The arctic curves of the four-vertex model arXiv: 2307.03076 (2023)

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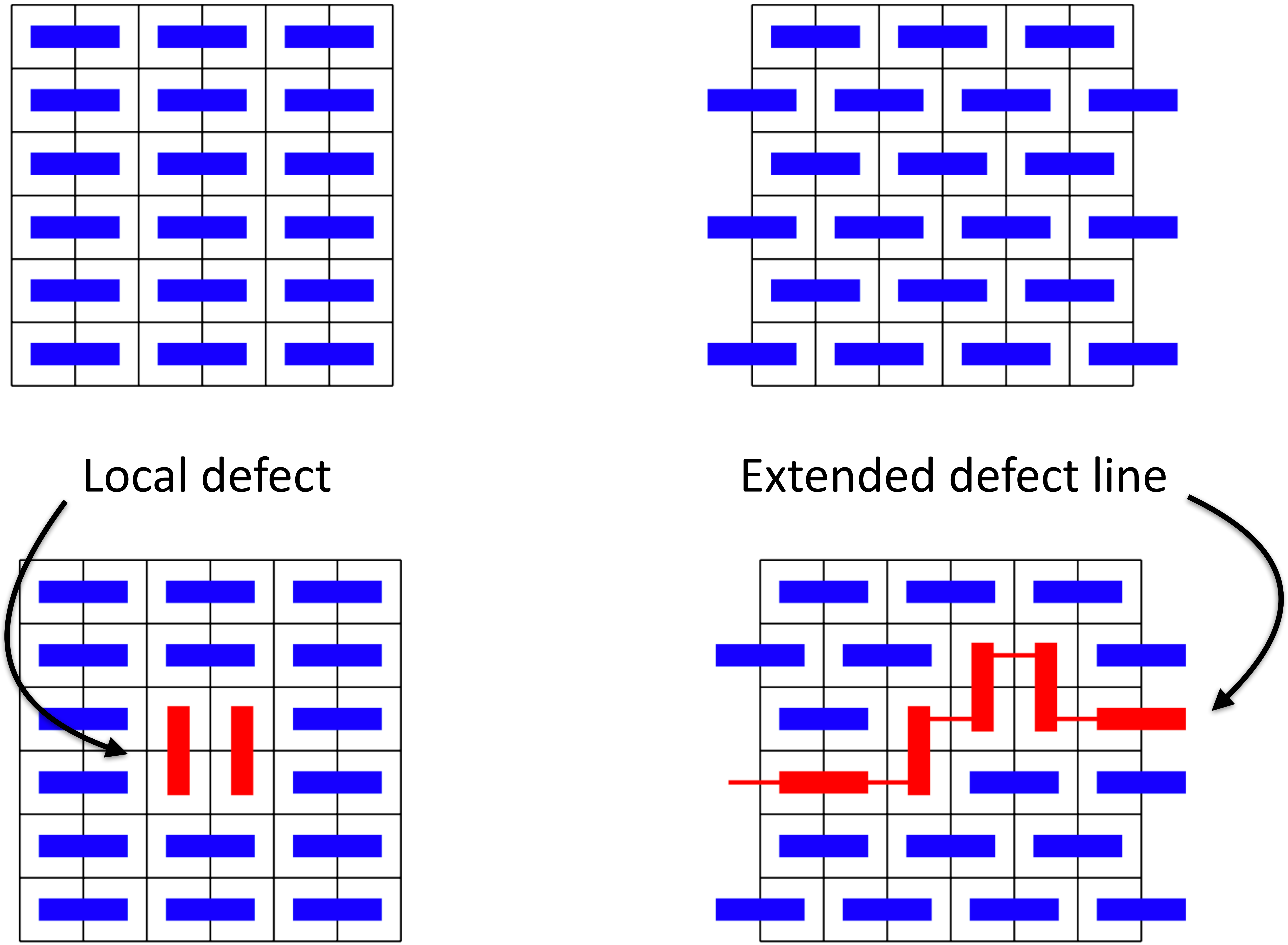
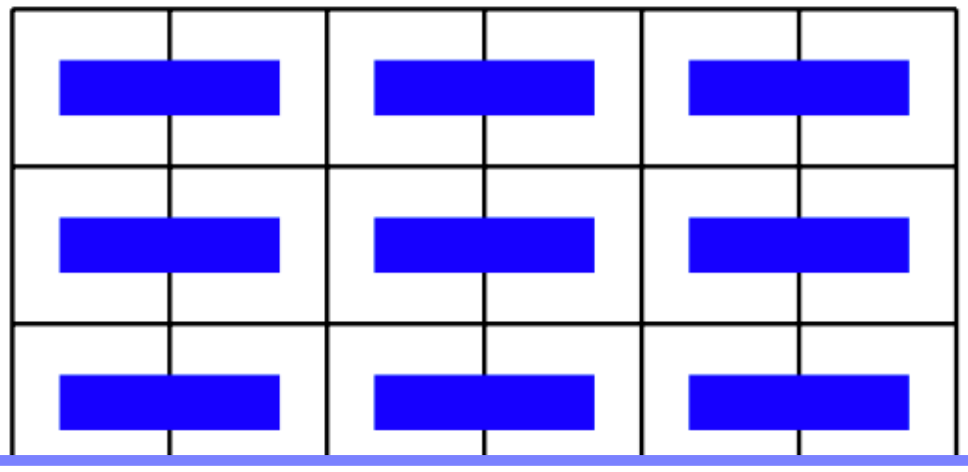
Contents

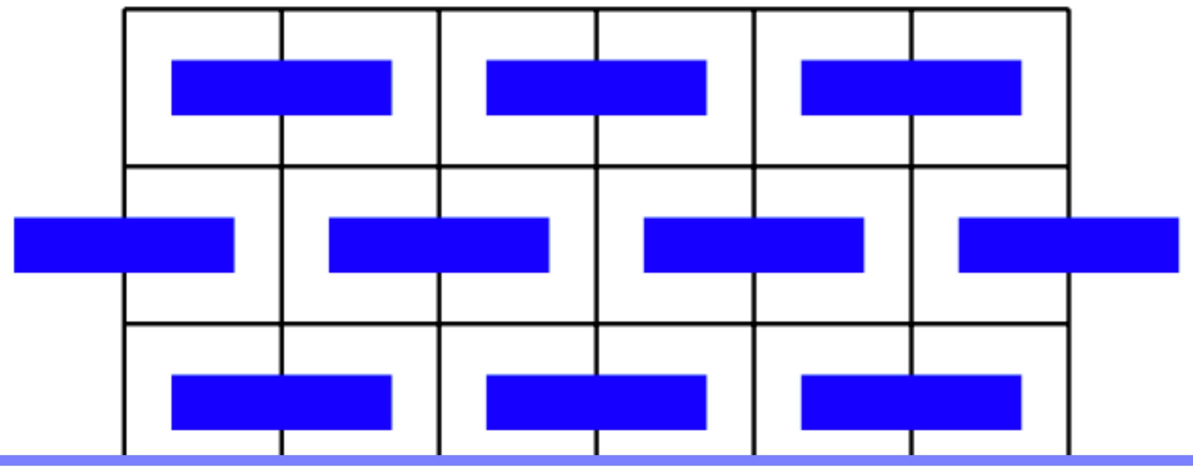
- General motivations
  - Tiling models
  - Adding the interactions
- The four-vertex model
  - Based on "*The arctic curves of the four-vertex model*"
    I. Burenev, F. Colomo, <u>AM</u>, A. Pronko (arXiv: 2307.03076)

Tiling of a square



Tiling of a square



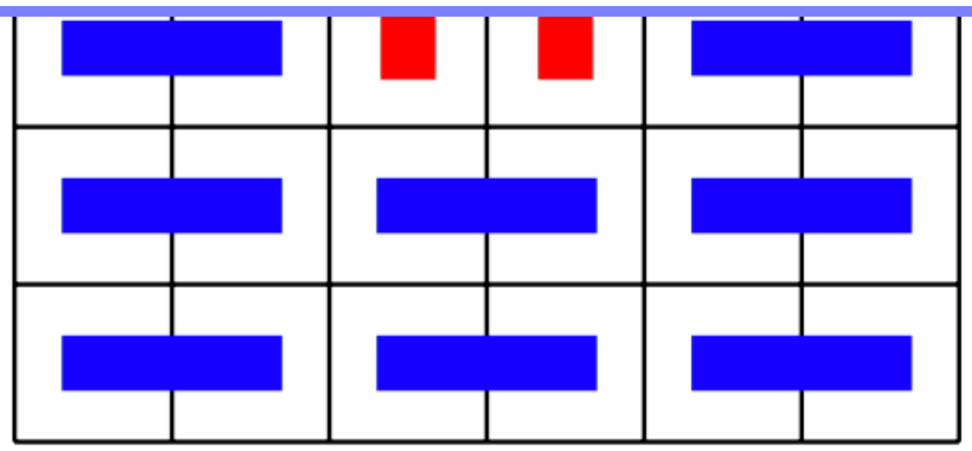


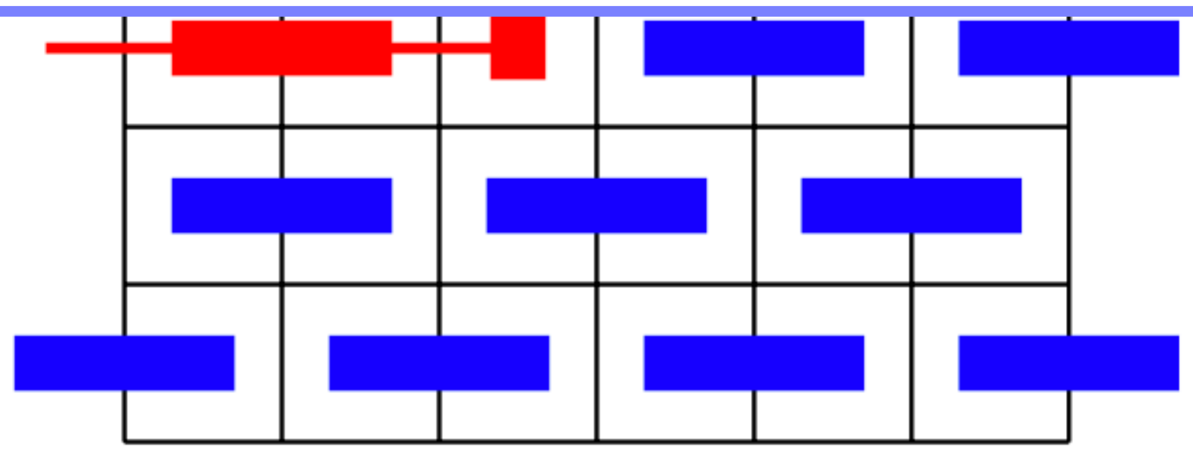
Geometric constraints can induce long range effects

