Entanglement along a massless flow: the tricritical Ising model

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Introduction

- The entanglement entropies

$$S = -\operatorname{Tr}\left[\rho_A \log \rho_A\right], \qquad S_n = \frac{1}{1-n}\log\left[\operatorname{Tr}\rho_A^n\right], \tag{1}$$

are known to be computable from the partition function of the replicated QFT, given by the correlator of *twist fields*, e.g. for an interval of lenght ℓ [1, 2]

$$\operatorname{Tr}\rho_A^n = \langle \mathcal{T}_n(0)\widetilde{\mathcal{T}}_n(\ell)\rangle. \tag{2}$$

- At criticality, for systems described by a CFT with central charge c, the entanglement entropies of an interval is

$$S_n = \frac{c}{6}\frac{n+1}{n}\log\frac{\ell}{\epsilon}. \tag{3}$$

- Using the form factor approach for twist fields [2], we compute the entanglement entropies out of criticality, in a massless renormalisation group flow.
- In presence of a, e.g, $\mathbb{Z}_2$ symmetry, the symmetry resolved entanglement entropy can instead be computed from the two point function of composite twist fields $\mathcal{T}_n^\mu$ given by the fusion of the standard twist field and the disorder field μ which implements the $\mathbb{Z}_2$ transformation.

Massless flow from tricritical to critical Ising

- The tricritical Ising model is described by the unitary minimal model $\mathcal{M}_4$ with central charge $c = \frac{7}{10}$.
- Deforming with the vacancy density field $\phi_{1,3}$ [3, 4]

$$\mathcal{A} = \mathcal{M}_4 + g\int \mathrm{d}^2 z\, \phi_{1,3}, \tag{4}$$

with $g > 0$, the theory interpolates between tricritical to critical Ising CFT $c_{\mathrm{UV}} = \frac{7}{10} \to c_{\mathrm{IR}} = \frac{1}{2}$.
- This renormalisation group flow possesses massless fermionic excitations, either left- or right-moving.
- Near the IR CFT, the model is described by $T\bar{T}$-deformed massless Majorana fermion [3, 4]

$$\mathcal{A} \approx \int \mathrm{d}^2 z \left[\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} - \frac{4}{M^2}\left(\psi\partial\psi\right)\left(\bar{\psi}\bar{\partial}\bar{\psi}\right) + \dots\right], \tag{5}$$

where M is a crossover mass scale.
- The renormalisation group flow has a $\mathbb{Z}_2$ spin-flip symmetry.

Twist fields form factors

- In a n-replicated QFT, the twist fields implement appropriate boundary conditions

$$\begin{aligned} \phi_i(\mathbf{y})\mathcal{T}_n(\mathbf{x}) &= \mathcal{T}_n(\mathbf{x})\,\phi_{i+1}(\mathbf{y}) && \text{for } y^1 > x^1, && (6)\\ &= \mathcal{T}_n(\mathbf{x})\,\phi_i(\mathbf{y}) && \text{otherwise,} && (7) \end{aligned}$$

$$\begin{aligned} \phi_i(\mathbf{y})\mathcal{T}^\mu_n(\mathbf{x}) &= e^{\mathrm{i}\pi\kappa_\phi}\,\mathcal{T}^\mu_n(\mathbf{x})\,\phi_{i+1}(\mathbf{y}) && \text{for } y^1 > x^1 \text{ and } i = n, && (8)\\ &= \mathcal{T}^\mu_n(\mathbf{x})\,\phi_i(\mathbf{y}) && \text{otherwise.} && (9) \end{aligned}$$

