$U(1)$ Entanglement asymmetry in the Ising CFT

How much is a defect non-topological?

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Outline

- Introduce entanglement asymmetry
- Provide interpretation in terms of (non) topological defects
- Show explicit calculation for Ising CFT

Entanglement asymmetry in general

Entanglement asymmetry

Ingredients: a **state** ρ and a **group action** on the Hilbert space $U_g, g \in G$.

Q: Is the state symmetric under U_g? "How much"? And on a subsystem?

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$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} \qquad \rho_A = \mathrm{tr}_{\bar{A}}\, \rho \qquad U_g = U_{A,g} \otimes U_{\bar{A},g} \qquad \rho_A \mapsto U_{A,g}^{-1} \rho_A U_{A,g}$$

Symmetrized state:

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Entanglement asymmetry = difference in Renyi entropy between $\rho_{A,G}$ and ρ_A

$$\Delta S_n = \frac{1}{1-n} \left(\log \mathrm{Tr}\, \rho_{A,G}^n - \log \mathrm{Tr}\, \rho_A^n \right)$$

Entanglement asymmetry

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Properties [1]

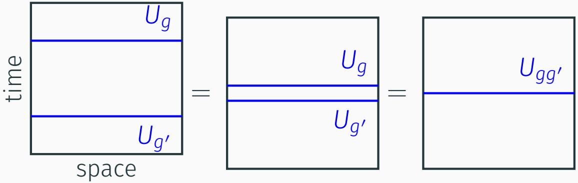
- $\Delta S_n \geq 0$
- $\Delta S_n = 0 \Leftrightarrow \rho_A = \rho_{A,G}$
- $\Delta S_n \leq \log |G|$ for finite groups, unbounded for G continuous

[1] Ares, Murciano, Calabrese *Nat. Comm.* (2023)
Ferro, Ares, Calabrese (2023)
Capizzi, Mazzoni (2023)

Interpretation with defects

"Modern language": symmetries $\leftrightarrow$ topological operators/defects[2]

$$[U_g, H] = 0 \Leftrightarrow$$



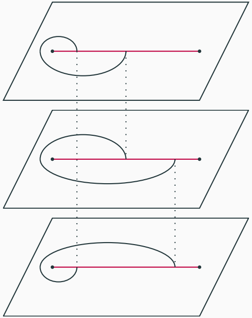
On the contrary, if the symmetry is broken (explicitely or spontaneously), the defects are not topological

[2] Gaiotto, Seiberg, Kapustin, Willett *JHEP* (2015)

Entanglement asymmetry with defects

$$\Delta S_n = \frac{1}{1-n} \log \frac{\mathrm{Tr}\, \rho_{A,G}^n}{\mathrm{Tr}\, \rho_A^n}$$

$\mathrm{Tr}\, \rho_A^n$



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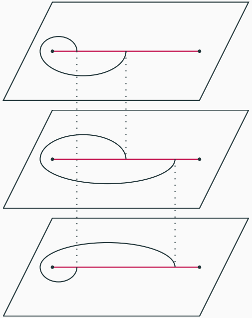

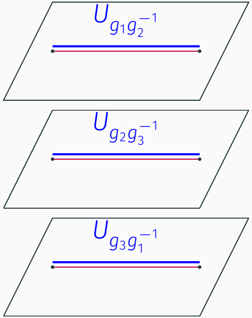

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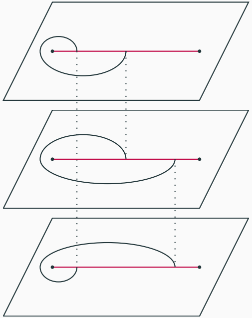
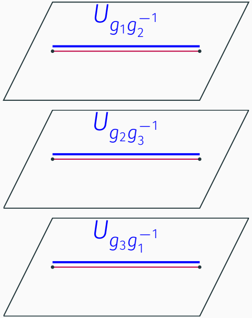




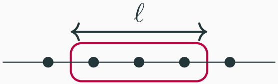
Entanglement asymmetry = how much the defects are not topological

Our case

Our system

Lattice

State: $|gs\rangle$ of critical Ising chain: $H = -\sum_{j\in\mathbb{Z}}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^z\right)$



Group action: $U_{A,\alpha} = \exp\left(i\frac{\alpha}{2}\sum_{j\in A}\sigma_j^z\right)$ rotation around $\hat{z}$.

