$\mathcal{N} = 8$ as a Theoretical Laboratory for Gravitational Scattering

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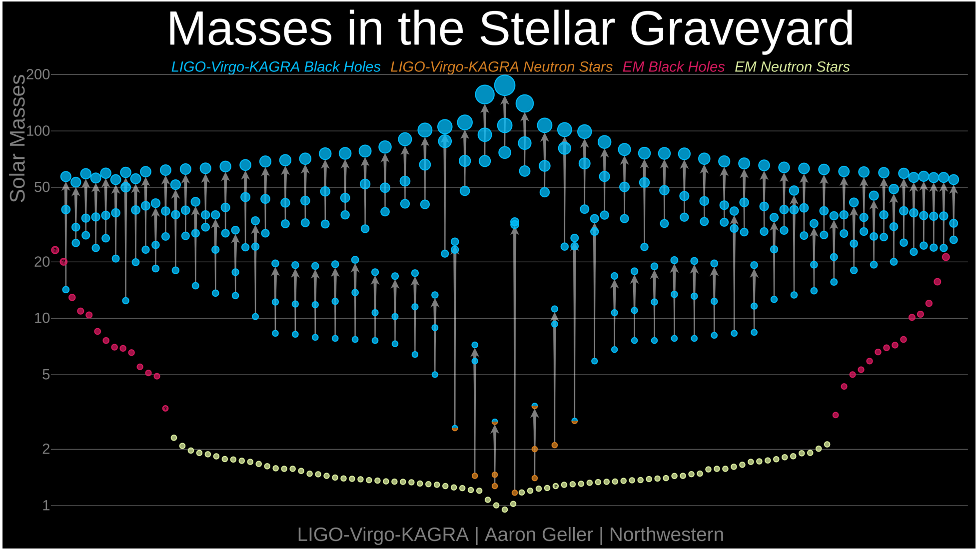
Based on
2008.12743, 2101.05772, 2104.03256,
+ in progress in collaboration with
P. Di Vecchia, R. Russo and G. Veneziano

Outline

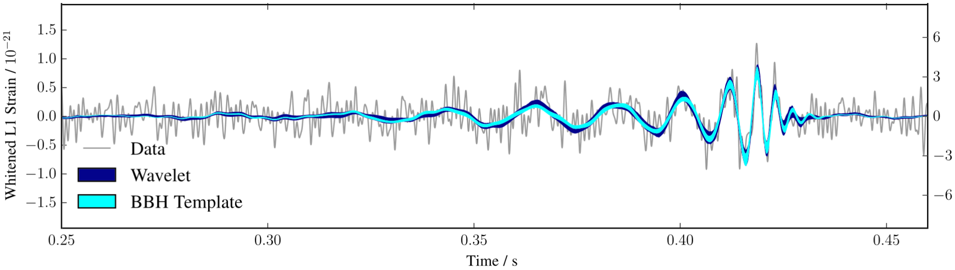
Outline

Gravitational Wave Astronomy



Waveform Templates

[LIGO Scientific Collaboration '16]



Inspiral
Weak gravity

Merger
Strong gravity

Ringdown
Small oscillations

Analytical Approximation Methods

- Post-Newtonian (PN): expansion "for small G and small v"

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1$$

- Post-Minkowskian (PM): expansion "for small G"

$$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}$$

- Self-Force: expansion in the near-probe limit $m_2 \ll m_1$ or

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \ll 1\,.$$

General Relativity from Scattering Amplitudes

Idea

Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion $\leftrightarrow$ PM expansion

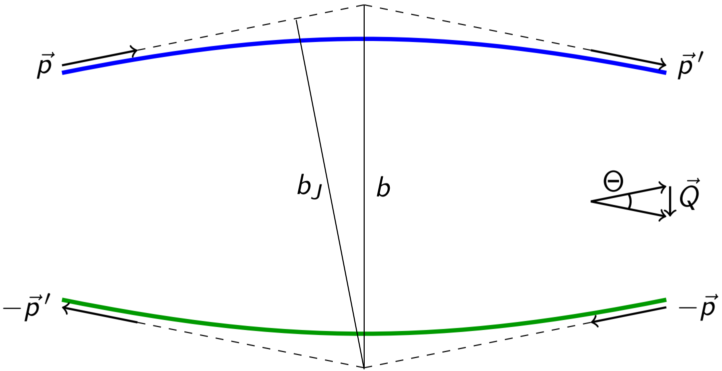
Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM This talk	4PM State of the art

- Lorentz invariance $\leftrightarrow$ generic velocities
- Study scattering events, then export to bound trajectories by analytic continuation.

