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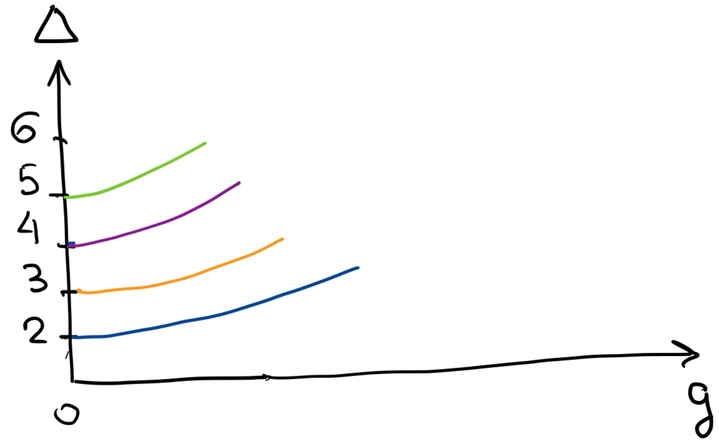
Fast QSC Solver for Excited States



10th Bologna Workshop on Conformal Field Theory and Integrable Models 2023

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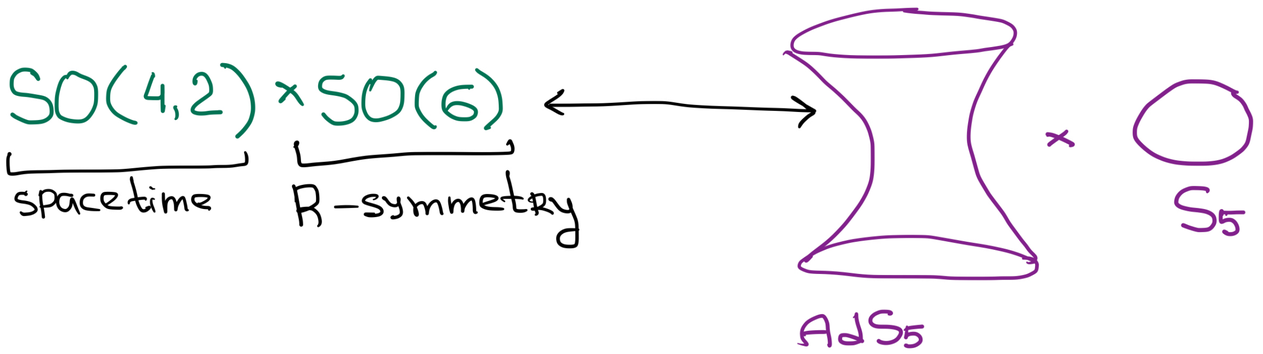
The theory which unites it all

What do we know about N=4 SYM?

- Many symmetries: 4d conformal symmetry and supersymmetry PSU$(2\,|\,2,4)$.
- In the large-N limit ($N \to \infty, g_{\mathrm{YM}} \to 0$) of $SU(N)$ colour group, the leading contribution to observables comes from planar diagrams. The spectrum is dominated by single-trace operators. Integrability: the spectrum of single-trace operators is known exactly.
- $\mathrm{AdS}_5/\mathrm{CFT}_4$: at strong coupling $\lambda \equiv g_{YM}^2 N$ ($g \equiv \sqrt{\lambda}/4\pi$) the theory is dual to string theory on the curved background.

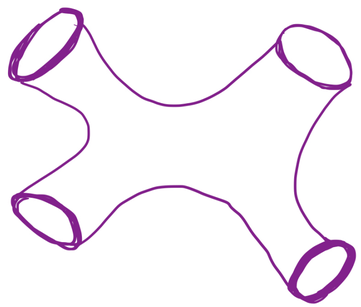


The theory which unites it all

What do we wish to know about N=4 SYM?

- Correlation functions of any sets of operators $\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$ at any coupling. Conformal symmetry already simplifies this task: it is sufficient to determine the spectrum and all OPE coefficients $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle \sim \sum_{\Delta} C_{12\Delta} C_{\Delta 34} G_{\Delta}(u, v)$. Is it possible to find OPE coefficients using integrability?

