Remarks on BPS Wilson loops in non-conformal $\mathcal{N} = 2$ gauge theories and localization

Alessandro Testa
University of Parma

in collaboration with Luca Griguolo and Marco Billo'

What's to come

1. Introduction and main motivations
2. Localization in $d = 4$ SYM theories
3. BPS Wilson loops in non-conformal $\mathcal{N} = 2$ QCD
4. Perturbative renormalization vs localization
5. Open questions

Introduction and main motivations

Introduction

Strongly coupled QFT's can be probed in the presence of a large amount of symmetry such as in ($4d$) $\mathcal{N} = 4$ super-Yang-Mills (SYM), where **exact formulae** have been obtained by means of localization, integrability and holography

$\mathcal{N} = 2$ SYM theories are less protected but the amount of supersymmetry is sufficient to apply **supersymmetric localization** on $\mathbb{S}^4$ which, in superconformal set-ups, was

- successfully tested against standard perturbative approaches for protected observables, such as 1/2 BPS Wilson loops [Andree (2010)] and special correlators of chiral operators [Baggio (2014), Komargodsky (2017), Billo' (2018)];
- employed in the study of the AdS/CFT correspondence in non-maximally supersymmetric theories [Pomoni (2016), Billo' (2021)];
- employed in the study of Brehmsstrahlung functions in $\mathcal{N} = 2$ SYM theories [Komargodsky (2015), Penati (2019), Bianchi (2019)];

Breaking conformal symmetry

When relaxing the condition on conformal symmetry the theory becomes highly non-trivial due to technical and conceptual issues (perturbative renormalization, computations on curved spacetimes...)

The localization mechanism on $\mathbb{S}^4$ **does not** require conformal symmetry but it is not obvious how this technique deals with ultraviolet divergent quantities which require a renormalization

Set-up: $\mathcal{N} = 2$ super-Yang-Mills theories with massless matter content and

$$\beta(g) \neq 0$$

Goals:

1. Understanding whether and how localization incorporates the renormalization procedure by computing the v.e.v. of the 1/2 BPS Wilson loop
2. Investigating how the conformal symmetry breaking occurs

Localization in $d = 4$ Super-Yang-Mills (SYM) theories

A quick look at supersymmetric localization

In (some) SYM theories, the partition function is *exactly* captured by the semi-classical expansion of an auxiliary quantity [Witten 1988]

$$\mathcal{Z}(t) = \int \mathrm{e}^{-S - t\mathcal{Q}V} \quad \overset{t\to\infty}{\longrightarrow} \quad \mathcal{Z} = \int_{\mathcal{F}} \mathrm{d}X_0 \; \mathrm{e}^{-S[X_0]} \; \underbrace{\left.\frac{\det(\mathrm{Fer})}{\det(\mathrm{Bos})}\right|_{X_0}}_{\mathcal{Z}_{1-\mathrm{loop}}}$$

with $\mathcal{Q}S = \mathcal{Q}^2 V = 0$ and $\mathcal{F}$ the space of critical configurations of $\mathcal{Q}V$. This mechanism also holds in the presence of an operator $\mathcal{O}_{BPS}$ such that $\mathcal{Q}\mathcal{O}_{BPS} = 0$

Note: Integrating out quadratic fluctuations about $\mathcal{F}$ gives rise to $\mathcal{Z}_{1-loop}$

SU(N) $\mathcal{N}=2$ SYM theories on $\mathbb{S}^4$

A vector multiplet (V) and (massive and/or massless) matter multiplets (H) in a representation $\mathcal{R}$ of SU(N)

$$\mathrm{V}_{\mathcal{N}=2} = \begin{cases} 1 \text{ vector } A_\mu \\ 2 \text{ real scalars } (\phi_1, \phi_2) \\ 2 \text{ chiral fermions } (\psi_\alpha, \lambda_\alpha) \end{cases} \qquad \mathrm{H}_{\mathcal{N}=2} = \begin{cases} 4 \text{ real scalars } h_{1,2}, \tilde{h}_{1,2} \\ 2 \text{ chiral fermions } (\eta_\alpha, \tilde{\eta}_\alpha) \end{cases}$$