Post-Minkowskian (PM) Scattering

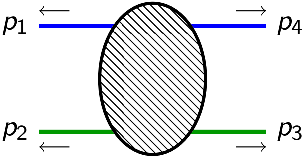
[see e.g. Kosower, Maybee, O'Connell '18; Bern et al. '19; Di Vecchia, CH, Russo, Veneziano '21; Bellazzini, Isabella, Riva '22]

$$\boxed{\frac{\hbar}{m} \ll Gm \ll b}$$



$$\frac{Gm^2}{\hbar} \underset{CL}{\gg} 1\,, \qquad \frac{Gm}{b} \underset{\text{PM}}{\ll} 1\,, \qquad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic)}, \qquad Q = 2p\sin\frac{\Theta}{2}\,.$$

The Elastic Eikonal



$$s = -(p_1 + p_2)^2 = E^2$$
$$= m_1^2 + 2m_1 m_2 \,\sigma + m_2^2 \,,$$
$$t = -(p_1 + p_4)^2 = -q^2 \,.$$

- From q to b: Fourier transform $[q \sim \mathcal{O}(\frac{\hbar}{b})]$

$$\tilde{\mathcal{A}}(b) = \text{FT}\,(b) = \int \frac{d^D q}{(2\pi)^D}\, 2\pi\delta(2p_1 \cdot q) 2\pi\delta(2p_2 \cdot q)\, e^{ib\cdot q} \mathcal{A}(s,q)\,, \qquad \boxed{1 + i\tilde{\mathcal{A}}(b) = e^{2i\delta(b)}}$$

with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim \frac{Gm^2}{\hbar}\left(\log b + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \cdots\right)$

- From b to Q: stationary-phase approximation $[Q \sim \mathcal{O}(p \cdot \frac{Gm}{b})]$

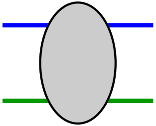
$$\int d^{D-2}b\, e^{-ib\cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial\, \text{Re}\, 2\delta}{\partial b^\mu}$$

Outline

Tree-Level $2 \to 2$ Amplitude

- $2 \to 2$ amplitude in momentum space

$$\mathcal{A}_0^{(4)} =$$



$$= \frac{32\pi G m_1^2 m_2^2 \sigma^2}{q^2} + \text{analytic in } q^2$$

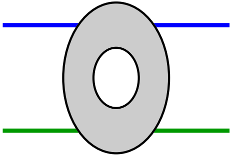
- In impact-parameter space $\boxed{1 + i\tilde{\mathcal{A}}^{(4)} \simeq e^{2i\delta_0}}$

$$\tilde{\mathcal{A}}_0^{(4)} = 2\delta_0 = \frac{4 G m_1 m_2 \sigma^2}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} + \text{ short-range}$$

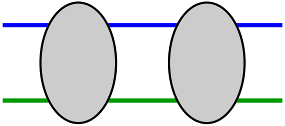
One-Loop $2 \to 2$ Amplitude

[Caron-Huot, Zahraee '18; Parra-Martinez, Ruf, Zeng '20] [Di Vecchia, CH, Russo Veneziano '21]

$$\mathcal{A}_1^{(4)} =$$



$$= \frac{i}{2}$$

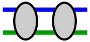


$$+ \mathcal{B}_1^{(4)} + \cdots$$

with

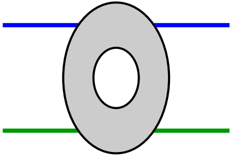
$$\mathcal{B}_1^{(4)} = 0 - \frac{128\pi^2 G^2 m_1^2 m_2^2 \left(m_1 + m_2\right) \sigma^4}{q \left(\sigma^2 - 1\right)} \epsilon + \mathcal{O}(\epsilon^2)$$

in $D = 4 - 2\epsilon$.

