New insights into quantum affine Gaudin models

Tommaso Franzini

10th Bologna Workshop on Conformal Field Theory and Integrable Models
September 2023

Based on work with **Charles Young**

[23xx.xxxxx] in preparation

Plan of the talk

Introduction

Finite-type Gaudin models

Affine-type Gaudin models

The Feigin-Frenkel homomorphism and generalisations

Conclusion and Outlooks

Gaudin Models

- **Quantum Gaudin models** [Gaudin '76] are defined by
  - a Lie algebra $\mathfrak{g}$ of **finite** or **affine** type
  - a set of points $\{z_1, \ldots, z_N\} \subset \mathbb{CP}^1$

  The Hamiltonians are

$$\Xi_i = \sum_{j \neq i = 1}^{N} \kappa_{ab} \frac{I^{a(i)} I^{b(j)}}{z_i - z_j} \tag{1}$$

Gaudin Models

- **Quantum Gaudin models** [Gaudin '76] are defined by
  - a Lie algebra $\mathfrak{g}$ of **finite** or **affine** type
  - a set of points $\{z_1, \dots, z_N\} \subset \mathbb{CP}^1$

  The Hamiltonians are

$$\Xi_i = \sum_{j \neq i = 1}^{N} \kappa_{ab} \frac{I^{a(i)} I^{b(j)}}{z_i - z_j} \tag{1}$$

- **Classical Gaudin models** can be obtained by a certain classical limit [Lacroix '18, ...]

Gaudin Models

- **Quantum Gaudin models** [Gaudin '76] are defined by
  - a Lie algebra $\mathfrak{g}$ of **finite** or **affine** type
  - a set of points $\{z_1, \ldots, z_N\} \subset \mathbb{CP}^1$

  The Hamiltonians are

$$\Xi_i = \sum_{j \neq i = 1}^{N} \kappa_{ab} \frac{I^{a(i)} I^{b(j)}}{z_i - z_j} \tag{1}$$

- **Classical Gaudin models** can be obtained by a certain classical limit [Lacroix '18, ...]

- **Why?** Integrable field theories [Vicedo '18], connection with 4D Chern-Simons theory [Costello-Witten-Yamazaki '19, Vicedo '19,...],...

Finite-type Gaudin models

Complete description for arbitrary $\mathfrak{g}$ in [Feigin Frenkel Reshetikhin '94]

Finite-type Gaudin models

Complete description for arbitrary $\mathfrak{g}$ in [Feigin Frenkel Reshetikhin '94]

- Characterisation of **local higher Hamiltonians**: there is a large commutative subalgebra $\mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$ which is in 1-1 correspondence with singular vectors of $\mathbb{V}_0^{\widehat{\mathfrak{g}}, -h^\vee}$, which are isomorphic to the polynomial algebra $\mathbb{C}[P_i]_{i=1,\dots,\mathrm{rank}\,\mathfrak{g}}$, and $\deg P_i = d_i + 1$ where d_i are the **exponents** of $\mathfrak{g}$.

Finite-type Gaudin models

Complete description for arbitrary $\mathfrak{g}$ in [Feigin Frenkel Reshetikhin '94]

- Characterisation of **local higher Hamiltonians**: there is a large commutative subalgebra $\mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$ which is in 1-1 correspondence with singular vectors of $\mathbb{V}_0^{\widehat{\mathfrak{g}}, -h^\vee}$, which are isomorphic to the polynomial algebra $\mathbb{C}[P_i]_{i=1,\dots,\mathrm{rank}\,\mathfrak{g}}$, and $\deg P_i = d_i + 1$ where d_i are the **exponents** of $\mathfrak{g}$.
- Description of their **spectrum**: The Gaudin subalgebra is isomorphic to the algebra of polynomial functions over the space of ${}^L\mathfrak{g}$-opers

$$\mathcal{Z}(\mathfrak{g}) \cong \mathrm{Fun}(\mathrm{Oper}_{^L\mathfrak{g}}(\mathbb{CP}^1)) \qquad (2)$$

Finite-type Gaudin models

Complete description for arbitrary $\mathfrak{g}$ in [Feigin Frenkel Reshetikhin '94]

- Characterisation of **local higher Hamiltonians**: there is a large commutative subalgebra $\mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$ which is in 1-1 correspondence with singular vectors of $\mathbb{V}_0^{\widehat{\mathfrak{g}}, -h^\vee}$, which are isomorphic to the polynomial algebra $\mathbb{C}[P_i]_{i=1,\ldots,\mathrm{rank}\,\mathfrak{g}}$, and $\deg P_i = d_i + 1$ where d_i are the **exponents** of $\mathfrak{g}$.
- Description of their **spectrum**: The Gaudin subalgebra is isomorphic to the algebra of polynomial functions over the space of ${}^L\mathfrak{g}$-opers

$$\mathcal{Z}(\mathfrak{g}) \cong \mathsf{Fun}(\mathsf{Oper}_{{}^L\mathfrak{g}}(\mathbb{CP}^1)) \tag{2}$$

- Is it possible to generalise this construction to the **affine case**?