Applying supersymmetric localization on a four-sphere of radius r [Pestun (2007)]

$$\mathcal{Z} = \int \mathrm{da} \; \underbrace{e^{-(8\pi^2 r^2/g_0^2)\,\mathrm{tr}\,\mathrm{a}^2}}_{\text{Classical term}} \times \underbrace{\mathcal{Z}^{(H,\mathcal{R})}_{\text{Perturbative}} \times \mathcal{Z}^{(H,\mathcal{R})}_{\text{Instantons}}}_{\equiv \; \mathcal{Z}_{1-\text{loop}}}$$

with $\mathrm{a} \in \mathfrak{su}(N)$ parametrizing the Coulomb Branch

General features of $\mathcal{Z}_{1-\text{loop}}$

$\mathcal{Z}_{1-\text{loop}}$ is the **key** feature of $\mathcal{N}=2$ and becomes trivial only in $\mathcal{N}=4$ set-ups

$$\mathcal{Z}_{1-\text{loop}} = 1 \quad \Leftrightarrow \quad \mathcal{R} = \text{Adj}$$

$\mathcal{Z}_{1-\text{loop}}$ does not spoil the convergence iff the representation $\mathcal{R}$ satisfies

$$i_{\mathcal{R}} = N \quad \Leftrightarrow \quad \beta(g) = 0$$

For $\beta \neq 0$ the one-loop determinants have to be regularized with a consistent prescription

BPS Wilson loops in non-conformal $\mathcal{N} = 2$ QCD

Localization results

Localization of massless $\mathcal{N} = 2$ QCD

A vector multiplet coupled to N_f hypermultiplets in the fundamental of $SU(N)$ makes the $\mathbb{S}^4$-partition function inconsistent:

$$i_{\mathcal{R}} = \frac{N_f}{2} \neq N \quad \rightarrow \quad \mathcal{Z}_{\mathrm{Pert}} \sim e^{(2N - N_f)\,\mathrm{tr}\,\mathrm{a}^2 \log \mathrm{a}} \qquad \text{for} \quad \mathrm{a} \rightarrow \infty$$

But there is no problem if we add $N_f' = 2N - N_f$ massive hypermultiplets, since $i_{\mathcal{R}} = N$ and $\beta(g)$ is vanishing

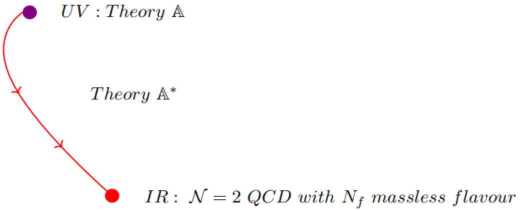
Note: we consider asymptotically free theories $2N > N_f$

A regulating flow

Consider theory $\mathbb{A}^*$ with a vector multiplet coupled to N_f massless and N'_f equally massive fundamental hypers, such that

$$2N = N_f + N'_f \quad \leftrightarrow \quad \beta(g) = 0$$

This theory defines a flow from superconformal QCD ($\mathbb{A}$) to $\mathcal{N} = 2$ QCD with N_f flavours and has a well-defined matrix model ($\mathcal{Z}_{1-\text{loop}} \sim \mathrm{e}^{M^2 \,\text{tr} \log \mathrm{a}}$)



Integrating out the massive fields

In the limit $M \to \infty$ the Gaussian term receives a contribution from $\mathcal{Z}_{\text{Pert}}^{M\to\infty}$:

$$\hat{S}_{\text{Cl}} = 8\pi^2 r^2 \overbrace{\left[\frac{1}{g_M^2}\right.}^{\textit{Gaussian term}} - \overbrace{\frac{(2N - N_f)}{8\pi^2} \log Mr}^{\textit{Infinitely-massive multiplets}} \left.\vphantom{\frac{1}{g_M^2}}\right] \operatorname{tr} \mathrm{a}^2$$

$$\equiv \frac{8\pi^2 r^2}{\widehat{\mathbf{g}}^2} \operatorname{tr} \mathrm{a}^2$$