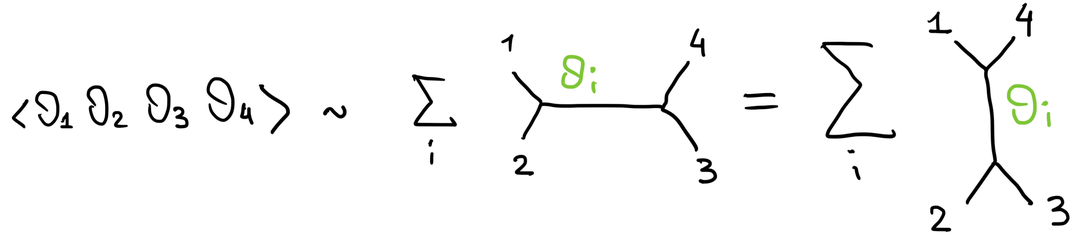
- By the holographic duality, at strong coupling this question corresponds to determining the scattering amplitudes of strings on the curved background.



Bootstrability

Hybrid approach: integrability and conformal bootstrap

- Conformal bootstrap is a set of consistency equations imposed on conformal field theory. For instance, the crossing equations:



- Numerical conformal bootstrap provides narrow bounds on the OPE coefficients and on the spectrum by excluding theories with are not compatible with crossing: $\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} F^{\Delta_\phi}_{\Delta,l}(u,v) = 0.$

[Ratazzi, Rychkov, Tonni, Vichi '08]

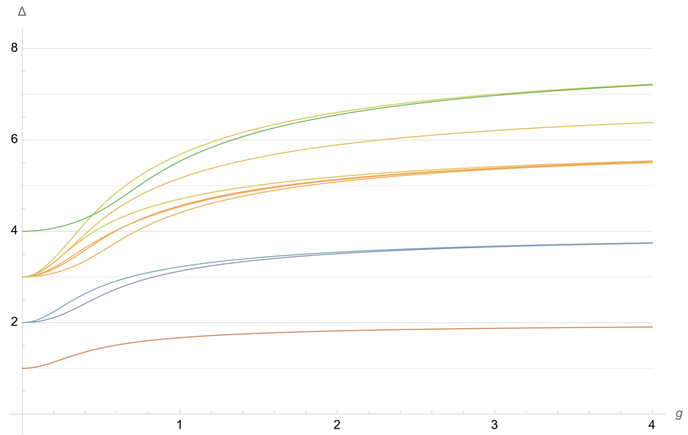
- Let us notice that the spectrum is usually unknown, and it is an additional parameter in the search (it is important not to have many parameters: the curse of dimensionality).

Bootstrability

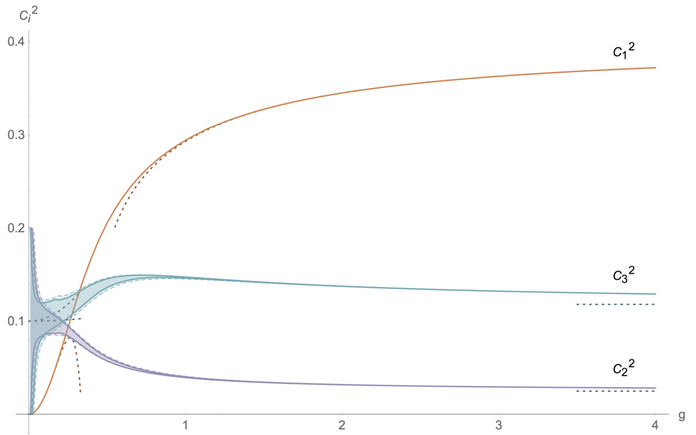
Hybrid approach: integrability and conformal bootstrap

Idea: use the knowledge of the spectrum and other consequences of integrability to combine with numerical/ analytical conformal bootstrap.