Suitable boundary conditions may modify thermodynamics

Order parameters (free energy, etc…) may acquire spatial dependence

Possibility of **spatial phase separation**

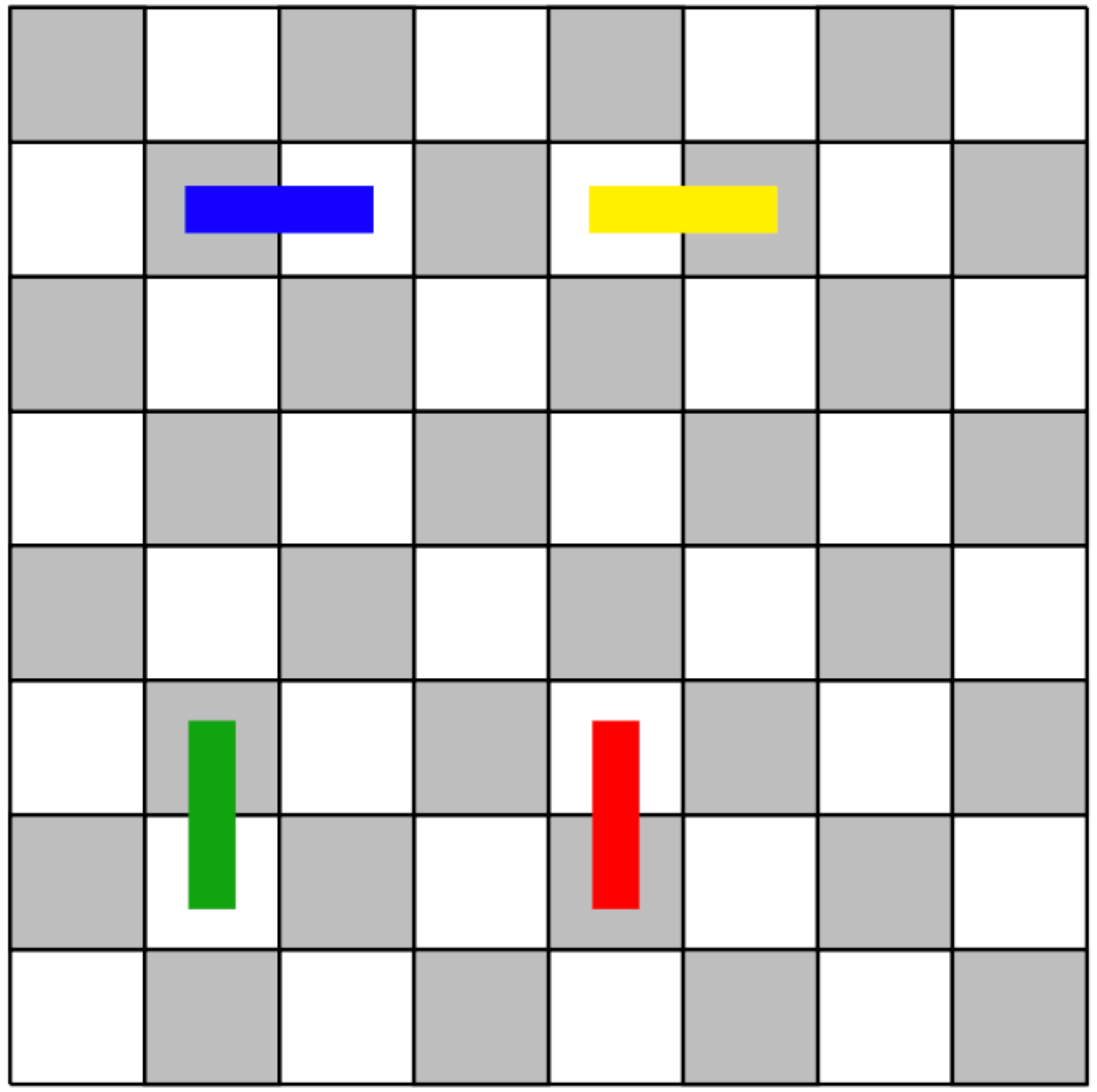




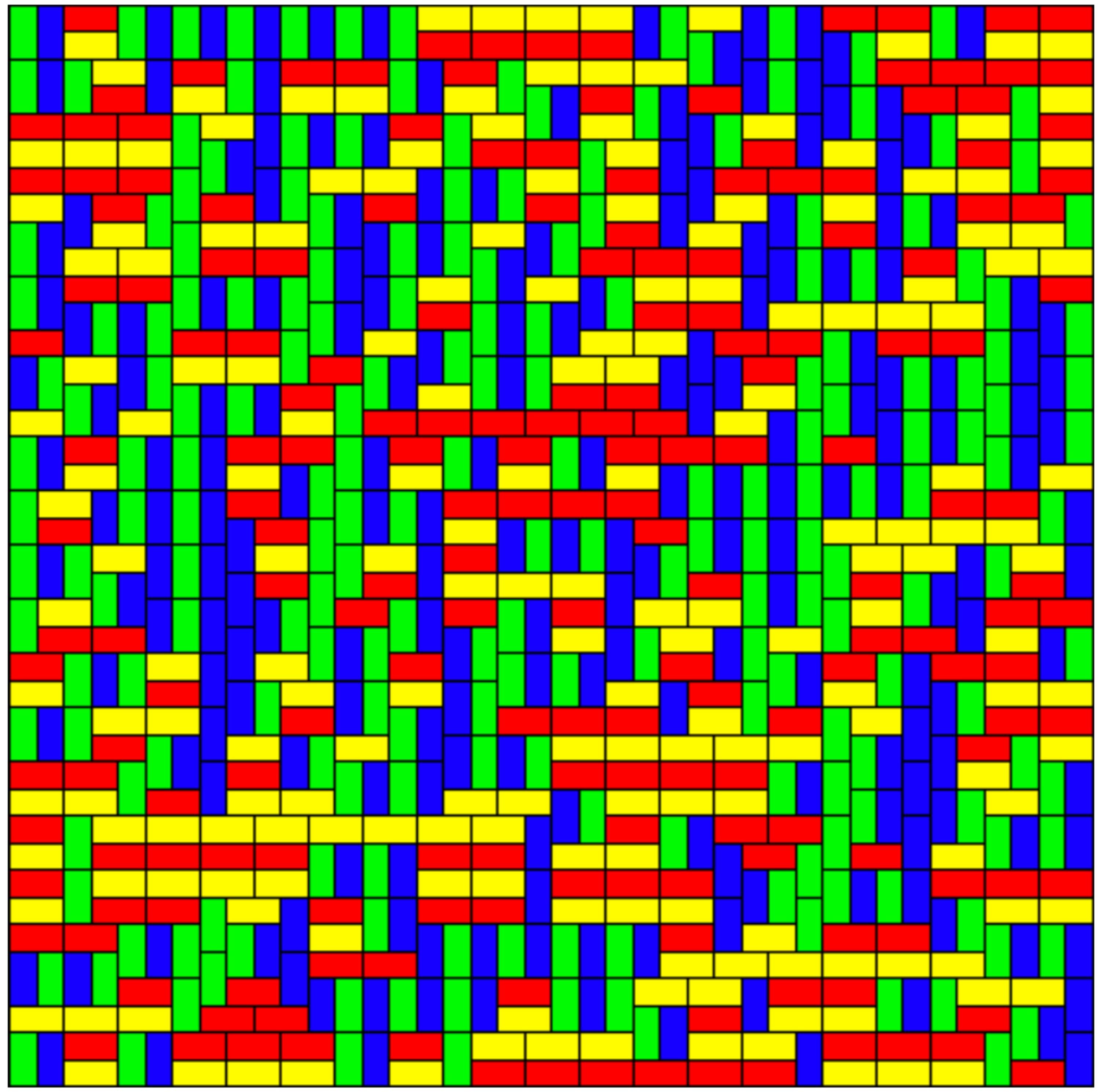
Tiling of a bipartite square

Square $N \times N$

Color coding of the tiles



Random tiling

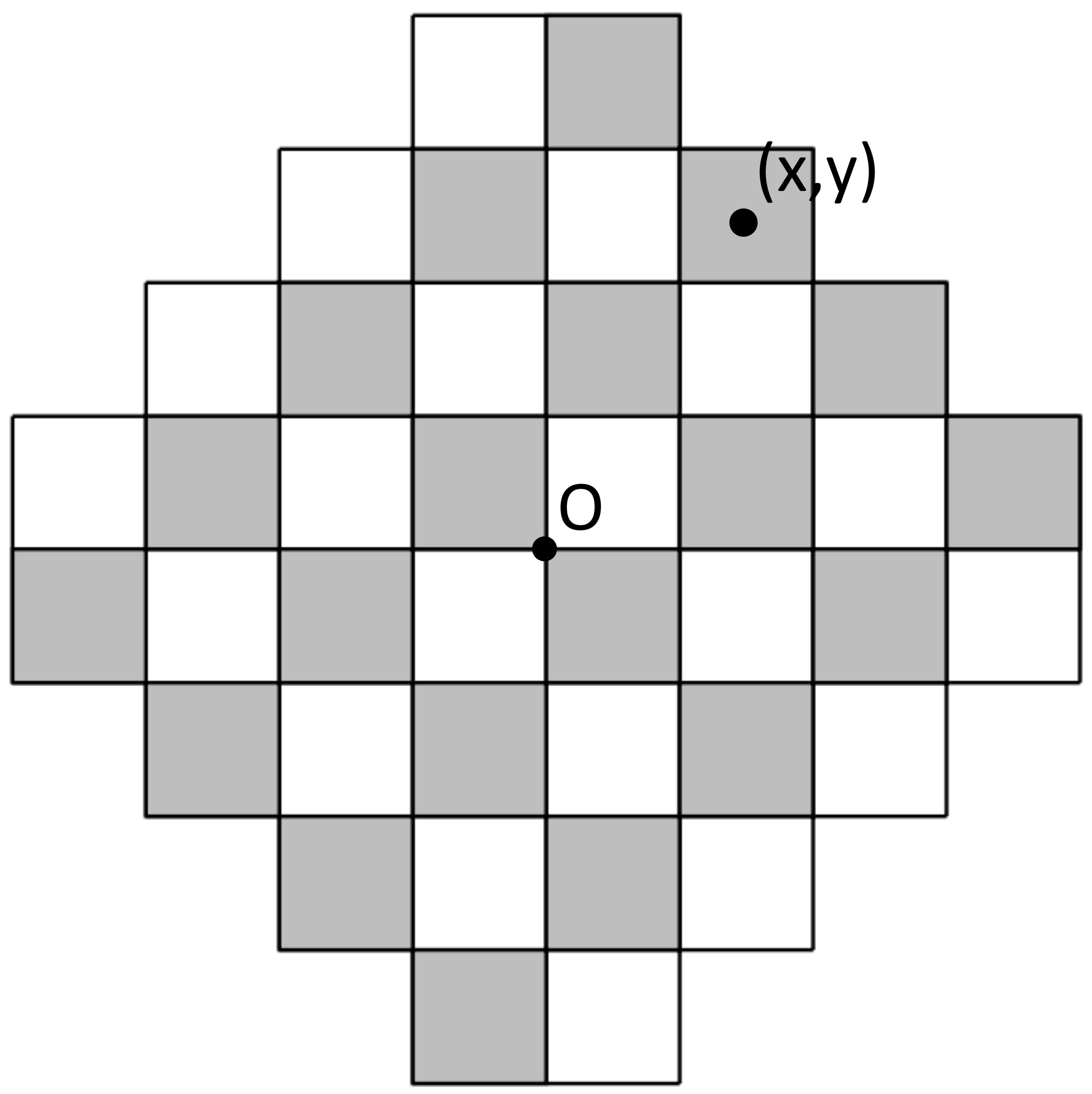


The arctic circle Theorem

Aztec diamond

of order *N*

$$|x| + |y| \leq N$$

