A fixed point approach to solve the generalized hydrodynamics equation

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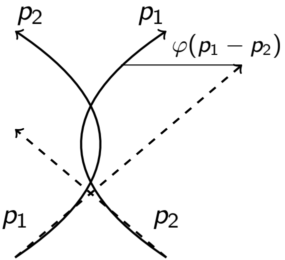
04.09.2023

Generalized Hydrodynamics (GHD)

- System of quasi-particles
  - No physical particles
  - Stable excitations
- Bare velocity $v(p)$
- Scattering: Position shift $\varphi(p_1 - p_2)$



- Density of quasi-particles $\rho(t, x, p)$
- GHD equation (Castro-Alvaredo, Doyon, and Yoshimura 2016; Bertini et al. 2016)

$$\partial_t \rho(t, x, p) + \partial_x (v^{\mathrm{eff}}(t, x, p)\rho(t, x, p)) = 0$$

- Effective velocity equation

$$v^{\mathrm{eff}}(p) = v(p) + \int \mathrm{d}q \, \varphi(p - q)\rho(q)(v^{\mathrm{eff}}(q) - v^{\mathrm{eff}}(p))$$

Motivation

- GHD is an effective description of an integrable model valid on large scales
- GHD equation is a highly non-linear PDE
- Mathematical structure behind GHD equation?
- Efficient algorithms to solve it?
- Mathematical questions:
  - Existence and uniqueness of solutions?
  - Shock formation?
- Connected to validity of the GHD approximation

The Cauchy-problem of the GHD equation

- Cauchy problem: Given $\rho(0, x, p)$, compute $\rho(t, x, p)$
- Coordinate transformation to free particles (Doyon, Spohn, and Yoshimura 2018):
  - $\partial_t \tilde{\rho} + \partial_{\tilde{x}}(v(p)\tilde{\rho}) = 0$
  - Allows to solve for $\rho(t, \cdot, \cdot)$ at arbitrary t
- New contribution: Given t and x solve for $\rho(t, x, \cdot)$
  - Efficient and precise solution algorithm
  - 'Ideal' to study mathematical properties of solutions

Functional fixed point equation

- Idea:
  - Quasi-particles follow GHD characteristics
  - Occupation function $n(t, x, p)$ constant along characteristics
  - $\partial_t n = 0$ in a suitable coordinate system $x \to k = K(t, x, p)$
- Result: Functional fixed point equation:

$$K(t, x, p) = F_{t,x}[K](p) = x - v(p)t + \int \frac{\mathrm{d}q}{2\pi} \varphi(p - q) \hat{N}_0(K(t, x, q), q)$$

- $\hat{N}_0(k, p)$ depends on initial data only
- Fixed point maps $F_{t,x}[K]$ are independent for different t, x

Lieb-Liniger model

- Fixed point map is contracting
- Banach fixed point theorem: Fixed point always exists and is unique
- Solution to the GHD equation exists and is unique for any initial state for all times t (under mild assumptions)
- Smooth initial conditions give rise to smooth solutions
  - First proof of absence of shock formation
- Fixed point iteration converges exponentially fast:
  - New efficient algorithm to solve GHD equation
  - Allows to compute solution at arbitrarily large t and x

Thank you