- In the content of 1d Maldacena-Wilson line in the defect CFT the bootstrability approach has shown incredible bounds on OPE coefficients [Cavaglia, Gromov, Julius, Preti ’21]. The spectrum of first operators is known from integrability as only single trace operators are exchanged:



Spectrum from integrability (QSC-based numerics) [Grabner, Gromov, Julius ’20] [Julius ’21] [Cavaglia, Gromov, Julius, Preti ’21]



Bounds on structure constants with numerical conformal bootstrap (SDPB) [Cavaglia, Gromov, Julius, Preti ’21 ’22]

QSC solver: our objective

- We want obtain the spectrum of local operators in the “bootstrability” spirit: all states up to some $\Delta_{\texttt{cutoff}}$, in a wide range of coupling

- Shortly after the formulation of the quantum spectral curve, the numerical algorithms were developed to compute spectrum of ground states [Gromov, Levkovich-Maslyuk, Sizov ’15].

- However, many of excited states were not accessible with the procedure.

- We have updated the algorithm such that it is possible to initialise all states, and that the computing time is fast enough to continue them to the strong coupling regime [$g \sim 1$].

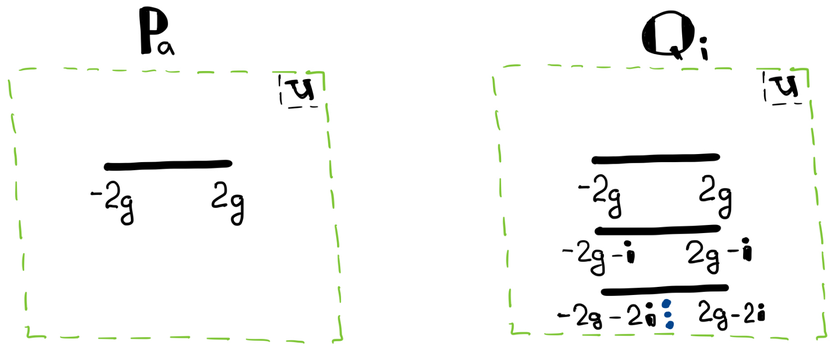
- The perturbative spectrum at weak coupling have been classified and computed up to higher loops [Marboe, Volin ’16, ’17]. This provides initial “seeds” for the solver.

Quantum Spectral Curve [Gromov, Kazakov, Leurent, Volin '14]

Any state in $\mathcal{N}=4$ SYM with quantum numbers $[\Delta, \ell_1, \ell_2, p_1, q, p_2]$ is described by a set of 256 "curves" with distinguished 16: $\mathbf{P}_a(u), \mathbf{P}^a(u), \mathbf{Q}_i(u), \mathbf{Q}^i(u), a=\{1,\dots,4\}, b=\{1,\dots,4\}$.

Asymptotic of the curves at large-u is given by the quantum numbers of the state which provides "initial" conditions:

$$\mathbf{P}_a \sim u^{\text{powP}_a}, \mathbf{P}^a \sim u^{-\text{powP}_a-1}, \mathbf{Q}_i \sim u^{\text{powQ}_i}, \mathbf{Q}^i \sim u^{\text{-powQ}_i-1}.$$



The curves are not independent: they are related by the set of functional relations called the QQ-relations, as well as by the "gluing" conditions: $\tilde{\mathbf{Q}}_i = G_{ij}\bar{\mathbf{Q}}^j, \tilde{\mathbf{Q}}^i = G^{ij}\bar{\mathbf{Q}}_j$.

Due to the single-cut structure of $\mathbf{P}_a(u)$ and $\mathbf{P}^a(u)$ curves, they can be parameterised by the Zhukovsky variable

$$x(u) \equiv \frac{u+\sqrt{u-2g}\sqrt{u+2g}}{2g}.$$

QSC solver: updated algorithm

Initial parameters

1. $\{\mathbf{P}_a, \mathbf{P}^a\}$ can be parametrised by the Zhukovsky variable $x(u)$ which gives a set of "starting" parameters $\{\Delta_{\texttt{weak coupling}}, c_{a,n}, c^{a,n}\}$.
Use functional relations to obtain $\{\mathbf{Q}_i, \mathbf{Q}^i\}$.