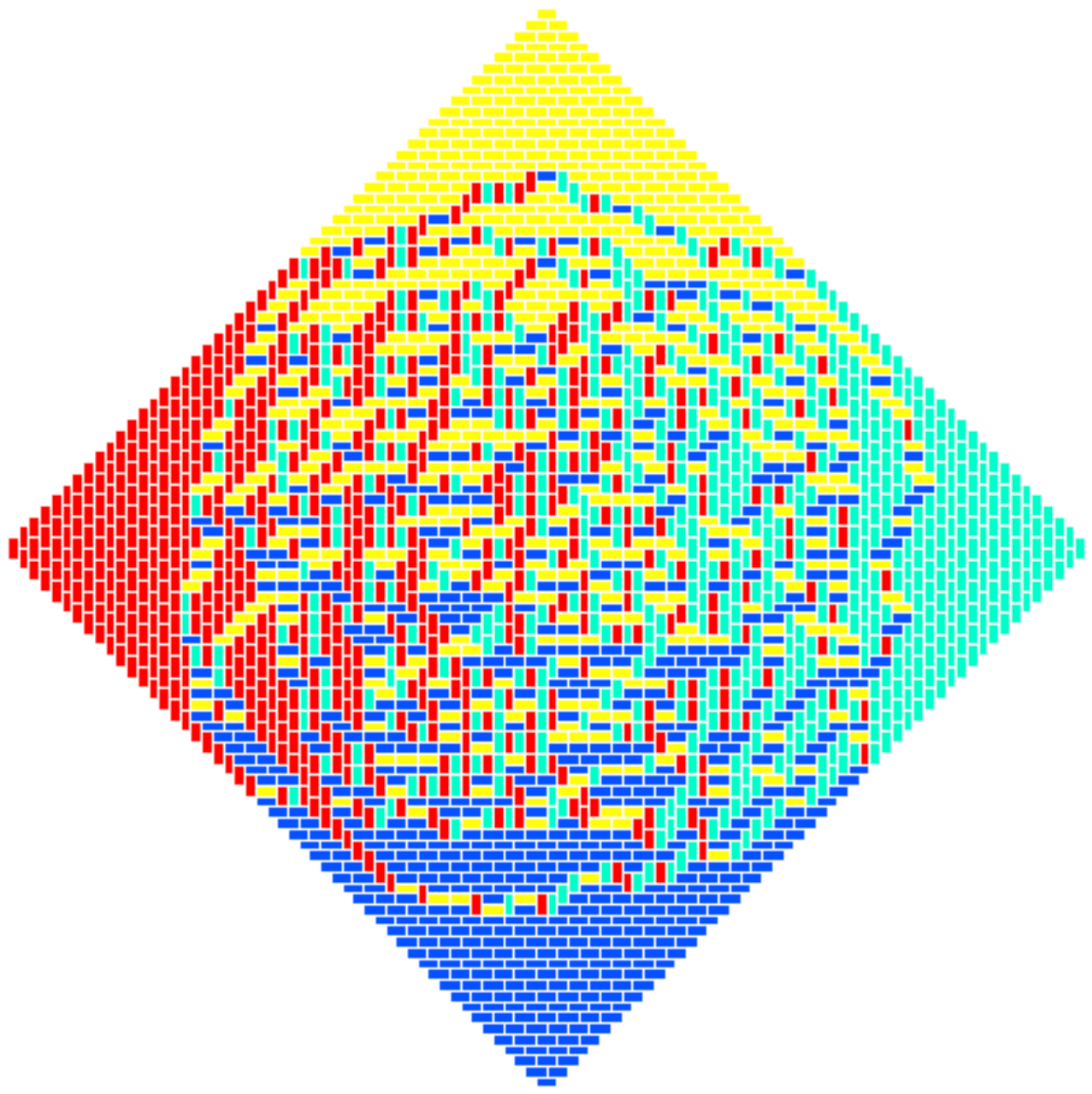


N = 4

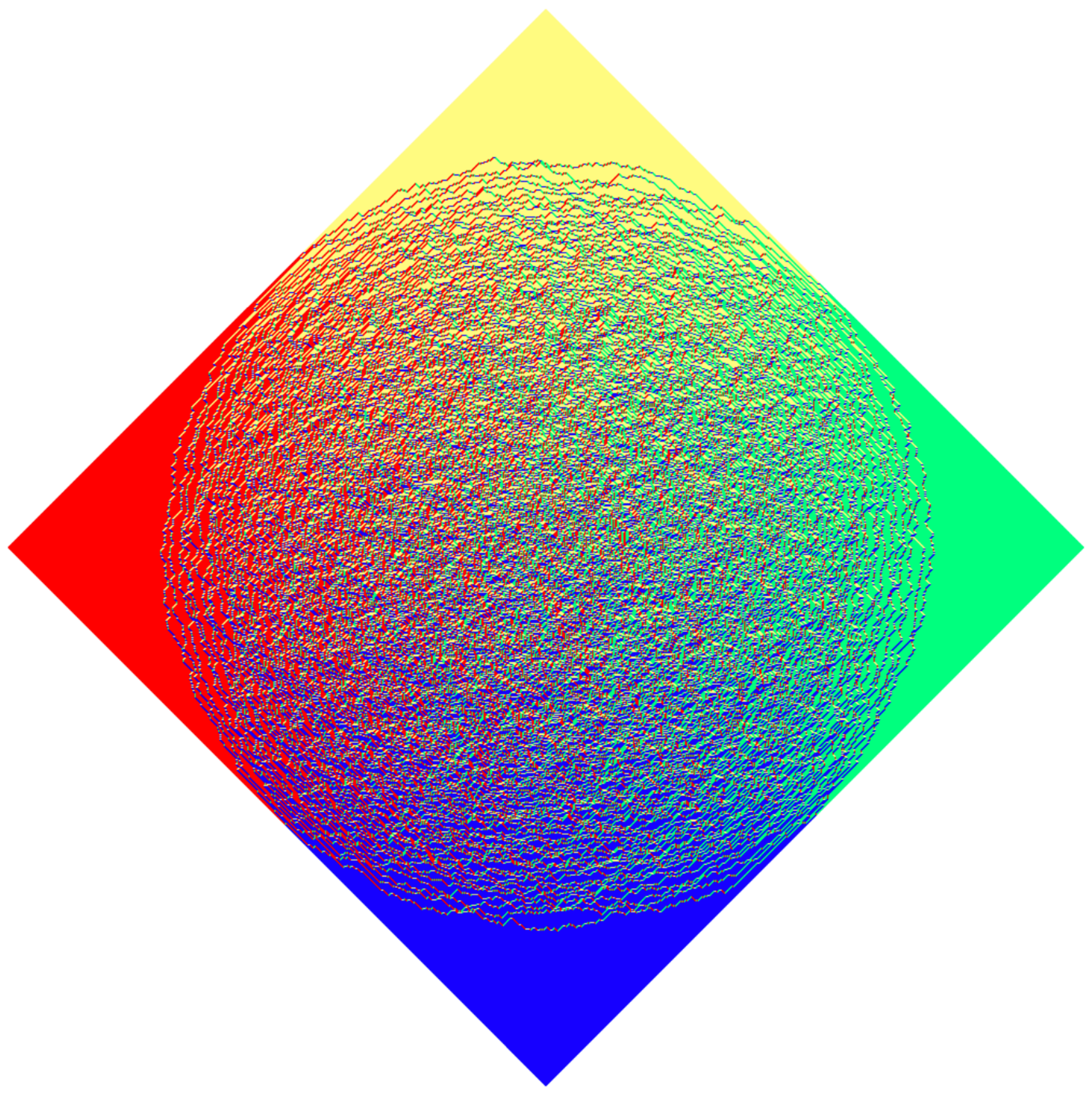
Arctic Circle Theorem

[Jockush - Shor - Propp, 1995]

“In the scaling limit the arctic curve is a circle”



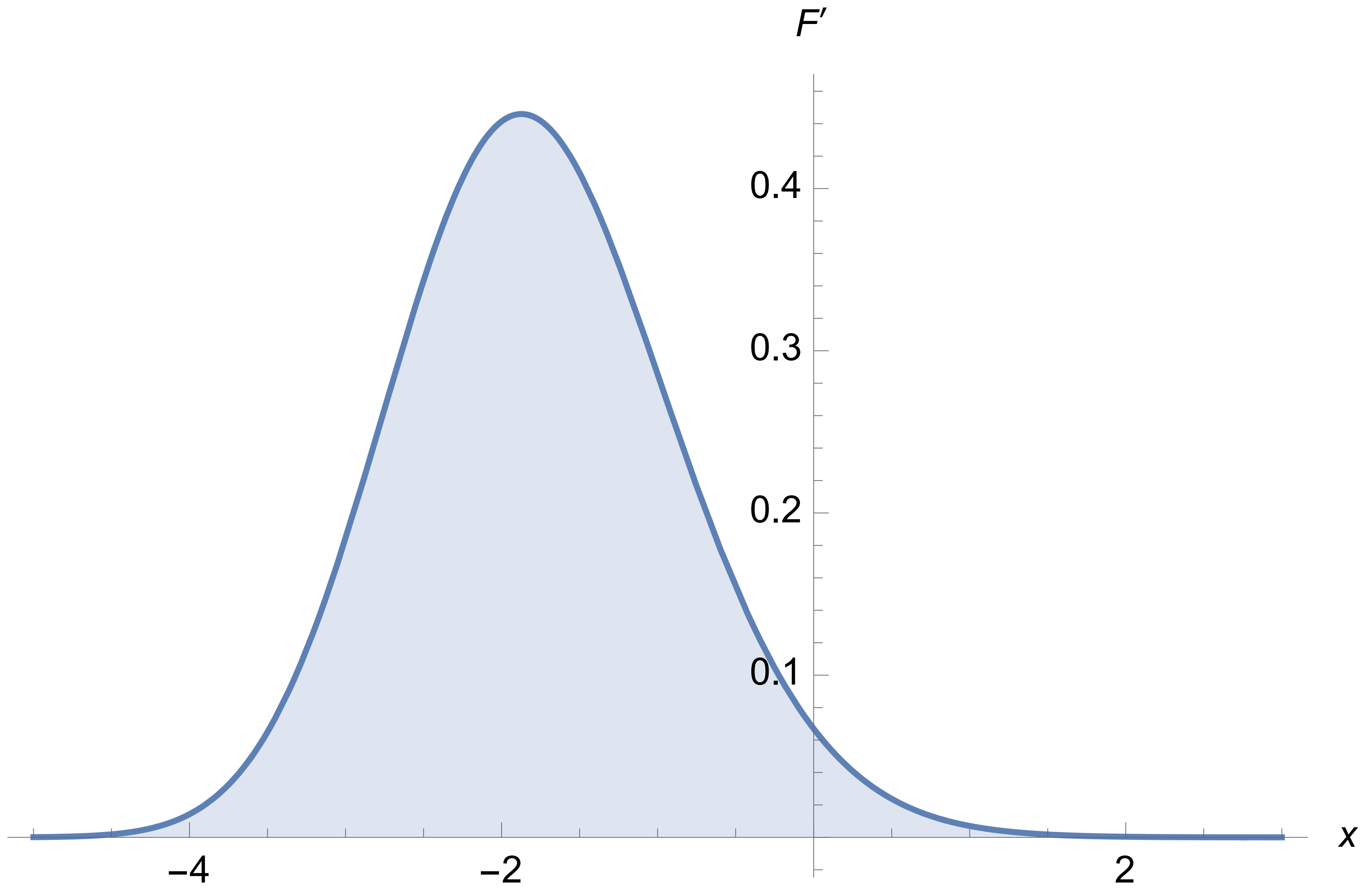
N = 64



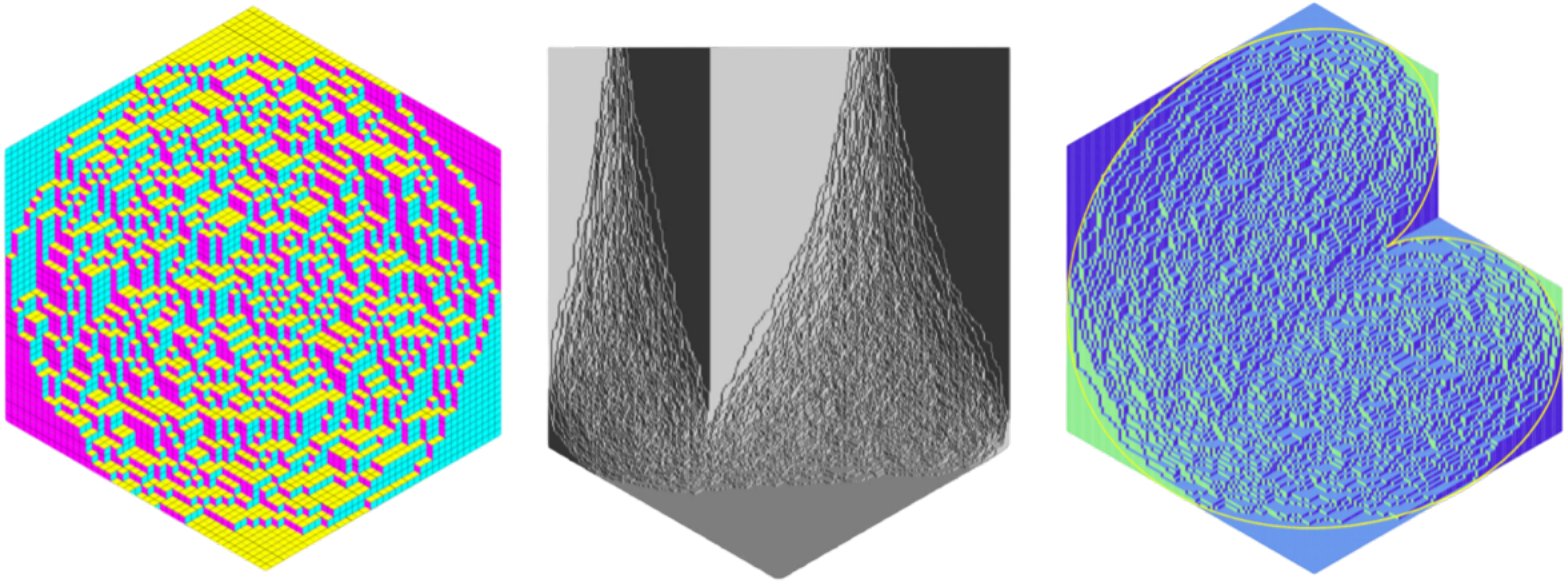
N = 500

Fluctuations around the arctic curve

- fluctuations $\propto N^{1/3}$ [Johansson, 2000]
- fluctuations along the diagonal are governed by Tracy-Widom distribution [Johansson, 2002]



