}(u), \mathcal{H}_{j+1,j+2}(u)].$$

The Method

- Parametrise a general Hamiltonian density $\mathcal{H}(u)$.
- Compute $\mathbb{Q}_3(u)$ using the boost operator.
- Impose $[\mathbb{Q}_2(u), \mathbb{Q}_3(u)] = 0$, and solve this for $\mathcal{H}(u)$.
- For each integrable Hamiltonian $\mathcal{H}(u)$, solve the Sutherland equations to find the R-matrix, subject to the boundary conditions

$$R(u,u) = P, \qquad \partial_u R(u,v)\Big|_{v\to u} = P\mathcal{H}(u).$$

- Verify that computed $R(u,v)$ indeed satisfies YBE.

Results and Outlook

Main results:

- We classified all regular 4×4 solutions of the YBE.
- Beyond usual six-/eight-vertex models we find several novel solutions.
- Many of the new models give rise to Hamiltonians which are non-diagonalisable.
- Found an interesting non-diagonalisable integrable deformation of the XXX model.

Outlook:

- Perform the classification for $n > 2$.
- Consider models on non-compact spaces, e.g. $\mathbb{C}[z]$.
- Classify integrable higher-range models.