- The two-particle cut $\frac{i}{2}$  is imaginary and infrared divergent
- It is also "superclassical" $\mathcal{O}(\frac{1}{\hbar^2})$
- $\mathcal{B}_1^{(4)}$ is real, finite and classical $\mathcal{O}(\frac{1}{\hbar})$.

Elastic Exponentiation at One Loop

[Caron-Huot, Zahraee '18; Parra-Martinez, Ruf, Zeng '20] [Di Vecchia, CH, Russo Veneziano '21]

$$\tilde{\mathcal{A}}_1^{(4)} = \text{FT}$$  $$= \frac{i}{2}(2\delta_0)^2 + \tilde{\mathcal{B}}_1^{(4)}$$

- This matches the exponential $\boxed{1 + i\tilde{\mathcal{A}}^{(4)} \simeq e^{i(2\delta_0 + 2\delta_1)}}$
- And it identifies

$$\tilde{\mathcal{B}}_1^{(4)} = 2\delta_1 = 0 - \frac{16\pi G^2 m_1 m_2 (m_1 + m_2)\,\sigma^4}{b\,(\sigma^2 - 1)^{3/2}}\,\epsilon + \mathcal{O}(\epsilon^2)$$

- Therefore, in $D = 4$ there is no 2PM correction to the deflection angle,

$$\Theta = \frac{4Gm_1 m_2 \sigma^2}{J\sqrt{\sigma^2 - 1}} + 0 + \mathcal{O}(G^3)\,.$$

Integrability in the Probe Limit

The vanishing of the 2PM eikonal phase can be seen as a consequence of integrability:

- Both the 1PM and the 2PM can be fixed by the mass scaling of the angle

$$\Theta \simeq \frac{Q}{p} = \frac{2Gm_1m_2}{J}\left[Q_{00} + Q_{10}\frac{2Gm_1}{b_J} + Q_{01}\frac{2Gm_2}{b_J} + \mathcal{O}(G^2)\right], \qquad J = b_J p$$

- In the strict probe limit $m_1 \gg m_2$ the deflection angle can be obtained exactly

$$\Theta = 2\arctan\left[\frac{2Gm_1}{b_J}\frac{\sigma^2}{\sigma^2-1}\right] = \frac{4Gm_1}{b_J}\frac{\sigma^2}{\sigma^2-1} + \mathcal{O}(G^3)$$

  and therefore

$$Q_{00} = \frac{2\sigma^2}{\sqrt{\sigma^2-1}}, \qquad Q_{10} = Q_{01} = 0\,.$$

- This stems from the fact that the $\sim 1/r$ potential does not receive one-loop corrections [Caron-Huot, Zahraee '18].

From the Deflection Angle to the Precession Angle

We introduce the effective potential $V(r)$

$$p^2 = p_r^2 + \frac{J^2}{r^2} + V(r)\,, \qquad V(r) = -\left(\frac{G}{r}\,f_1 + \frac{G^2}{r^2}\,f_2 + \frac{G^3}{r^3}\,f_3 + \cdots\right)$$

to extract information about the bound system as well.

- Expanding the deflection angle for small G,

$$\Theta = -\pi + 2J\int_{r_*}^{\infty}\frac{dr}{r^2\sqrt{p^2 - \frac{J^2}{r^2} - V(r)}} = \frac{G}{pJ}\,f_1 + \frac{G^2\pi}{2J^2}\,f_2 + \mathcal{O}(G^3)$$

  fixes $f_1 = 4m_1^2m_2^2\sigma^2/E$ and of course $f_2 = 0$.
- Analytically continuing the integral to $\sigma < 1$ (**bound case**) we obtain the precession angle