Affine-type Gaudin models

There is **no general construction**

- There is no analogue of the vacuum Verma module at critical level (new directions from Higher Kac-Moody algebras [Faonte Kapranov Hennion '17, Alfonsi Young '22])
- No isomorphism between the Bethe algebra and a space of opers

Affine-type Gaudin models

There is **no general construction**

- There is no analogue of the vacuum Verma module at critical level (new directions from Higher Kac-Moody algebras [Faonte Kapranov Hennion '17, Alfonsi Young '22])
- No isomorphism between the Bethe algebra and a space of opers

There are several **conjectures** [Feigin Frenkel '11, Lacroix Vicedo Young '18] based of the definition of **affine opers**.

Some higher local Hamiltonians have been constructed [Lacroix Vicedo Young '20, Kotousov Lacroix Teschner '21, TF Young '21], for example in $\widehat{\mathfrak{sl}}_2$

$$\varsigma_3(z) = \Big[\delta_{(ab}\delta_{cd)} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z) I^d_{-1}(z) + \frac{20}{3} f_{abc} I^a_{-2}(z) I^{b\prime}_{-1}(z) I^c_{-1}(z) + \frac{40}{9} I^a_{-3}(z) I^{a\prime\prime}_{-1}(z) - \frac{20}{3} I^{a\prime\prime}_{-2}(z) I^a_{-2}(z) + \frac{40}{9} I^{a\prime}_{-3}(z) I^{a\prime}_{-1}(z) - \frac{10}{3} I^{a\prime}_{-2}(z) I^{a\prime}_{-2}(z) - \frac{20}{3} I^a_{-3}(z) I^a_{-1}(z) \mathsf{k}'(z)\Big] |0\rangle$$

Vertex algebras

A **vertex algebra** [Borcherds '86, Frenkel Huang Lepowsky '93, Frenkel Ben-Zvi '01] is a vector space V, with a vacuum vector $|0\rangle \in V$ and a map $T \in \mathrm{End}(V)$ endowed with an **infinite number of products**

$$\cdot_{(n)} \cdot : V \times V \to V \qquad n \in \mathbb{Z} \tag{3}$$

that can be packed up into **fields**

$$Y(A, x) := \sum_{n \in \mathbb{Z}} A_{(n)} x^{-n-1}, \qquad A_{(n)} \in \mathrm{End}(V) \tag{4}$$

satisfying a series of **axioms** (translation covariance, vacuum, locality, Borcherds identities).

They are the mathematical **framework to describe CFTs** (state field correspondence, OPEs, . . .).

The Feigin-Frenkel homomorphism

In particular, the relation between Hamiltonians and opers is based on the **homomorphism of vertex algebra** [Wakimoto '86, Feigin Frenkel '90]

$$\mathbb{V}_0^{\widehat{\mathfrak{g}},-h^\vee} \to \mathsf{M}(\mathfrak{n}_+) \tag{5}$$

The Feigin-Frenkel homomorphism

In particular, the relation between Hamiltonians and opers is based on the **homomorphism of vertex algebra** [Wakimoto '86, Feigin Frenkel '90]

$$\mathbb{V}_0^{\widehat{\mathfrak{g}},-h^\vee} \to \mathsf{M}(\mathfrak{n}_+) \tag{5}$$

Procedure:

- Realise the Lie algebra in terms of **differential operators** on the algebra $\mathcal{O}(\mathfrak{n}_+) = \mathbb{C}[X^\alpha]_{\alpha\in\Delta_+}$

$$\mathfrak{g} \to \mathrm{Der}\,\mathcal{O}(\mathfrak{n}_+), \qquad A \mapsto \sum_{\alpha\in\Delta_+} P_A^\alpha(X) D_\alpha \tag{6}$$

  For $\mathfrak{sl}_2$: $E \mapsto D$, $H \mapsto -2XD$, $F \mapsto -XXD$.