→

Gluing conditions on the cut

2. $\{\mathbf{Q}_i, \mathbf{Q}^i\}$ have a branch cut between $[-2g, 2g]$. Sample them at a set of problem points $\{u_0\}$, close to the branch cut. Impose a set of "gluing" conditions on $\{\mathbf{Q}_i, \mathbf{Q}^i\}$and their analytic continuations $\{\tilde{\mathbf{Q}}_i, \tilde{\mathbf{Q}}^i\}$, which ensure correct analytic properties. Use functional relations to compute $\{\mathbf{P}_a, \mathbf{P}^a, \tilde{\mathbf{P}}_a, \tilde{\mathbf{P}}^a\}$ on the probe points $\{u_0\}$. Perform a Fourier transform to get the updated parameters $\{\Delta, c_{a,n}^{\texttt{Updated}}, c^{a,n,\texttt{Updated}}\}$.

↓

Search in the space of parameters

3. Numerical search: Newton's method to solve $\vec{F}(\{c\}, \Delta) = \left\{c_{a,n}^{\texttt{Updated}} - c_{a,n}, c^{a,n\,\texttt{Updated}} - c^{a,n}\right\} = 0$.

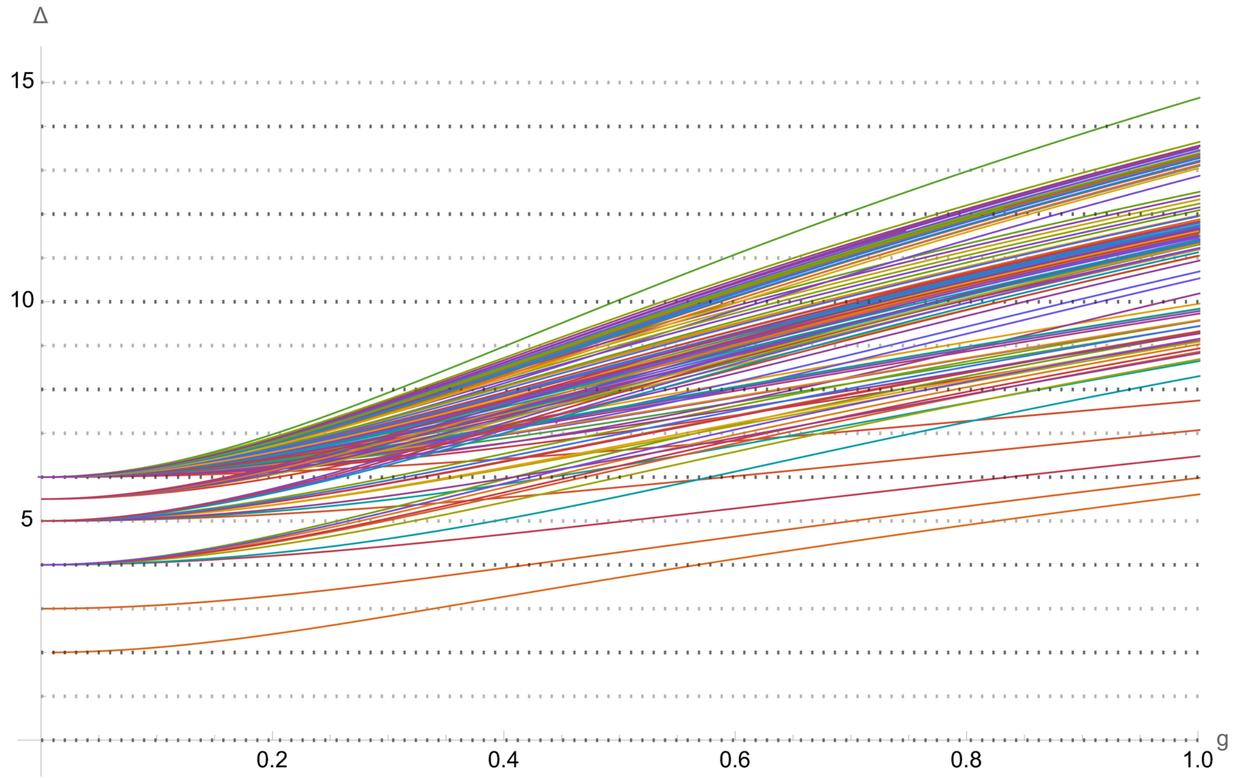
←

Read the spectrum

4. Precision control is achieved by seeing how close to zero, certain parameters, which are "gauge fixed" to be zero, are.
Once the desired precision is reached, read off the value of Δ obtained.