Consistent result: the gauge coupling constant runs under the *RG* accordingly to the β-function of $\mathcal{N} = 2$ QCD with N_f massless flavours and endowed with a *UV* cut-off given by M

Note: this procedure is valid in any non-conformal set-ups

Localization of a 1/2 BPS Wilson loops on $\mathbb{S}^4$

Localization expresses the BPS circular Wilson loop on the equator of $\mathbb{S}^4$ as

$$W(C) = \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\,\exp\Big\{\int_C \Big[\mathrm{i}\;\mathrm{d}A + \mathrm{d}\tau\;\, r\;\phi_1\Big]\Big\} \quad\rightarrow\quad W(\mathrm{a}) = \frac{1}{N}\,\mathrm{tr}\;\,\mathrm{e}^{2\pi r \mathrm{a}}$$

The weak-coupling prediction ($\mathcal{Z}_{\mathrm{Inst}} = 1,\; g_M << 1$) of localization is

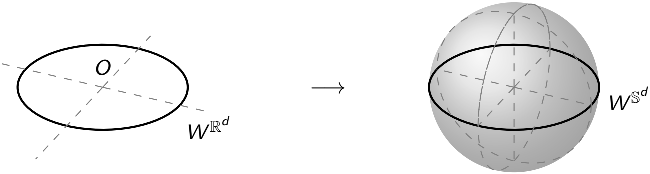
$$\langle W(C)\rangle = \underbrace{\frac{C_F g_M^2}{4} + \frac{(2N^2-3)C_F g_M^4}{192N}}_{\text{Ladder contributions},\; C_F \,=\, (N^2-1)/(2N)} + \frac{C_F g_M^4}{32\pi^2}(2N - N_f)\;\underbrace{(1+\gamma_E+\log rM)}_{\text{Typical of ren. quantities}}$$

Field theory approach

Embedding formalism

Consider a set of inertial coordinates $x^M \in \mathbb{R}^{d+1}$ and identify a d-dimensional hyperplane ($x^0 = 0$) with $\mathbb{R}^d$, i.e. the flat space where we define the dimensionally regularized theory. Then $x^M \to X^M(x)$ via the stereographic projection (conformal map in $\mathbb{R}^{d+1}$!)

$\mathbb{R}^d \simeq \{x^0 = 0\}$ $\qquad$ $\mathbb{S}^d = \{X^M X_M = r^2\}$

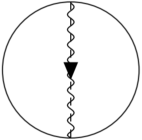


Propagators and perturbative one-loop diagrams are expressed on $\mathbb{S}^d$ in terms of the **embedding coordinates** X^M and acquire simple expressions

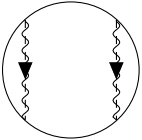
$$\langle \phi_i(x_1)\phi_j(x_2) \rangle = \frac{\delta_{ij}\Gamma(d/2-1)}{4\pi^{d/2}[X_{12}^2(x_1,x_2)]^{d/2-1}}$$

1/2 BPS supersymmetric Wilson loops

Ladder diagrams on $\mathbb{S}^d$ and $\mathbb{R}^d$ are regular for $d \to 4$ and **identical**

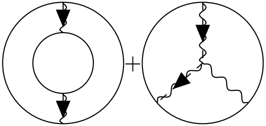


$$= \frac{g_0^2 C_F}{4}$$



$$= \frac{g_0^4 (2N^2 - 3)}{192N}$$

Ultraviolet divergent contributions



$$= \begin{cases} -\dfrac{C_F(2N - N_f) g_0^4 \alpha(d)}{(2 - d/2)} + (d/2 - 2) g_0^4 C_F A(d) \text{ on } \mathbb{S}^d \\ \underbrace{-\dfrac{C_F(2N - N_f) g_0^4 \alpha(d)}{(2 - d/2)}}_{\text{Identical to } \mathbb{S}^d!} + (d/2 - 2) g_0^4 C_F B(d) \text{ on } \mathbb{R}^d \end{cases}$$

with $B(d) \neq A(d)$. The evanescent terms in $\mathbb{R}^d$ and on $\mathbb{S}^d$ **are different**