The Feigin-Frenkel homomorphism

In particular, the relation between Hamiltonians and opers is based on the **homomorphism of vertex algebra** [Wakimoto '86, Feigin Frenkel '90]

$$\mathbb{V}_0^{\widehat{\mathfrak{g}},-h^\vee} \to \mathsf{M}(\mathfrak{n}_+) \tag{5}$$

Procedure:

- Realise the Lie algebra in terms of **differential operators** on the algebra $\mathcal{O}(\mathfrak{n}_+) = \mathbb{C}[X^\alpha]_{\alpha\in\Delta_+}$

$$\mathfrak{g} \to \mathrm{Der}\,\mathcal{O}(\mathfrak{n}_+), \qquad A \mapsto \sum_{\alpha\in\Delta_+} P_A^\alpha(X) D_\alpha \tag{6}$$

  For $\mathfrak{sl}_2$: $E \mapsto D$, $H \mapsto -2XD$, $F \mapsto -XXD$.

- Promote this to **vertex algebras**:

$$\mathbb{V}_0^{\widehat{\mathfrak{g}},k} \to \mathsf{M}(\mathfrak{n}_+), \qquad A[-1]\,|0\rangle \mapsto \sum_{\alpha\in\Delta_+} P_A^\alpha(\gamma[0])\beta_\alpha[-1]\,|0\rangle \tag{7}$$

The Feigin-Frenkel homomorphism

In particular, the relation between Hamiltonians and opers is based on the **homomorphism of vertex algebra** [Wakimoto '86, Feigin Frenkel '90]

$$\mathbb{V}_0^{\widehat{\mathfrak{g}},-h^\vee} \to \mathsf{M}(\mathfrak{n}_+) \qquad (5)$$

Procedure:

- Realise the Lie algebra in terms of **differential operators** on the algebra $\mathcal{O}(\mathfrak{n}_+) = \mathbb{C}[X^\alpha]_{\alpha\in\Delta_+}$

$$\mathfrak{g} \to \mathrm{Der}\,\mathcal{O}(\mathfrak{n}_+), \qquad A \mapsto \sum_{\alpha\in\Delta_+} P_A^\alpha(X) D_\alpha \qquad (6)$$

  For $\mathfrak{sl}_2$: $E \mapsto D$, $H \mapsto -2XD$, $F \mapsto -XXD$.

- Promote this to **vertex algebras**:

$$\mathbb{V}_0^{\widehat{\mathfrak{g}},k} \to \mathsf{M}(\mathfrak{n}_+), \qquad A[-1]\,|0\rangle \mapsto \sum_{\alpha\in\Delta_+} P_A^\alpha(\gamma[0])\beta_\alpha[-1]\,|0\rangle \qquad (7)$$

- In order to be a **homomorphism of vertex algebras**, k must be the **critical level** $-h^\vee$

Affine analogue of Feigin-Frenkel homomorphism

Generalising to the **affine case**, one obtains a homomorphism

$$\widehat{\mathfrak{g}} \to \widetilde{\mathrm{Der}}\mathcal{O}(\mathfrak{n}_+), \qquad A \mapsto \sum_{(\alpha,n)\in \mathsf{A}} P_A^{\alpha,n}(X) D_{\alpha,n} \tag{8}$$

For example, in $\widehat{\mathfrak{sl}}_2$:

$$\begin{aligned} J_{E,1} \mapsto & D_{E,1} - \sum_{k\geq 3} X^{F,k-1} D_{H,k} + 2\sum_{k\geq 3} X^{H,k-1} D_{E,k} \\ & \quad + (-2X^{F,2}X^{E,2} + \dots) D_{E,5} + (X^{F,2}X^{F,2} + \dots) D_{F,5} \\ & \qquad + (-2X^{F,3}X^{E,3} - 2X^{H,3}X^{H,3} + \dots) D_{E,7} + \dots \end{aligned}$$

Affine analogue of Feigin-Frenkel homomorphism

Generalising to the **affine case**, one obtains a homomorphism

$$\widehat{\mathfrak{g}} \to \widetilde{\mathrm{Der}}\mathcal{O}(\mathfrak{n}_+), \qquad A \mapsto \sum_{(\alpha,n)\in \mathsf{A}} P_A^{\alpha,n}(X) D_{\alpha,n} \tag{8}$$

For example, in $\widehat{\mathfrak{sl}}_2$:

$$\begin{aligned} J_{E,1} \mapsto & D_{E,1} - \sum_{k\geq 3} X^{F,k-1} D_{H,k} + 2 \sum_{k\geq 3} X^{H,k-1} D_{E,k} \\ & + (-2X^{F,2}X^{E,2} + \dots) D_{E,5} + (X^{F,2}X^{F,2} + \dots) D_{F,5} \\ & + (-2X^{F,3}X^{E,3} - 2X^{H,3}X^{H,3} + \dots) D_{E,7} + \dots \end{aligned}$$

Can this still be **promoted to vertex algebra homomorphism**?