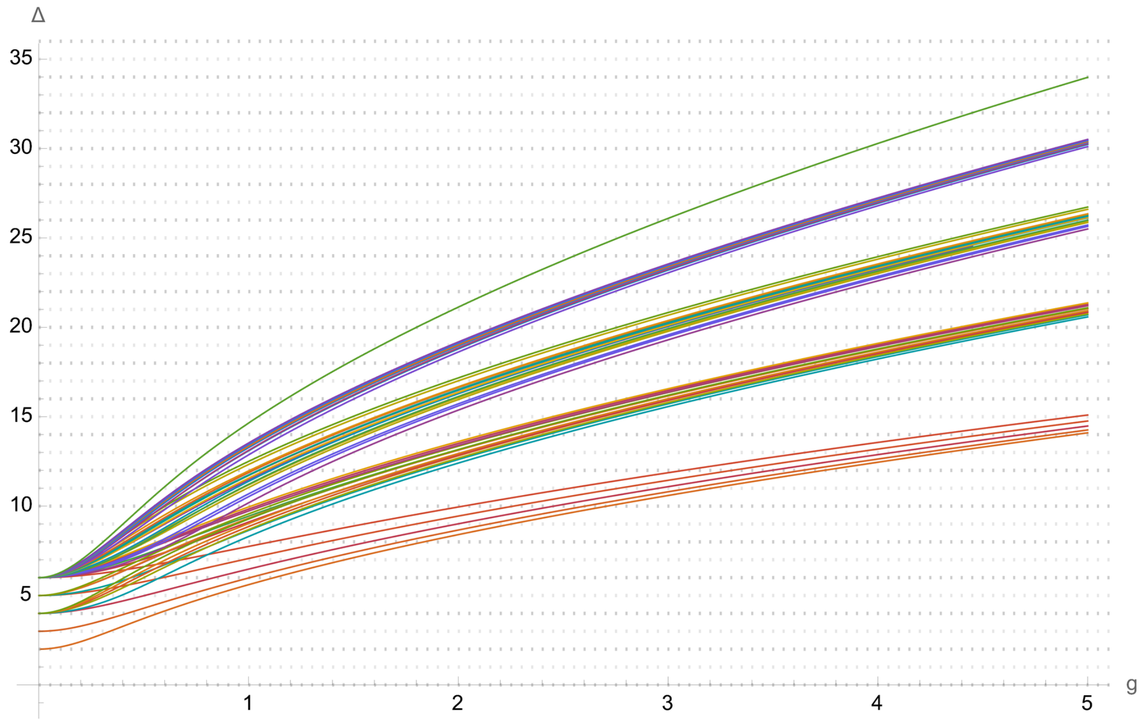
Outcome: following the states

We have computed all states of $\mathcal{N} = 4$ SYM up to $\Delta_{\text{bare}} < 7$. The running time of states depends on symmetries of the state, which results in symmetries of $\mathbf{P}_a$ and $\mathbf{P}^a$.



The states are computed up to the following couplings. “Left-right” and parity symmetric: $g \in [0,5]$; “Left-right” symmetric and general parity: $g \in [0,2]$; General and parity symmetric: $g \in [0,2]$; General: $g \in [0,1]$; Konishi: $g \in [0,13]$.

Outcome: strong coupling analysis



- At strong coupling spectrum forms “bands” with slopes corresponding to “string mass levels” [Gubser, Klebanov, Polyakov ’98]: $\Delta \simeq 2\sqrt{\delta}\lambda^{\frac{1}{4}}$.