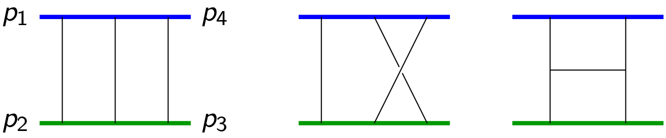
$$\Delta\Phi = -2\pi + 2J\int_{r_-}^{r_+}\frac{dr}{r^2\sqrt{p^2 - \frac{J^2}{r^2} - V(r)}} = 0$$

No precession! [Caron-Huot, Zahraee '18]

Outline

Two-Loop $\mathcal{N} = 8$ Supergravity

- Tackle the problem in a theory with a simpler amplitude integrand.
- Topologies entering the two-loop $\mathcal{O}(G^3)$ calculation (+ crossed topologies):



- Same families of integrals as in the GR case.
- We can calculate them via Method of Regions + Master Integrals + Differential Equations

Differential Equations and Boundary Conditions

- Reduce all *soft* integrals in a given family to a basis $\vec{I} = (I_1, I_2, \dots)$ (master integrals) via integration by parts. [Parra-Martinez, Ruf, Zeng '20] (`LiteRed`, `FIRE6`).

- In a "pure" basis (`Epsilon`), they satisfy simple differential equations. M_j constant, ϵ-independent matrices and $x \simeq \sigma - \sqrt{\sigma^2 - 1}$,

$$d\vec{I}(\epsilon; x) = \epsilon \sum_{j=0,\pm 1} M_j \, \vec{I}(\epsilon; x) \, d \log(x - j)$$

- Solve perturbatively $\vec{I}(\epsilon; x) = \vec{I}^{(0)}(x) + \epsilon \, \vec{I}^{(1)}(x) + \cdots$ as $\epsilon \to 0$.

- Boundary conditions fixed in the $x \to 1^-$ (small velocity) limit.

The 3PM Eikonal in $\mathcal{N}=8$ Supergravity

[Parra-Martinez, Ruf, Zeng '20], [Di Vecchia, CH, Russo, Veneziano '20, '21]

- Full eikonal phase:

$$\mathrm{Re}\,2\delta_2 = 2\tilde{\delta}_2 + 2\delta_2^{\mathrm{RR}}\,.$$

- Conservative part: time-reversal *even* contributions to Θ

$$2\tilde{\delta}_2 = \frac{16G^3 m_1^2 m_2^2}{b^2}\left[-\frac{\sigma^4}{\sigma^2-1}\,\mathrm{arccosh}\,\sigma\right]$$

- Radiation-Reaction part: time-reversal *odd* contributions to Θ,

$$2\delta_2^{\mathrm{RR}} = \frac{16G^3 m_1^2 m_2^2}{b^2}\left[\frac{\sigma^6}{(\sigma^2-1)^2} + \frac{\sigma^5(\sigma^2-2)}{(\sigma^2-1)^{\frac{5}{2}}}\,\mathrm{arccosh}\,\sigma\right].$$

- Infrared divergent exponential suppression:

$$\mathrm{Im}\,2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon}+\log(\sigma^2-1)\right]\mathrm{Re}\,2\delta_2^{\mathrm{RR}} + \cdots$$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit:

- the *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$,
- its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

$$\mathrm{Re}\, 2\delta_2 \sim Gs \, \frac{\Theta^2}{4} \,, \qquad \Theta \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

Beyond the Probe Limit: 3PM Deflection and Precession

Armed with the 3PM eikonal...