Not a symmetry, unless $\alpha = \pi$ ($\mathbb{Z}_2$ symmetry)

[3] Ares, Murciano, Calabrese *Nat. Comm.* (2023)
Murciano, Ares, Klich, Calabrese *in preparation*

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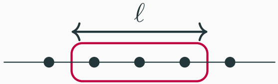

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Known results: from lattice calculations[3]

$$\Delta S_n = \frac{1}{2}\log\ell - \frac{1}{2}\log\frac{4}{\pi} + o(1)$$

Goal: access subleading terms with CFT

[3] Ares, Murciano, Calabrese *Nat. Comm.* (2023)
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Our system

Continuum

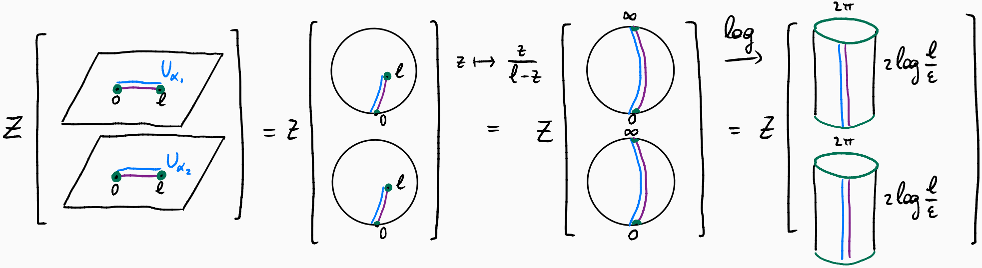
State: ground state of the Majorana CFT: $H = \frac{1}{2i} \int_{\mathbb{R}} dx \left(\psi \partial_x \psi - \bar{\psi} \partial_x \bar{\psi}\right)$

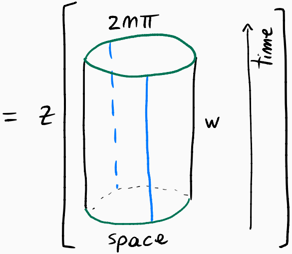
Group action: $U_{A,\alpha} = \exp\left(-\alpha \int_A dx\, \psi \bar{\psi}\right)$

$$U_\alpha \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} U_\alpha^{-1} = \underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}_{\in SO(2)} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

again, not a symmetry unless $\alpha = \pi : \psi \mapsto -\psi,\ \bar{\psi} \mapsto -\bar{\psi}$

Conformal mappings



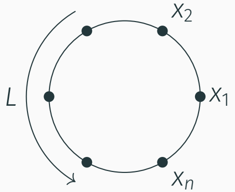


$\sim e^{-E_{gs} w}$

$$U_\alpha = \exp\left(-\alpha \int dx\, \overbrace{\psi(x)\bar{\psi}(x)}^{\Delta=1}\right)$$

transforms as a scalar

Hamiltonian with defects



$$H = \frac{1}{2i}\int_x (\psi\partial\psi - \bar{\psi}\partial\bar{\psi}) + \sum_j \overbrace{\alpha_j \psi(x_j)\bar{\psi}(x_j)}^{\text{anti-Hermitian}} = \frac{1}{2i}\int_x \Psi^\dagger D \Psi$$

analytic continuation $\alpha_j = i\lambda_j$

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \qquad\qquad D = \begin{pmatrix} \partial_x & -\sum_j \lambda_j \delta(x - x_j) \\ \sum_j \lambda_j \delta(x - x_j) & -\partial_x \end{pmatrix}$$

Hamiltonian is quadratic $\longrightarrow$ diagonalize kernel D $\longrightarrow$ compute E_{gs}