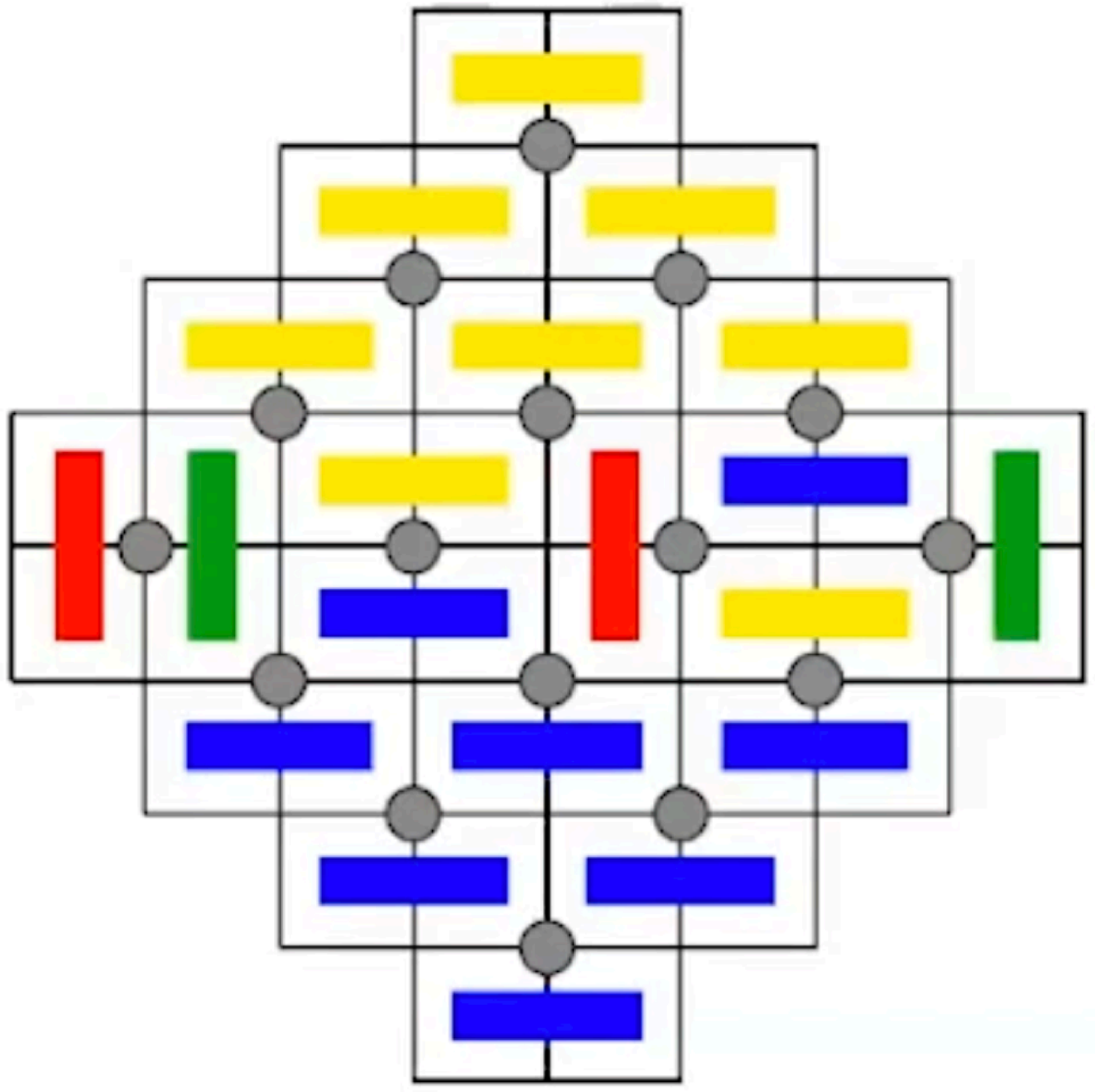
Other tiling models



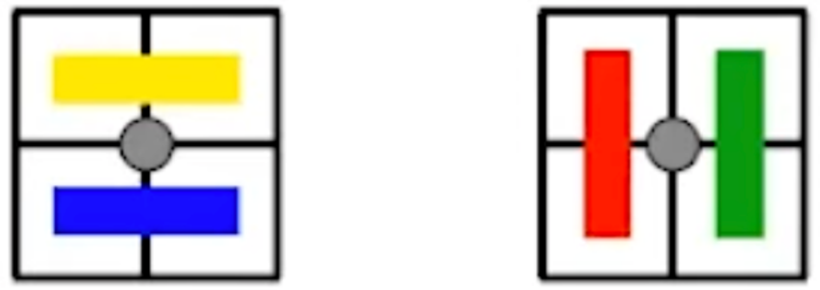
Kenyon, Okounkov and Sheffield formulated a general theory for all dimer models on bipartite graphs with generic boundary conditions and domains. However, it appears that all these models may be viewed as discrete free fermionic models.

[Kenyon - Okounkov - Sheffield, 2006]

Adding the interactions



Assign a Boltzmann weight e^{δ} only to

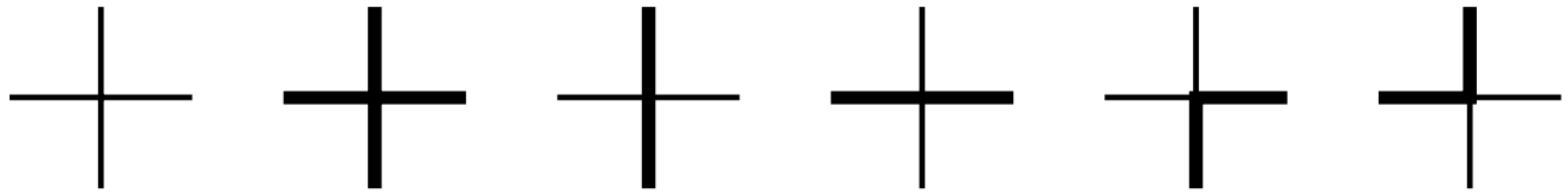


and you get an exactly solvable model:

the six-vertex model, with $\Delta = 1 - e^{\delta} (<1)$

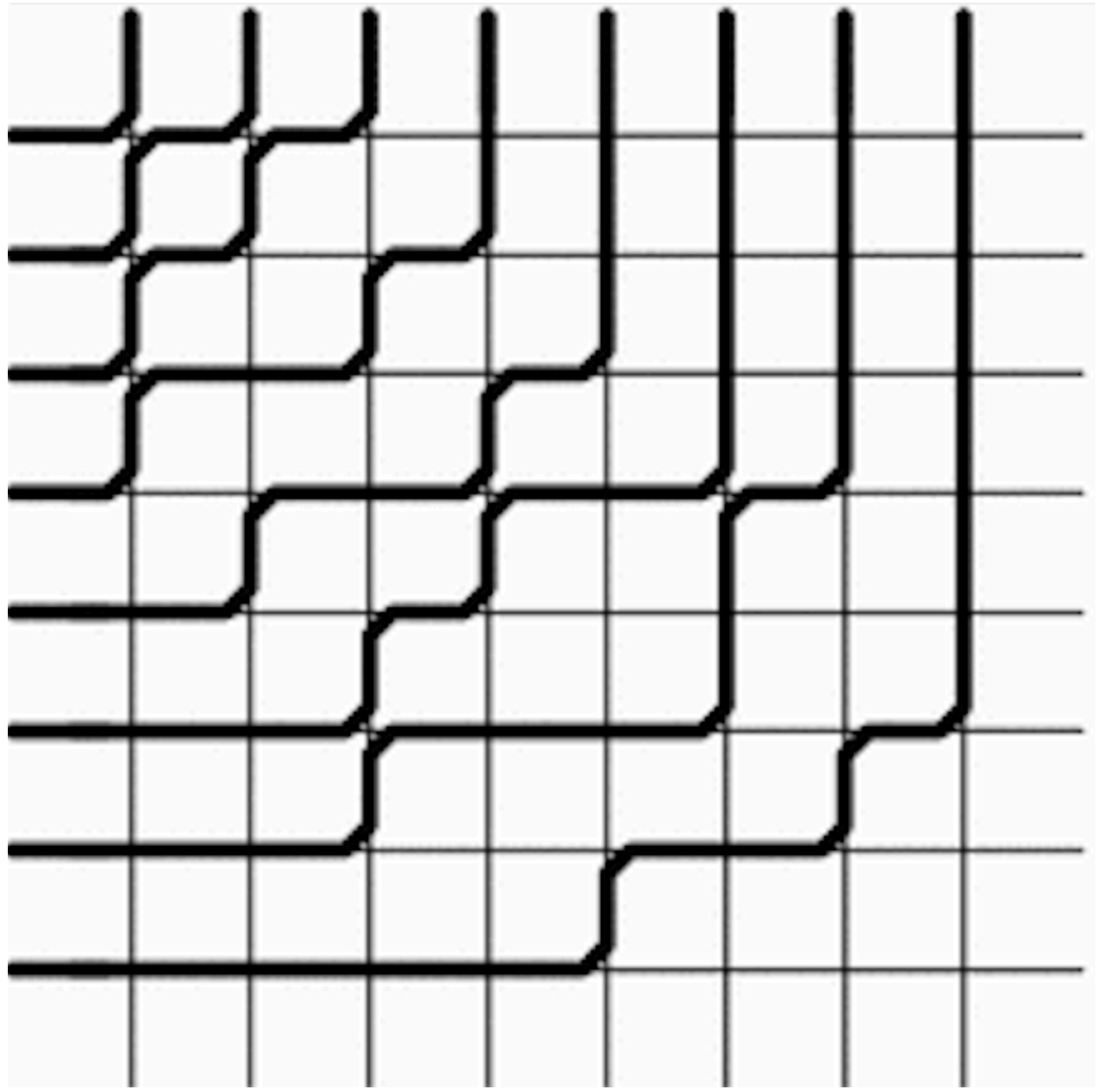
[Kuperberg, 1996]

The six-vertex model [Lieb, 1967] [Sutherland, 1967]