- Conservative 3PM deflection angle

$$\tilde{\Theta} = \frac{4Gm_1m_2\sigma^2}{J\sqrt{\sigma^2-1}} - \frac{16G^3m_1^3m_2^3\sigma^6}{3J^3(\sigma^2-1)^{\frac{3}{2}}} + \frac{32G^3m_1^4m_2^4\sigma^6}{J^3E^2}\left[-\sigma^4 \text{ arccosh}\,\sigma\right]$$

- Coefficient f_3 in the effective potential,

$$f_3 = \frac{16m_1^3m_2^3}{E\sqrt{\sigma^2-1}}\left[-\sigma^4 \text{ arccosh}\,\sigma\right]$$

- Precession angle in the Post-Newtonian limit $v_\infty = \sqrt{1-\sigma^2} \to 0$ (for fixed Gm_1m_2/v_∞^2),

$$\Delta\Phi = -2\pi + 2J\int_{r_-}^{r_+} \frac{dr}{r^2\sqrt{p^2 - \frac{J^2}{r^2} - V(r)}} = -48v_\infty^4\left(\frac{Gm_1m_2}{Jv_\infty}\right)^4 \nu$$

Precession to order ν! (beyond the probe limit).

Summary and Outlook

- Eikonal phase: $\mathrm{Re}\, 2\delta_2$ is smooth and universal at high energy thanks to the interplay between conservative and radiation-reaction effects.
- Integrability/the no-precession conjecture only holds in the strict probe limit, and thus fails from two loops onward. [cf. Caron-Huot, Zahraee '18; Davis, Melville '23]

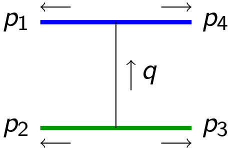
For the future:

- Learning more at **three loops** $\mathcal{O}(G^4)$ using $\mathcal{N} = 8$ as a theoretical laboratory
- Extending the boundary-to-bound analytic continuation to **dissipative** effects
- Study a resummation in G at fixed ν (**self-force** approach).

ADDITIONAL MATERIAL

Example: the 1PM Eikonal

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



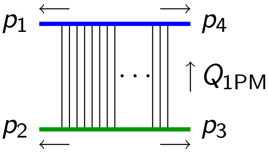
$$\mathcal{A}_0(s,q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0(s,b) = \frac{4 G m_1 m_2 (\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2-1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} .$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al.'18]

$$e^{2i\delta_0} \underset{\text{"small } G\text{"}}{\to} 1 + i\tilde{\mathcal{A}}_0 \implies 2\delta_0 = \tilde{\mathcal{A}}_0 .$$

- From $Q = \partial_b 2\delta$, we obtain the leading-order deflection



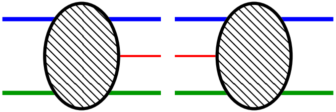
$$Q_{1PM} = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2-1}}$$

$$\Theta_{1PM} = \frac{4 G E \left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2-1)} .$$

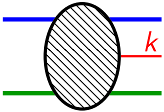
3PM Radiation-Reaction from Soft Theorems [2101.05772]

- Analyticity: $i\log(1-\sigma^2-i0) = i\log(\sigma^2-1)+\pi$
- Unitarity: $\mathrm{Im}\,2\delta_2 = [\mathrm{Im}\,\tilde{\mathcal{A}}_2]_{3p.c.}$ and

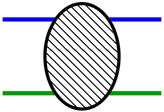
$$[\mathrm{Im}\,2\mathcal{A}]_{3p.c.} = \int d(\mathrm{LIPS})$$



- Soft theorem: [Weinberg '64,'65]



$$\mu\nu \sim \left[\sum_n \frac{\sqrt{8\pi G}\, p_n^\mu p_n^\nu}{p_n \cdot k}\right]$$



For $\mathrm{Im}\,2\delta_2$ this gives $\frac{1}{\pi}\mathrm{Re}\,2\delta_2^{\mathrm{RR}}$ times

$$\int_0^{\omega_{\max} b} \frac{2\,d\omega}{\omega^{1+2\epsilon}} \sim -\frac{1}{\epsilon} + 2\log(\omega_{\max} b) \sim -\frac{1}{\epsilon} + \log(\sigma^2-1)\,.$$

3PM Radiation-Reaction from Soft Theorems [2101.05772]

The IR divergence in $\operatorname{Im} 2\delta_2$ determines $\operatorname{Re} 2\delta_2^{RR}$

$$\operatorname{Re} 2\delta_2^{RR} = \lim_{\epsilon \to 0} \left[-\pi\epsilon \operatorname{Im} 2\delta_2 \right] .$$