Perturbative renormalization vs localization

Renormalized Wilson loops on $\mathbb{S}^4$

The *UV* divergences are removed by means of $g_0 = Z_g^{SQCD}\mu^{2-d/2}g(\mu)$ to all orders in perturbation theory [Korchemsky (1987)] for smooth contours

$$\tilde{W}_C^{\mathbb{S}^4} \equiv \lim_{d\to 4} \overbrace{\langle W^{\mathbb{S}^d}(C)\rangle}^{Dim.\ regularized}\Big|_{g_0=g(\mu)(\ldots)} \quad \rightarrow \quad \underbrace{\left(\beta(g)\frac{\partial}{\partial g} - E\frac{\partial}{\partial E}\right)\tilde{W}_C^{\mathbb{S}^4} = 0}_{Callan-Symanzik\ equation}$$

The solution of the C.S. equation is given by in terms of an arbitrary function F

$$\tilde{W}_C^{\mathbb{S}^4} = F(\widehat{\mathbf{g}}(E, g(\mu))) \qquad \text{with} \qquad \frac{\mathrm{d}\ \hat{g}}{\mathrm{d}\log\mu/E} = \beta(\hat{g})$$

Note: $E = 1/r$ and $\widehat{\mathbf{g}}$ is the running coupling encountered in localization

Comparing with localization and flat space

The renormalized observable matches the localization prediction if $\mu = M\sqrt{e^{\gamma_E}/\pi}$

$$\tilde{W}_C^{\mathbb{S}^4} = \frac{C_F g_\mu^2}{4} + \frac{C_F g_\mu^4 (2N^2-3)}{192N} + \underbrace{\frac{(2N-N_f)g_\mu^4}{64\pi^2}(\log r^2\mu^2}_{\text{The same in loc.approach}} + 1 + 2\gamma_E + \log\pi)$$

Repeating the procedure in flat space leads to the renormalized observable $\tilde{W}_C^{\mathbb{R}^4}$

$$\underbrace{\tilde{W}_C^{\mathbb{R}^4} = H(\widehat{\mathbf{g}}(E, g(\mu)))}_{\text{Due to C.S.equation}} \qquad \text{with} \qquad \boxed{\tilde{W}_C^{\mathbb{R}^4} = \tilde{W}_C^{\mathbb{S}^4} + \mathcal{O}(g^6)}$$

Surprisingly even if conformal symmetry is broken at the quantum level, the theory does not distinguish between the flat space and the sphere at order g_μ^4

Evanescent terms are not evanescent

The poles in Z_g activate the evanescent terms at subsequent perturbative orders and are the responsible for the **expected** mismatch between the flat-space and the sphere

$$\mathbb{S}^d \ : \ (d/2-2)(2N-N_f)g_0^4 C_F A(d) \quad \longrightarrow \quad \mathbb{S}^4 : \ g(\mu)^6 C_F (2N-N_f)^2 \mathcal{A}$$
$$\mathbb{R}^d \ : \ (d/2-2)(2N-N_f)g_0^4 C_F B(d) \quad \longrightarrow \quad \mathbb{R}^4 : \ g(\mu)^6 C_F (2N-N_f)^2 \mathcal{B}$$

We find that the two numerically coefficients are different

$$\boxed{\mathcal{B} \neq \mathcal{A}}$$

This non-trivial mechanism poses two interesting questions

1. Can we predict this effect by means of first principles ?
2. Does supersymmetric localization capture $\mathcal{A}$ at order g^6 ?

Open questions

Open questions

Our analysis highlights that localization naturally ties nicely in with the RG machinery and therefore, this technique seems to be extremely powerful also in non-conformal set-ups. **However** this analysis poses different questions and suggest future directions

- are the evanescent terms at high orders in perturbation theory on $\mathbb{S}^4$ captured by localization ?
- can we predict the anomaly in the change of coordinates connecting the flat space and the four-sphere ?
- in the decompatification limit $r \to \infty$ we have a breakdown of perturbation theory and non-perturbative effects, such as the the instanton corrections, should be predictable from localization
- exploring the agreement between localization and field theory approaches in non-conformal theories different dimensions

Thank you for your attention