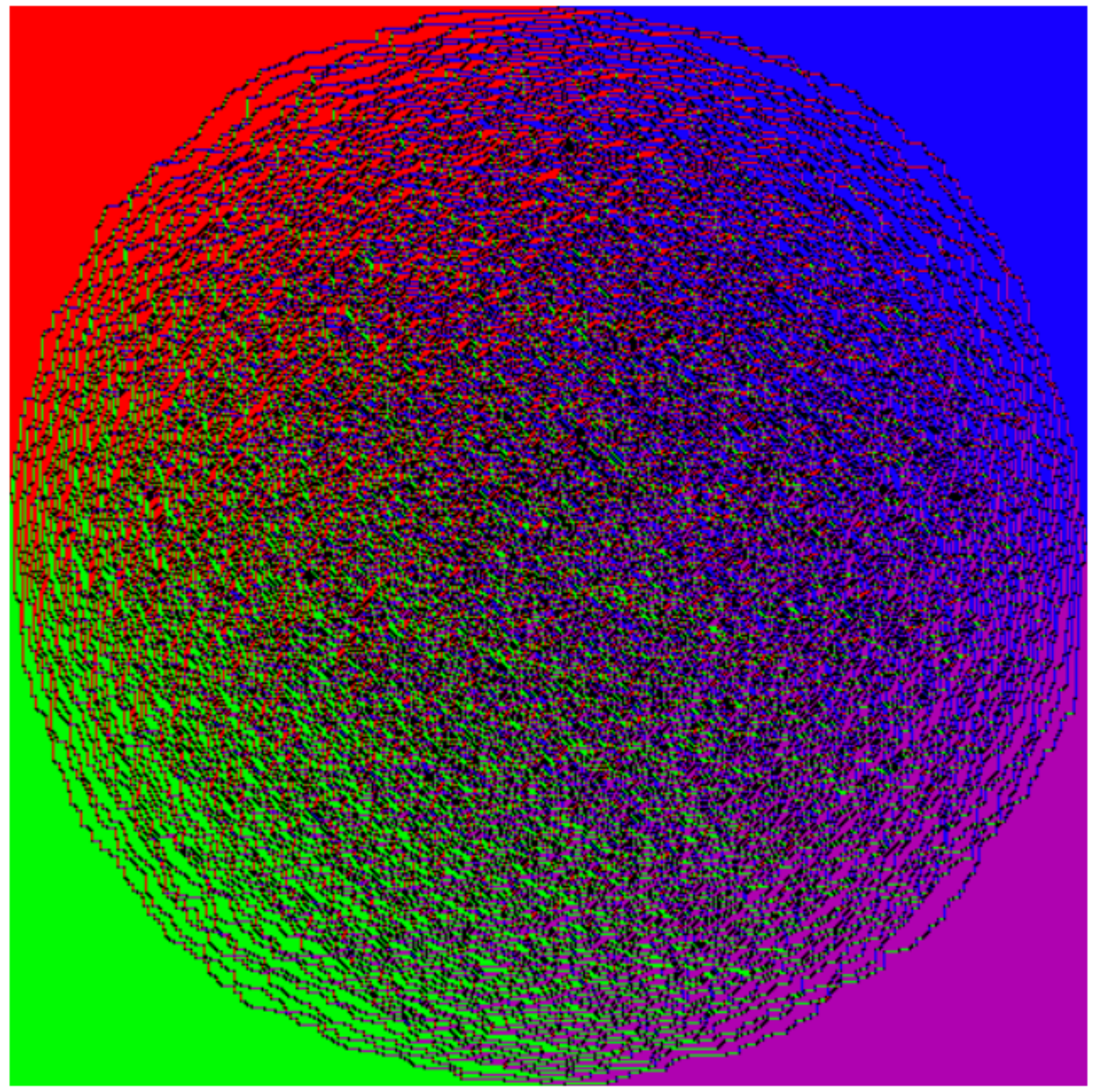


a a b b c c

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$



$\Delta \neq 0$: Interface fluctuations

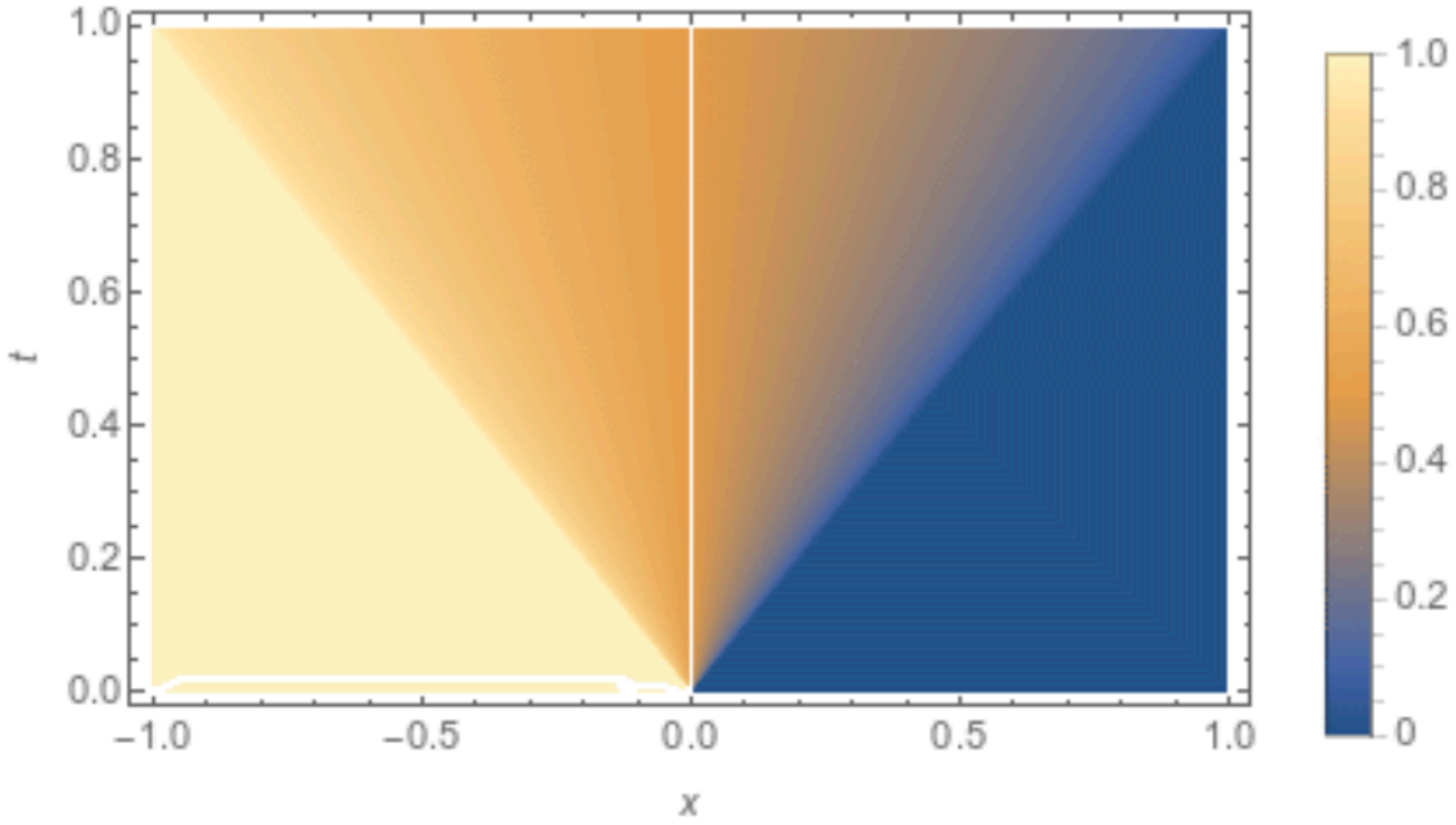


Recent numerics:

T-W scaling is observed
with excellent accuracy

[Prauhofer - Spohn, 2023]

[Korepin - Lyberg - Viti, 2023]

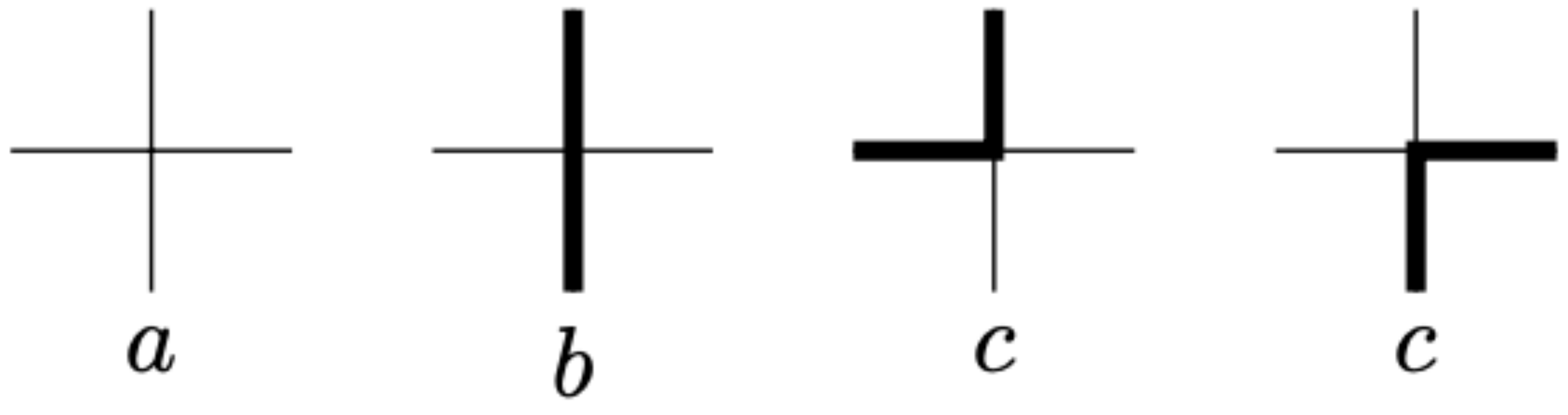


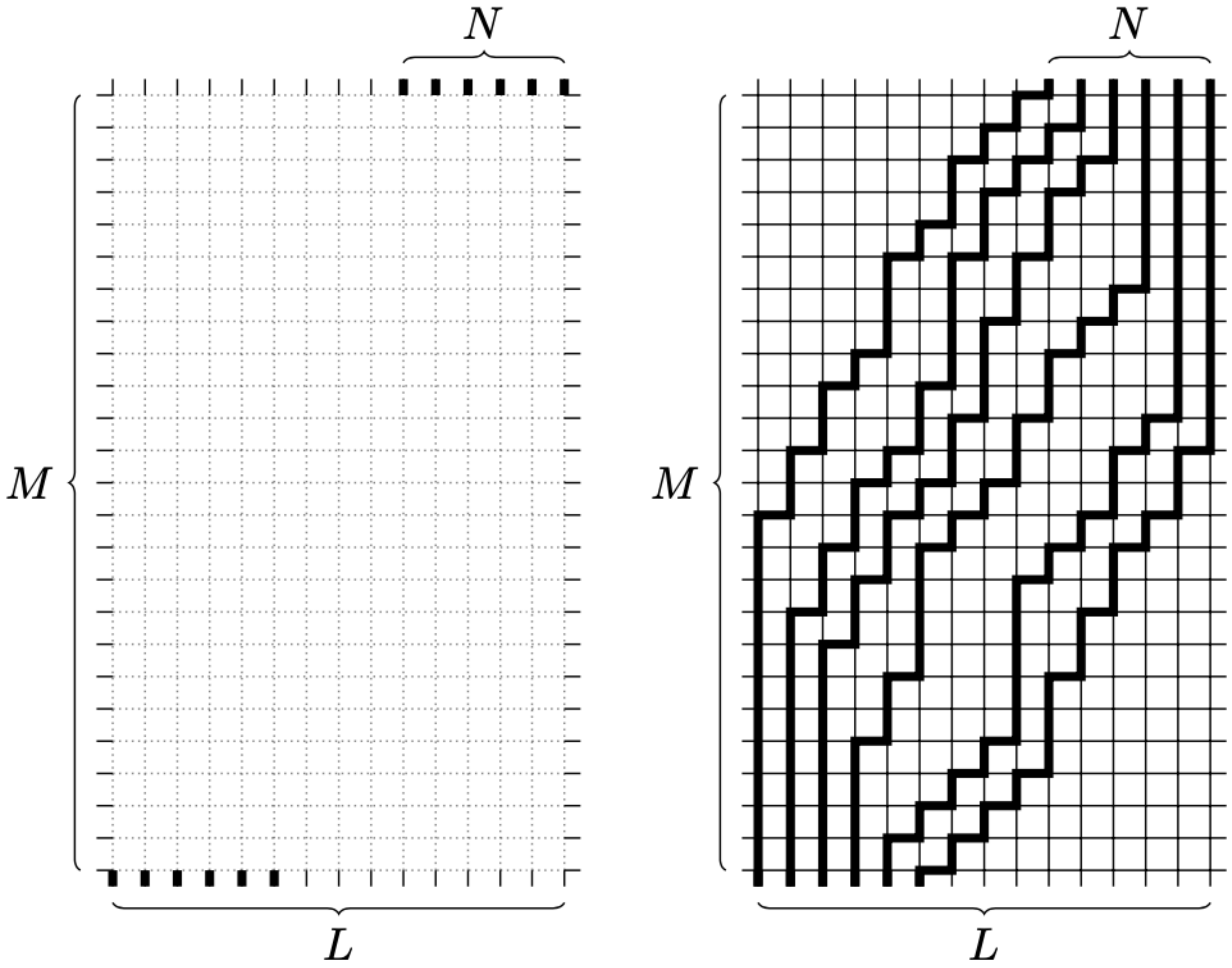
Analytic prediction:

T-W scaling is destroyed

[Collura - De Luca - Viti, 2018]

The four-vertex model [I. Burenev, F. Colomo, AM, A. Pronko, arXiv: 2307.03076]





The number of vertices of each type does not depend on the configuration:

$\#a = (L - N)(M - N)$

$\#b = N(M - L + N)$

$\#c = 2N(L - N)$

So we can set $a = b = c = 1$ without loss of generality.