No, since the (properly completed) Fock module $\widetilde{\mathsf{M}}$ **has not the structure of a vertex algebra**.

Why? Consider $J_{H,0} \in \widehat{\mathfrak{sl}}_2 \mapsto a \in \widetilde{\mathsf{M}}$

$$\begin{aligned}
a_{(1)}a &= 4\sum_{k\geq 0}\sum_{j\geq 0}(\gamma^{E,k}[0]\beta_{E,k}[-1]\,|0\rangle)_{(1)}\gamma^{E,j}[0]\beta_{E,j}[-1]\,|0\rangle \\
&\qquad + 4\sum_{k\geq 1}\sum_{j\geq 1}(\gamma^{F,k}[0]\beta_{F,k}[-1]\,|0\rangle)_{(1)}\gamma^{F,j}[0]\beta_{F,j}[-1]\,|0\rangle \\
&= 4\sum_{k\geq 0}\sum_{j\geq 0}\gamma^{E,k}[1]\beta_{E,k}[0]|\gamma^{E,j}[0]|\beta_{E,j}[-1]\,|0\rangle \\
&\qquad + 4\sum_{k\geq 1}\sum_{j\geq 1}\gamma^{F,k}[1]\beta_{F,k}[0]|\gamma^{F,j}[0]\beta_{F,j}[-1]\,|0\rangle \\
&= (-4\sum_{k\geq 0}1 - 4\sum_{k\geq 1}1)\,|0\rangle = (-4 - 8\sum_{k\geq 1}1)\,|0\rangle\,.
\end{aligned}$$

Why? Consider $J_{H,0} \in \widehat{\mathfrak{sl}}_2 \mapsto a \in \widetilde{\mathsf{M}}$

$$
\begin{aligned}
a_{(1)}a = 4\sum_{k\geq 0}\sum_{j\geq 0}&(\gamma^{E,k}[0]\beta_{E,k}[-1]\,|0\rangle)_{(1)}\gamma^{E,j}[0]\beta_{E,j}[-1]\,|0\rangle \\
&+4\sum_{k\geq 1}\sum_{j\geq 1}(\gamma^{F,k}[0]\beta_{F,k}[-1]\,|0\rangle)_{(1)}\gamma^{F,j}[0]\beta_{F,j}[-1]\,|0\rangle \\
= 4\sum_{k\geq 0}\sum_{j\geq 0}&\gamma^{E,k}[1]\beta_{E,k}[0]|\gamma^{E,j}[0]|\beta_{E,j}[-1]\,|0\rangle \\
&+4\sum_{k\geq 1}\sum_{j\geq 1}\gamma^{F,k}[1]\beta_{F,k}[0]|\gamma^{F,j}[0]\beta_{F,j}[-1]\,|0\rangle \\
= (-4\sum_{k\geq 0}1 &- 4\sum_{k\geq 1}1)\,|0\rangle = (-4-8\sum_{k\geq 1}1)\,|0\rangle\,.
\end{aligned}
$$

One **possible solution** [Young '20]: realise the Lie algebra as two glued copies of the algebra of derivations $\mathfrak{g} \to \widetilde{\mathrm{Der}\mathcal{O}}(\mathfrak{n}_+) \oplus \widetilde{\mathrm{Der}\mathcal{O}}(\mathfrak{n}_-)$. Regard the new infinite sums as abstract generators of $\mathfrak{gl}(\mathfrak{g})[t,t^{-1}] \rtimes \mathbb{C}\mathsf{D}$, and define a homomorphism $\mathbb{V}_0^{\widehat{\mathfrak{g}},0} \to \overline{\mathbf{M}} := \overline{\mathsf{M}} \otimes \mathbb{V}_0^{\mathfrak{gl}(\mathfrak{g})[t,t^{-1}]\rtimes\mathbb{C}\mathsf{D},0}$.