Diagonalized Hamiltonian

$$H = \frac{1}{2}\sum_k k(\eta_k^\dagger \eta_k + \bar{\eta}_k^\dagger \bar{\eta}_k) \qquad E_{gs} = \sum_{k<0} k$$

n. defects	momentum space	E_{gs}
0	$\frac{2\pi}{L}(\mathbb{Z} + \frac{1}{2})$	$-\frac{\pi}{12L}$
2	$\frac{2\pi}{L}(\mathbb{Z} + \frac{1}{2} + \frac{\beta_1 - \beta_2}{2\pi})$	$\frac{1}{L}\left[-\frac{\pi}{12} + \frac{(\beta_1-\beta_2)^2}{\pi}\right]$
3	$\bigcup_{j=1}^{3} \frac{6\pi}{L}(\mathbb{Z} + \frac{\theta_j}{2\pi})$	$\frac{6\pi}{L}\left[\frac{1}{4} - \sum_{j=1}^{3}\left(\frac{\theta_j}{4\pi} - \frac{1}{2}\left(\frac{\theta_j}{2\pi}\right)^2\right)\right]$

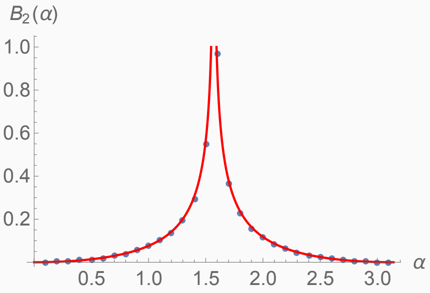
$$\beta_j = 2\arctan\tanh\frac{\lambda_j}{2}$$

$$\theta_1 = \mathrm{Arg}\left[\frac{1}{2}\left(1 - \sqrt{-s} - \sqrt{-3 - s - 2\sqrt{-s}}\right)\right] \qquad s = \sin\beta_1 \sin\beta_2 + \text{cycl. perm.}$$

2-asymmetry ΔS_2

$$E_{gs} \rightsquigarrow \text{partition function}$$

$$\frac{\text{Tr}\left[\rho_A e^{i\alpha Q_A} \rho_A e^{-i\alpha Q_A}\right]}{\text{Tr}\,\rho_A^2} = \underbrace{e^{A_2(\alpha)\ell}}_{\text{from lattice}} \overbrace{\ell^{B_2(\alpha)}}^{\text{from CFT}}$$



$$B_2(\alpha) = \frac{2}{\pi^2} \,\text{arctanh}^2\left(\tan \frac{\alpha}{2}\right)$$

$$\Delta S_2 = -\log \int_\alpha e^{A_2(\alpha)\ell} \ell^{B_2(\alpha)} \rightarrow \text{saddle point around } \alpha = 0$$

$$A_2(\alpha) = a_2\alpha^2 + \dots \qquad B_2(\alpha) = b_2\alpha^2 + \dots$$

$$\boxed{\Delta S_2 = \frac{1}{2}\log \ell + \frac{1}{2}\log(-a_2\pi) + \frac{b_2}{2a_2}\frac{\log \ell}{\ell} + o\left(\frac{\log \ell}{\ell}\right)}$$

3-asymmetry ΔS_3

$$\frac{\mathrm{Tr}\left[\rho_A e^{i\alpha_1 Q_A}\rho_A e^{i\alpha_2 Q_A}\rho_A e^{i(-\alpha_1-\alpha_2)Q_A}\right]}{\mathrm{Tr}\,\rho_A^3} = \underbrace{e^{A_3(\boldsymbol{\alpha})\ell}}_{\text{from lattice}}\;\overbrace{\ell^{B_3(\boldsymbol{\alpha})}}^{\text{from CFT}}$$

$$\Delta S_3 = -\frac{1}{2}\log\int_{\boldsymbol{\alpha}} e^{A_3(\boldsymbol{\alpha})\ell}\ell^{B_3(\boldsymbol{\alpha})} \to \text{saddle point around } \boldsymbol{\alpha} = 0$$

$$A_3(\boldsymbol{\alpha}) = a_3\cdot(\alpha_1^2+\alpha_2^2+\alpha_1\alpha_2)+\dots \qquad B_3(\boldsymbol{\alpha}) = b_3\cdot(\alpha_1^2+\alpha_2^2+\alpha_1\alpha_2)+\dots$$

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Same structure as $n = 2$

Conclusions

- Entanglement asymmetry: how much the defects are not topological
- CFT approach $\longrightarrow E_{gs}$ of a system with defects
- $\frac{\log l}{l}$ corrections in ΔS_n

Future directions

- Dirac CFT, same group action (again, not a symmetry)
- Extend to $SU(N)$ (e.g. XXZ spin chain)

Thank You!