The arctic curves of the four-vertex model

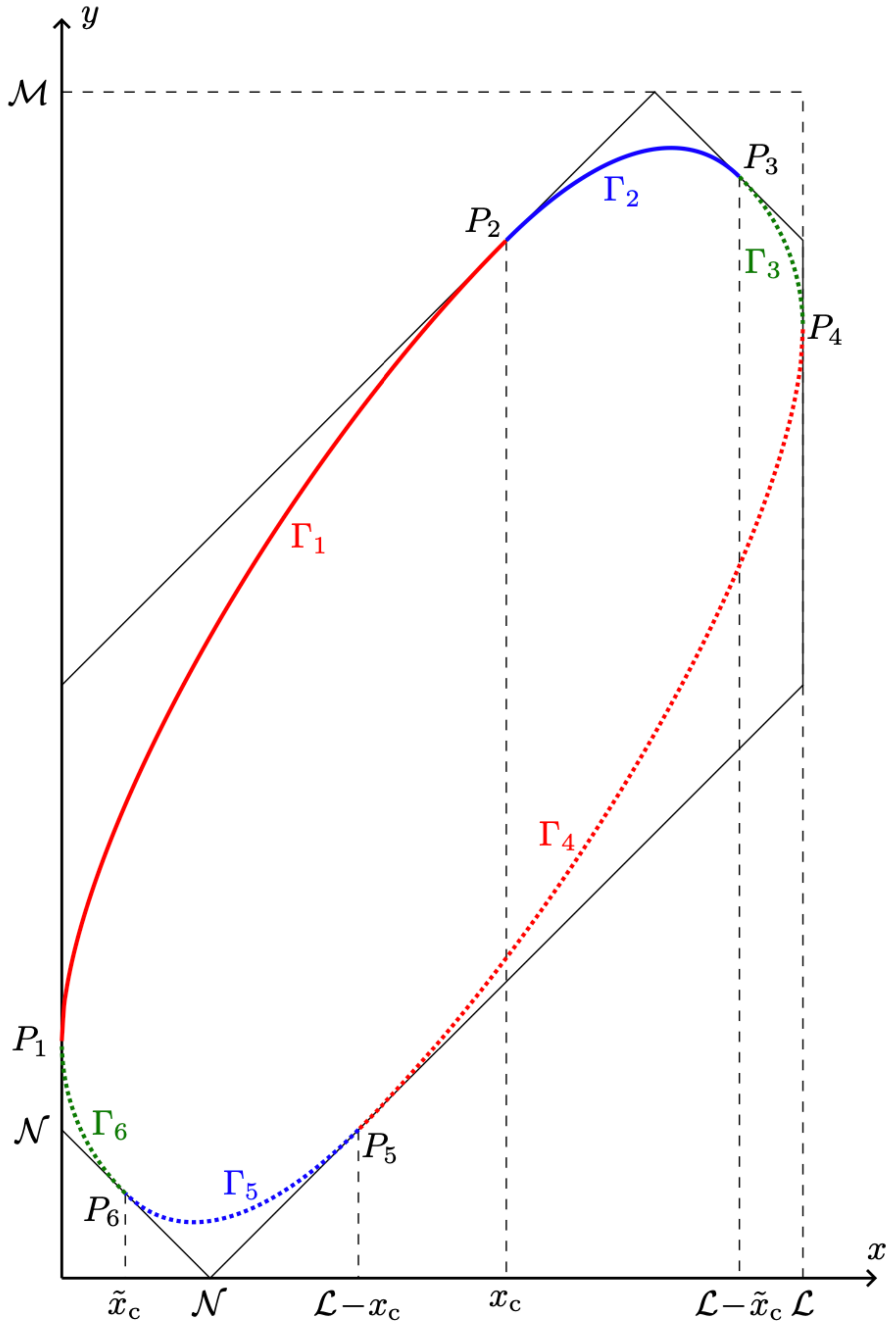
The scaling limit is achieved with:

$$L = [\mathcal{L}\ell], \quad M = [\mathcal{M}\ell], \quad N = [\mathcal{N}\ell], \quad \ell \to \infty.$$

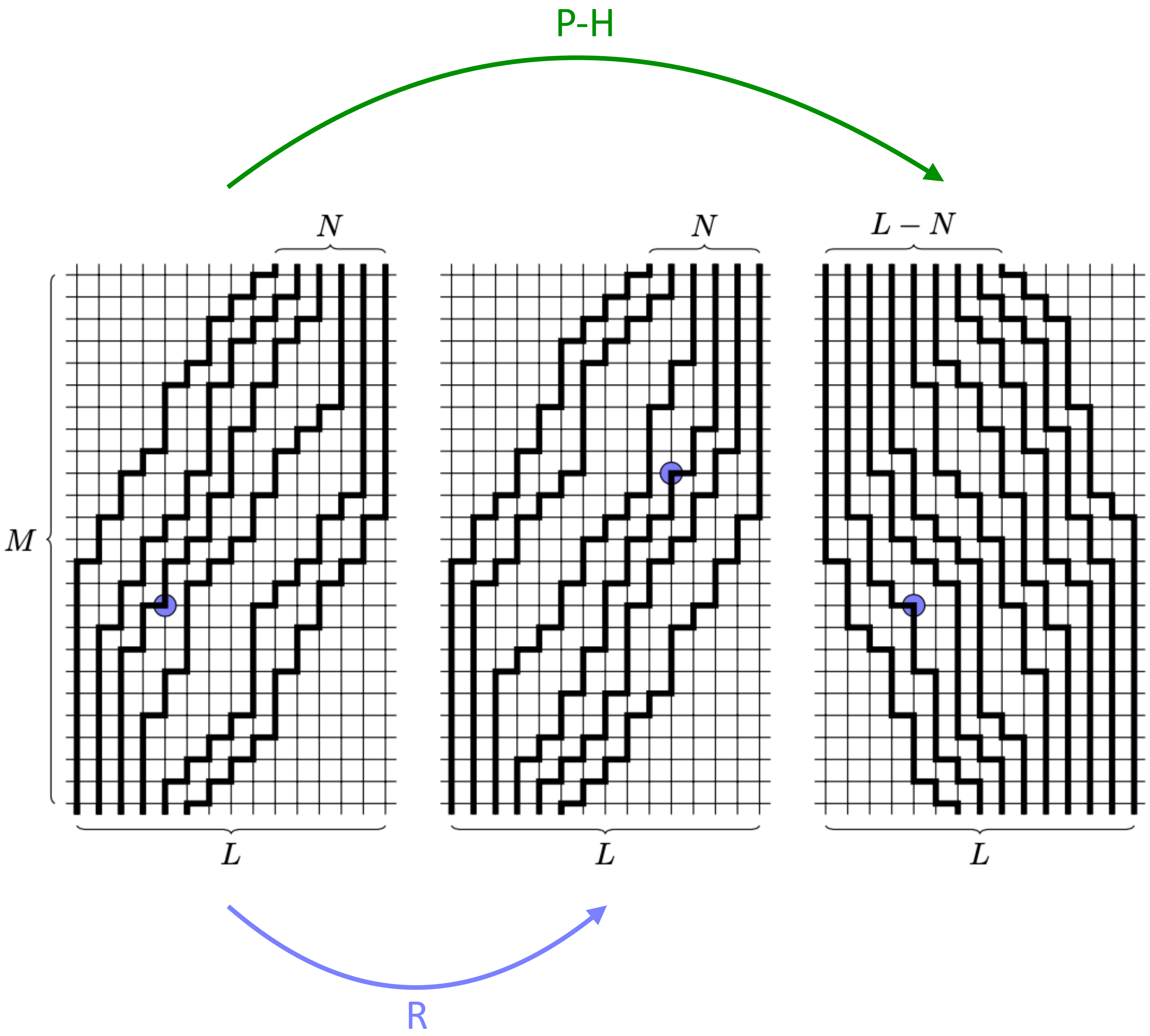
The arctic curve is made of six consecutive arcs, joined end by end at six contact points P_i.

$$\Gamma_1 : \; y_1 = \frac{\mathcal{M}\mathcal{N}(\mathcal{L}-2x)+(\mathcal{M}+\mathcal{N})\mathcal{L}x}{\mathcal{L}^2} + 2\frac{\sqrt{\mathcal{M}\mathcal{N}(\mathcal{L}-\mathcal{N})(\mathcal{M}-\mathcal{L})(\mathcal{L}-x)x}}{\mathcal{L}^2}$$

$$\Gamma_2 : \; y_2 = (\mathcal{L} - \mathcal{M} - \mathcal{N} - x) + 2y_1$$



Particle-hole and reflection symmetries



P-H

Swapping the state (thick $\leftrightarrow$ thin) of each vertical edge, and reflecting with respect a vertical axis, we obtain a four-vertex model with L-N lines.

$$y_3(\mathcal{L}, \mathcal{M}, \mathcal{N}; x) = y_1(\mathcal{L}, \mathcal{M}, \mathcal{L} - \mathcal{N}; \mathcal{L} - x)$$

R

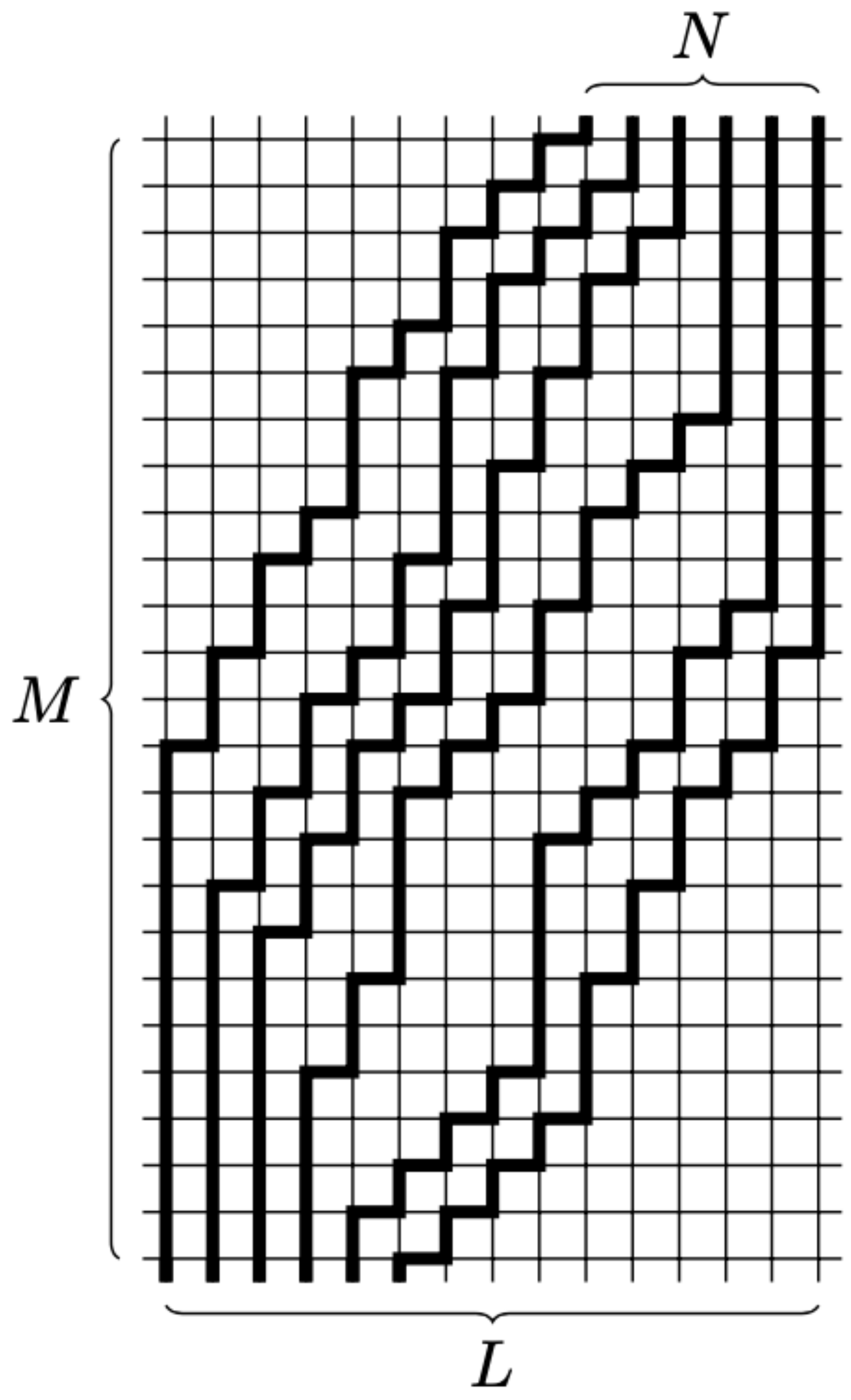
Another symmetry is the simultaneous reflection under the vertical and horizontal axes.