- Degeneracies of states with the same δ and ℓ are lifted. The subleading coefficient d_1 for the most of the states is determined: $\Delta \simeq 2\sqrt{\delta}\lambda^{1/4} + \Delta_{\texttt{const}} + \dfrac{d_1}{\sqrt{\delta}\lambda^{1/4}}$ which turns out to be a simple rational number.

- We provide a heuristic argument that $\Delta_{\texttt{const}} = -2$ for all states in planar $\mathcal{N} = 4$ SYM and give credence to it by fitting our data.

Outcome: Kaluza-Klein towers

It is possible to count states with given δ in the flat space limit by counting the representations of $SO(9)$, the massive little group of $\mathbb{R}^{1,9}$ [Alday, Hansen, Silva '23]. To capture the $AdS_5 \times S_5$ structure, one should count the representations of $SO(4) \times SO(5)$.

How to identify this spectrum with $\mathcal{N} = 4$ SYM at strong coupling? One can consider the compactification of five directions into S_5 such that representations of $SO(5)$ are identified with the tower of $SO(6)$ representations. For example,

$$[0;0] = \sum_{p=0}^{\infty} [0,p,0].$$

We were able to assign states of $\mathcal{N} = 4$ SYM with quantum numbers $[\Delta, \ell_1, \ell_2, q_1, p, q_2]$ to the corresponding KK-towers by noticing that the subleading coefficient of the Casimir operator $J^2 \simeq 2\delta\sqrt{\lambda} + j_1$ is the same! If so, we can compute the subleading coefficient of the spectrum d_1 by computing just one state at the top:

$$d_1 = \frac{p^2}{2} + p + 1 + \frac{j_1}{2}$$

Structure constants at strong coupling

Consider the four-point function of 20' operators: operators with different twists $\tau \equiv \Delta - \ell$ are exchanged in the operator product expansion: $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle = \sum_{\tau,\ell} C^2_{\tau,\ell} G_{\tau+4,\ell}(u,v)$.

At strong coupling, under a simple change of variables, $C^2_{\tau,\ell} \to f_{\delta,\ell}$, [Alday, Hansen, Silva '22] derived conformal bootstrap constraints on CFT-data combinations of type $\sum_k f^{(n)}_{\delta,\ell;\,k}$ and $\sum_k f^{(n)}_{\delta,\ell;\,k}\, \tau^{(m)}_{\delta,\ell;\,k}$. The superscript refers to order in the strong coupling expansion. These sums are over all operators with same δ and ℓ, and need to be "unmixed" in order to extract individual predictions.

For $\delta = 2$ and $\ell = 0$, there are two such operators, which we can unmix using our spectral data. Therefore, we obtain

$$f^{(0)}_{2,0;\,1} = 0\,, \qquad f^{(0)}_{2,0;\,2} = \frac{1}{4}\,,$$

which are the first Bootstrability predictions at strong coupling for $\mathcal{N} = 4$ SYM!

Bootstrability for $\mathcal{N} = 4$ SYM?

- We have computed the spectrum of local operators in the “bootstrability” spirit: all states up to some $\Delta_{\texttt{cutoff}}$, in a wide range of coupling. How can we implement the numerical conformal bootstrap effectively? [Niarchos, Papageorgakis, Richmond, Stapleton, Woolley ’23].
- How to tame contribution from double-trace operators? [Caron-Huot, Coronado, Trinh, Zahraee ’22] [Alday, Hansen, Silva ’22].
- We can extract more OPE coefficients at strong coupling with the dispersive sum rules [Alday, Hansen, Silva ’22 ’23].
- We can provide building blocks for resumming wrapping corrections [Basso, Georgoudis, Klemenchuk Sueiro ’22] in Hexagon approach [Basso, Komatsu, Vieira ’15].

QSC Solver: open source

The Solver is available on GitHub (https://github.com/julius-julius/qsc) and includes:

- C++ core code.
- Auxiliary packages which allow to initialise from perturbative QSC solver of [Marboe, Volin ’17 ’18].
- Python script which manages the parameters of the run and does not require control from the user.
- Mathematica notebook with the code prototype.
- Numerical data for all already computed states, so that it is possible for a user to run own states and continue those already computed.

Thank you!