A regularisation procedure

Modify the commutation relations by introducing a **regulator** $z \in \mathbb{C}$

$$[\beta_{a,m}[M], \gamma^{b,n}[N]] = \boldsymbol{z}^n \delta_{N+M,0} \delta_a^b \delta_m^n \mathbf{1} \tag{9}$$

then,

$$a_{(1)} a = (-4 - 8 \sum_{n \geq 1} z^{2n}) |0\rangle \tag{10}$$

A regularisation procedure

Modify the commutation relations by introducing a **regulator** $z \in \mathbb{C}$

$$[\beta_{a,m}[M], \gamma^{b,n}[N]] = z^n \delta_{N+M,0} \delta_a^b \delta_m^n \mathbf{1} \tag{9}$$

then,

$$a_{(1)}a = (-4 - 8 \sum_{n \geq 1} z^{2n}) |0\rangle \tag{10}$$

One can now try to **regularise** the sum using ζ-**function regularisation** [Lepowski '00, Doyon Lepowsky Milas '06]

- regard infinite sum as small-z expansion of rational function in z;
- perform the transformation $z \to e^y$;
- power expand the resulting term for small values of y;
- regard result as the ratio of Laurent series, which is again a Laurent series;
- extract the **constant term** of the series obtained.

For example,

$$
\begin{aligned}
&-4-8\sum_{n\geq 1} z^{2n} \xrightarrow{i} -4-\frac{8z^2}{1-z^2} \xrightarrow{ii} -4-\frac{8e^{2y}}{1-e^{2y}} \\
&\quad \xrightarrow{iii} -4-\frac{8(1+2y+2y^2+\dots)}{1-1-2y-2y^2-\dots} = -4+\frac{8}{2y}\frac{(1+2y+\dots)}{(1+y+\dots)} \\
&\qquad \xrightarrow{iv} -4+\frac{4}{y}(1+2y+\dots)(1-y+\dots) = \frac{4}{y}+\mathcal{O}(y) \xrightarrow{v} 0
\end{aligned}
\tag{11}
$$

For example,

$$
\begin{aligned}
-4-8\sum_{n\geq 1} z^{2n} &\xrightarrow{i} -4-\frac{8z^2}{1-z^2} \xrightarrow{ii} -4-\frac{8e^{2y}}{1-e^{2y}} \\
&\xrightarrow{iii} -4-\frac{8(1+2y+2y^2+\dots)}{1-1-2y-2y^2-\dots} = -4+\frac{8}{2y}\frac{(1+2y+\dots)}{(1+y+\dots)} \\
&\xrightarrow{iv} -4+\frac{4}{y}(1+2y+\dots)(1-y+\dots) = \frac{4}{y}+\mathcal{O}(y) \xrightarrow{v} 0
\end{aligned}
\tag{11}
$$

We can define a **new space** $\widehat{\mathsf{M}}_z \subset \widetilde{\mathsf{M}}_z$, spanned by $\overline{\mathsf{M}}_z$ and the infinite sums of the form $\sum_{k\geq L} \gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle$ ($L \in \mathbb{Z}_{\geq 0}, a, b \in \mathcal{I}, n \in \mathbb{Z}$).

Theorem
$\widehat{\mathsf{M}}_z$ *has the **structure of a vertex algebra**, after regularisation.*

For example,

$$-4-8\sum_{n\geq 1} z^{2n} \xrightarrow{i} -4-\frac{8z^2}{1-z^2} \xrightarrow{ii} -4-\frac{8e^{2y}}{1-e^{2y}}$$
$$\xrightarrow{iii} -4-\frac{8(1+2y+2y^2+\dots)}{1-1-2y-2y^2-\dots} = -4+\frac{8}{2y}\frac{(1+2y+\dots)}{(1+y+\dots)}$$
$$\xrightarrow{iv} -4+\frac{4}{y}(1+2y+\dots)(1-y+\dots) = \frac{4}{y}+\mathcal{O}(y) \xrightarrow{v} 0 \tag{11}$$

We can define a **new space** $\widehat{\mathsf{M}}_z \subset \widetilde{\mathsf{M}}_z$, spanned by $\overline{\mathsf{M}}_z$ and the infinite sums of the form $\sum_{k\geq L} \gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle$ ($L \in \mathbb{Z}_{\geq 0}, a, b \in \mathcal{I}, n \in \mathbb{Z}$).

Theorem

$\widehat{\mathsf{M}}_z$ *has the* ***structure of a vertex algebra****, after regularisation.*

In particular one can define a map $\vartheta : \mathbb{V}_0^{\mathfrak{g},0} \to \widehat{\mathsf{M}}_z$ such that for all $A, B \in \mathbb{V}_0^{\mathfrak{g},0}$

$$\vartheta(A)_{(0)}\vartheta(B) = \vartheta([A,B]) \tag{12}$$

$$\mathtt{reg}[\vartheta(A)_{(1)}\vartheta(B)] = 0 \tag{13}$$

Outlooks

- Lift the construction to a homomorphism of **double loop algebras**
- Proceed with the construction of **Wakimoto modules**
- Make contact with new constructions of **higher Kac-Moody algebra** and understand the meaning of the regularisation procedure

Thank you!