$y_4,\ y_5,\ y_6$ from $y_1,\ y_2,\ y_3$ through:

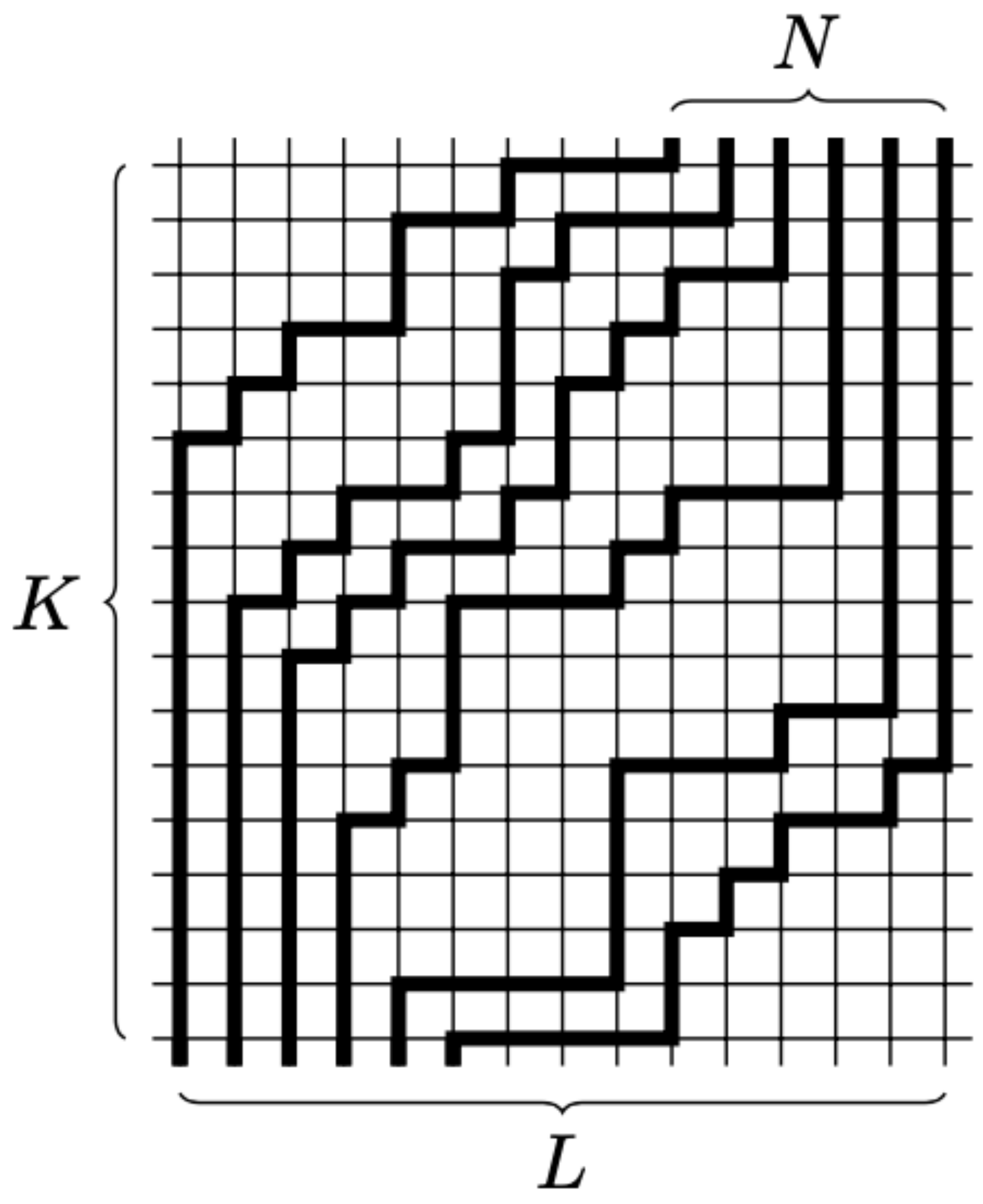
$$x \to \mathcal{L} - x,\ y \to \mathcal{M} - y$$

Methods: bijection to non-intersecting lattice paths

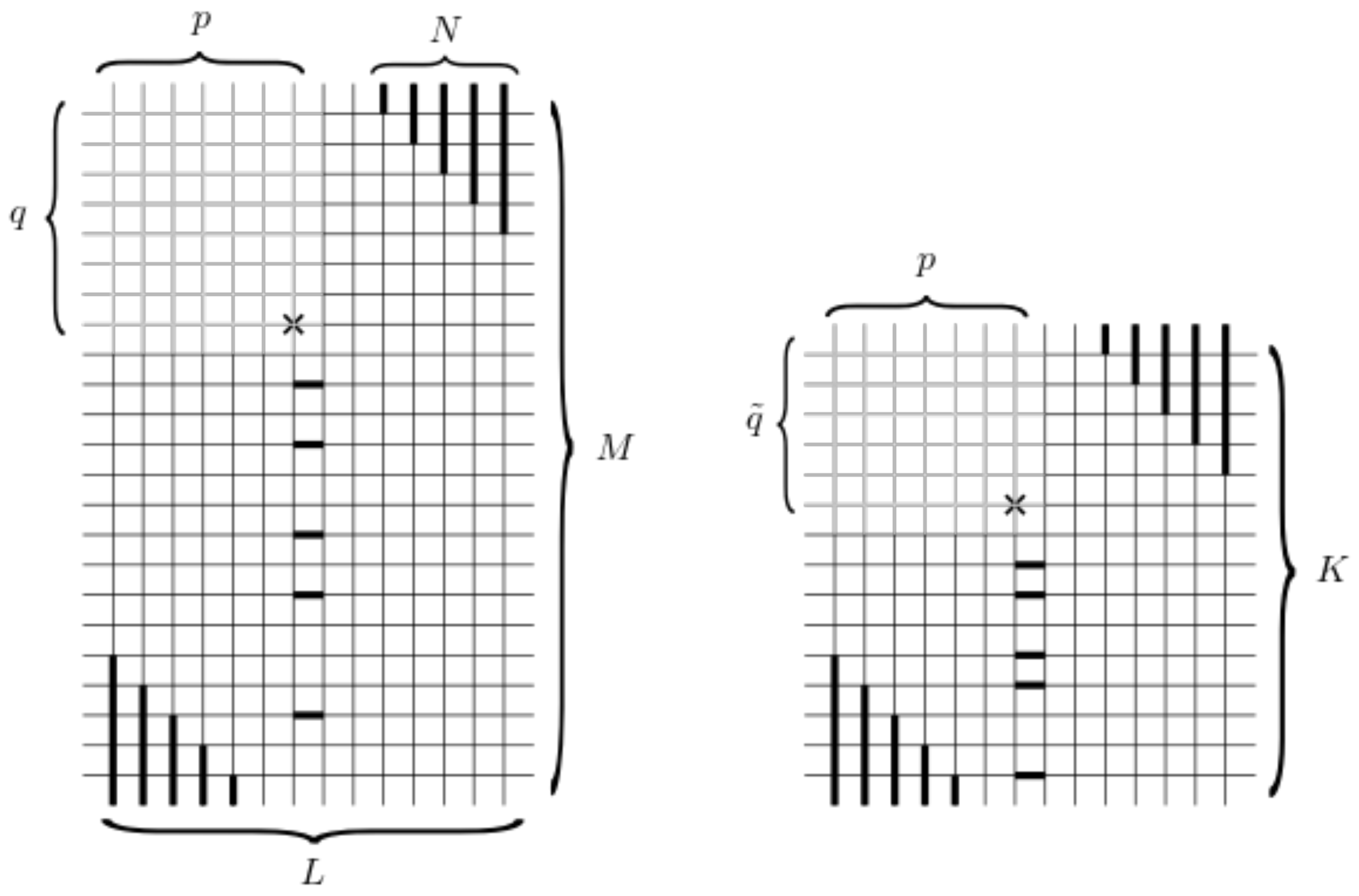
If we shift the *i*-th horizontal edges (*i*=1,…,L-N) of each path by *i*-1 lattice spacing southward, then we have a non-intersecting lattice paths model



$$K := M - L + N + 1$$



The emptiness formation probability (EFP)



The EFP is the probability to have at least q vertices of type *a* starting from the point (p,M)

After the bijection, it is the probability that no paths pass through the point $(p, K-\tilde{q}+1)$, with $\tilde{q} = p+q-L+N$

$$H(d,\alpha,\beta,s,n) := \frac{1}{H_0(\alpha,\beta,s,n)} \sum_{0\le x_1,\dots,x_s\le d} \prod_{1\le i<j\le s} (x_j-x_i)^2 \prod_{i=1}^{s} \binom{\alpha+x_i}{x_i}\binom{\beta+n-x_i}{n-x_i}$$

Hahn measure

$$d = M-N+\min(p,N)-p-q, \qquad \alpha = |N-p|, \qquad \beta = L-N-p,$$
$$s = \min(p,N), \qquad n = M-L+\min(p,N).$$

EFP in a log-gas representation

Let $\mathbf{x} = \{x_1, \dots, x_s\}$ with $0 \leq x_1 < \cdots < x_s \leq n$ denote the position of s particles on the interval $[0, n]$ and consider the probability measure on $[0, n]^s$:

$$P_{n,s}^{(\alpha,\beta)}[\mathbf{x}] = \frac{1}{Z(\alpha,\beta,s,n)} \prod_{1 \leq i < j \leq s} (x_i - x_j)^2 \prod_{i=1}^{s} w_n^{(\alpha,\beta)}(x_j)$$

$$w_n^{(\alpha,\beta)}(x) = \binom{\alpha + x}{x}\binom{\beta + n - x}{n - x}$$

$$\boxed{H(d,\alpha,\beta,s,n) := \sum_{0 \leq \mathbf{x} \leq d} P_{n,s}^{(\alpha,\beta)}[\mathbf{x}].}$$

The **EFP is the probability that**, in a discrete log-gas of *s* particles, associated to the Hahn measure, **no particle has coordinate larger than *d*.**

The arc of the arctic curve

The scaling limit is achieved with: $d = [d_0 \ell], \ \alpha = [\alpha_0 \ell], \ \beta = [\beta_0 \ell], \ s = [s_0 \ell], \ n = [n_0 \ell], \ \ell \to \infty$

Following standard methods from random matrix models, we rescale $x_j = [\mu_j \ell]$.

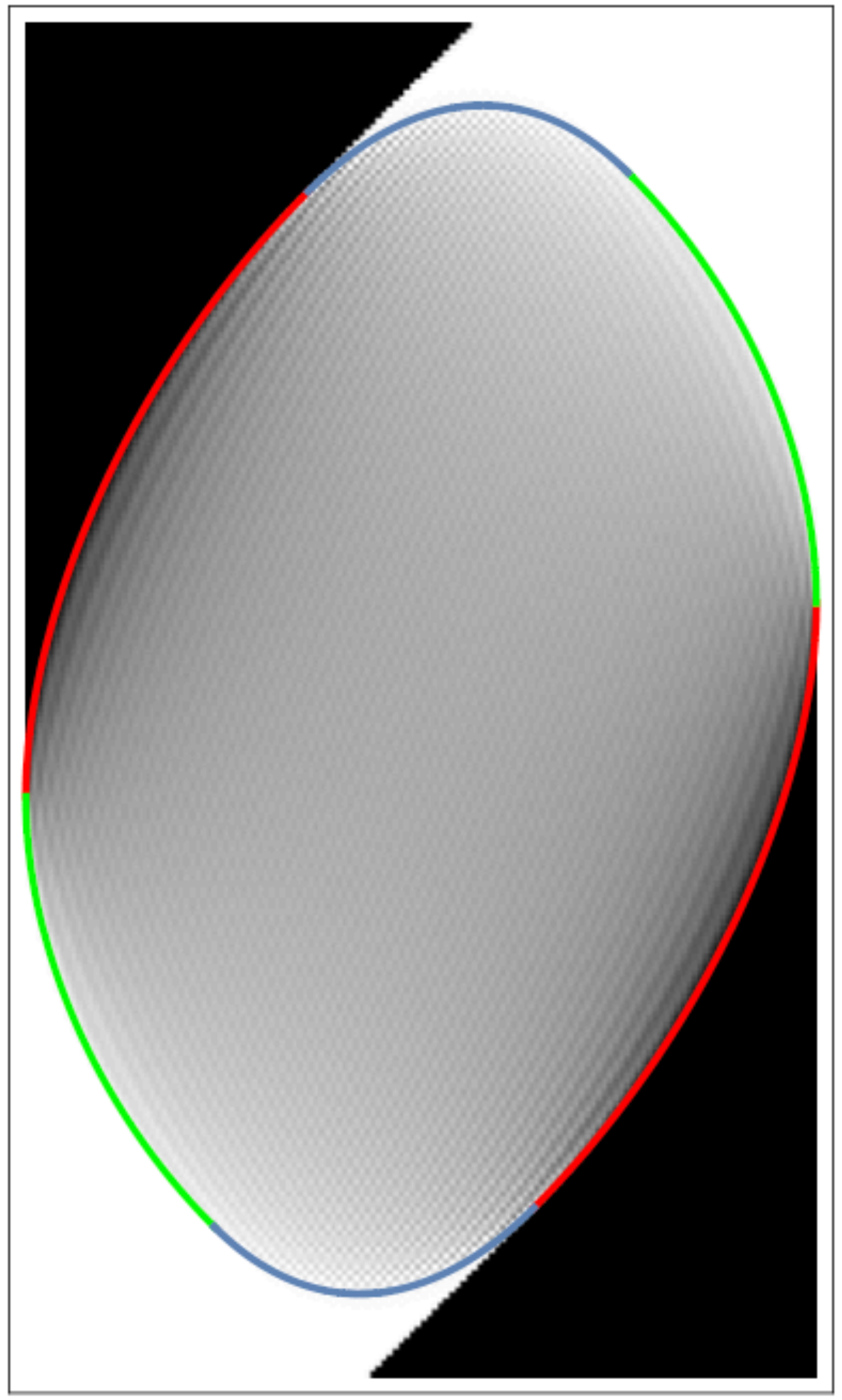
In the scaling limit, the EFP is known to tend to one in a frozen region, and zero otherwise. To find the arctic arc, it is sufficient the support [L,R] of the density.

We have the arctic curve when:

$$R(\alpha_0, \beta_0, s_0, n_0) = d_0 \longrightarrow y_1 = \frac{\mathcal{M}\mathcal{N}(\mathcal{L}-2x)+(\mathcal{M}+\mathcal{N})\mathcal{L}x}{\mathcal{L}^2} + 2\frac{\sqrt{\mathcal{M}\mathcal{N}(\mathcal{L}-\mathcal{N})(\mathcal{M}-\mathcal{L})(\mathcal{L}-x)x}}{\mathcal{L}^2}$$

(setting $x = p_0, \ y = \mathcal{M} - q_0$)

Fluctuations of the arctic curve



Taking the position ξ of the last thick edge at some column p and comparing it with the rightmost point R of the interval:

$$\lim_{\ell\to\infty} \mathbb{P}\left(\frac{\xi - \ell R(\alpha_0,\beta_0,s_0,n_0)}{(t\ell)^{1/3}} \leq x\right) = \det\left[1 - A|_{(x,\infty)}\right]$$

where $A|_{(x,\infty)}$ acts on $L^2[x,\infty)$ with the Airy kernel.

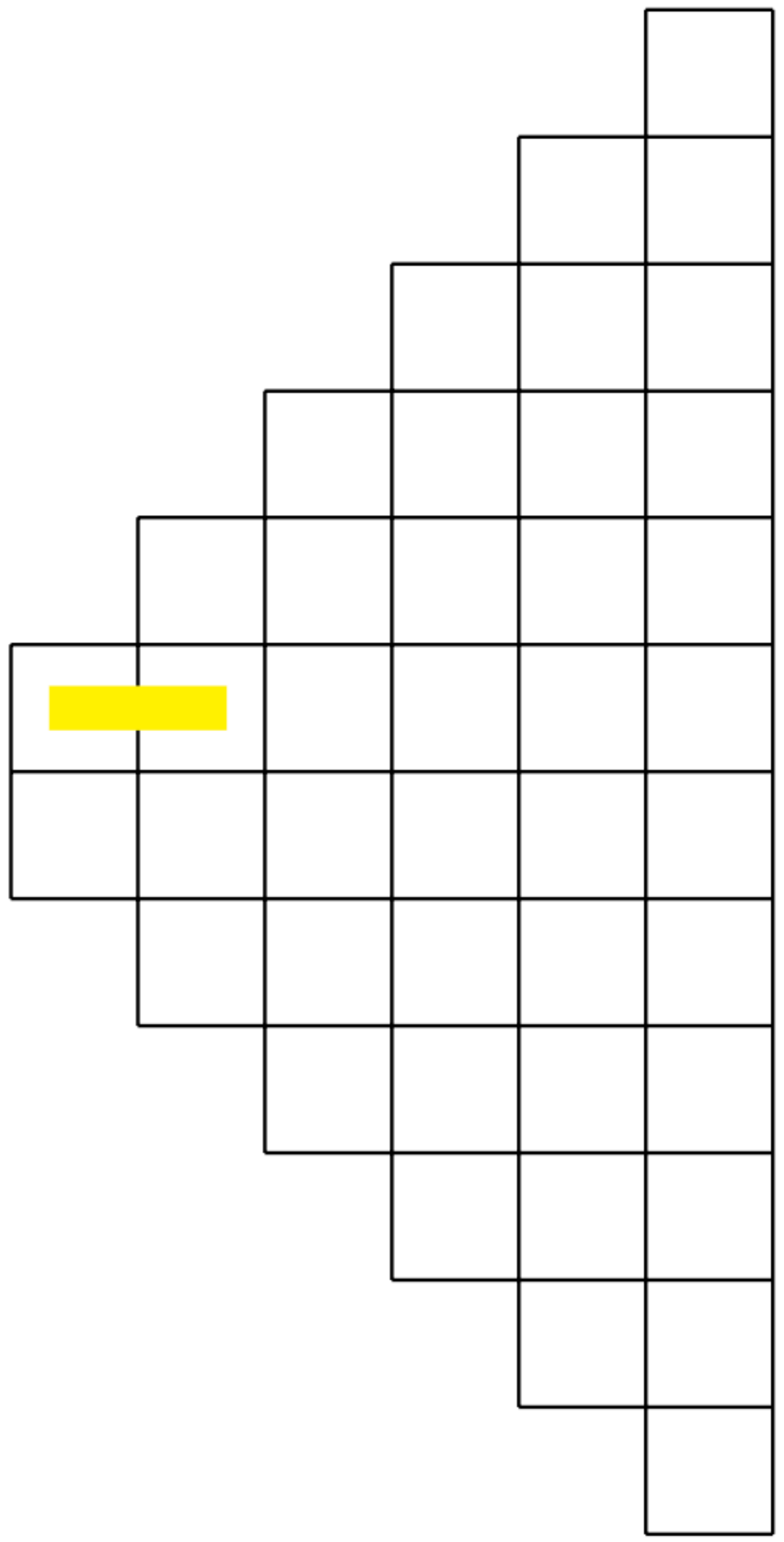
This quantity is exactly $F_2(x)$ (T-W in the GUE).

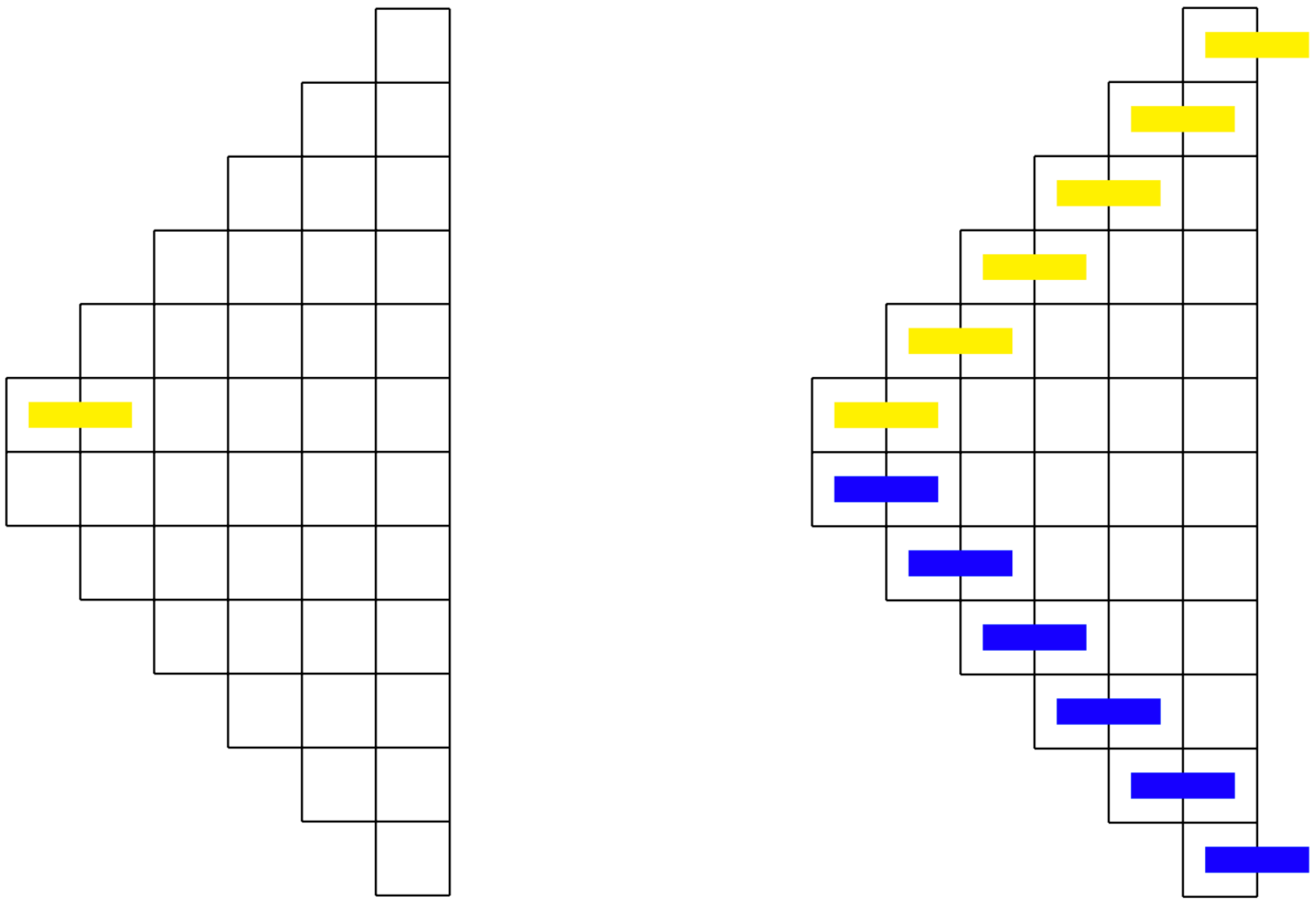
Conclusions and Outlooks

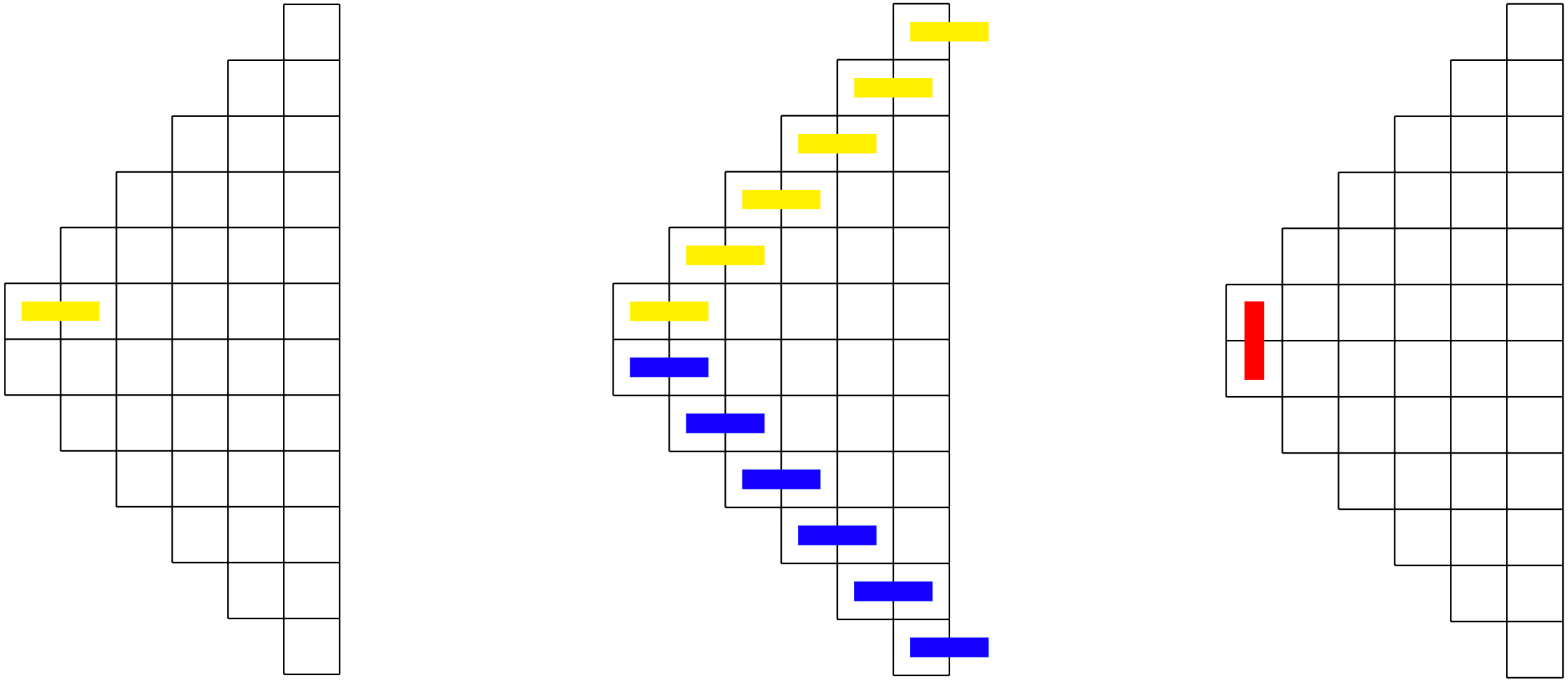
- Main Results
  - Calculation of the arctic curve for the four-vertex model
  - Observation that the fluctuations are governed by T-W

- Further investigations
  - Generalization to the five-vertex model

Thank you for your time!







Mapping the *AD(N)* to the $N \times N$ six-vertex model with DWBC