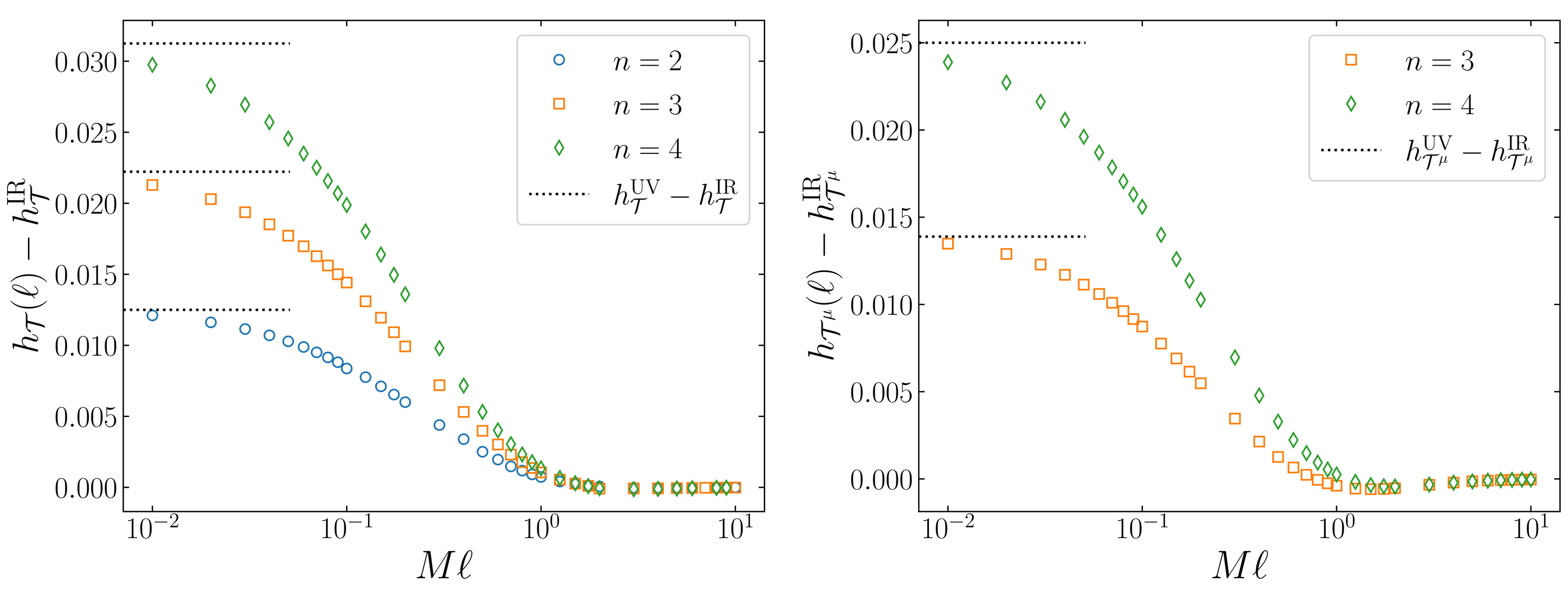
- Using the bootstrap relations for form factors [2], we find the form factors of twist fields in the massless flow up to the four particle level.
- Due to $\mathbb{Z}_2$ symmetry only form factors with an even number of left and of right movers are non vanishing.
- Form factors with only left- or right-movers are identical to the one in the massive Ising model found in [2].
- At the four particle level the only form factor which is different from the Ising ones is $F_{2,2}^{\mathcal{T}|j_1 j_2 j_1' j_2'}$ ($F_{2,2}^{\mathcal{T}^\mu|j_1 j_2 j_1' j_2'}$ for the composite twist field), coupling two left- and two right-movers.

Running conformal dimension

- A running dimension of both the standard and symmetry resolved twist field can be computed using the Δ-sum rule [5].
- In [6] it was argued that the running dimension of the standard twist field provides an entropic c-function, being monotonically decreasing along a renormalisation group flow.
- At the 4-particle order, the running conformal dimension of the twist field is given by

$$\begin{aligned} h(\ell) - h^{\mathrm{IR}} \approx -\frac{n}{2\langle\mathcal{T}_n\rangle}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_1'\mathrm{d}\theta_2'}{2\times 2\,(2\pi)^4}\frac{(1+\ell\,E)\,e^{-\ell\,E}}{2\,E^2}\times \\ \times F_{2,2}^{\Theta}(\theta_1,\theta_2,\theta_1',\theta_2')\left(F_{2,2}^{\mathcal{T}|1111}(\theta_1,\theta_2,\theta_1',\theta_2';n)\right)^*, \end{aligned} \tag{10}$$

where the form factor of the trace of the stress-energy tensor Θ was found in [7].



(a) Branch point twist field $\mathcal{T}_n$ (b) Composite twist field $\mathcal{T}_n^\mu$

Figure 1. Running conformal dimension of both the standard and the composite twist fields from the Δ-sum rule.

- In Fig. 1a we see that already at the 4-particle order the running dimension of the standard twist field $\mathcal{T}_n$ is monotonically decreasing.

Cumulant expansion of the entanglement entropy

- The form factor expansion of the entanglement entropies is expressed in terms of the cumulants

$$(1-n)S_n(M\ell) = \log\langle\mathcal{T}_n(0)\widetilde{\mathcal{T}}_n(\ell)\rangle \approx \sum_{r,l \text{ even}} c_{r,l}^{\mathcal{T}}(M\ell;n) + \text{const}, \tag{11}$$

$$c_{r,l}^{\mathcal{T}}(M\ell;n) = \sum_{j,j'}\int_{-\infty}^{+\infty}\frac{\prod_{i=1}^r \mathrm{d}\theta_i \prod_{k=1}^l \mathrm{d}\theta_k'}{r!\,l!\,(2\pi)^{r+l}} f_{r,l}^{\mathcal{T}|j_1\dots j_1'\dots}(\theta_1,\dots,\theta_1',\dots;n)\, e^{-\frac{M\ell}{2}\left(\sum_i e^{\theta_i}+\sum_k e^{\theta_k'}\right)}. \tag{12}$$

where $f_{r,l}^{\mathcal{T}|j_1\cdots}$ is the connected part of $\left|F_{r,l}^{\mathcal{T}|j_1\cdots}\right|^2$, obtained subtracting all possible clusterisations.
- We distinguish non-interacting cumulants $c_{r,0}^{\mathcal{T}}$ with only either left- or right-movers, and interacting ones which couple them.
- The non-interacting cumulants are identical to the ones of massive Ising model up to the energy and, after introducing an IR cut-off Λ, are logarithmic in the length of the interval

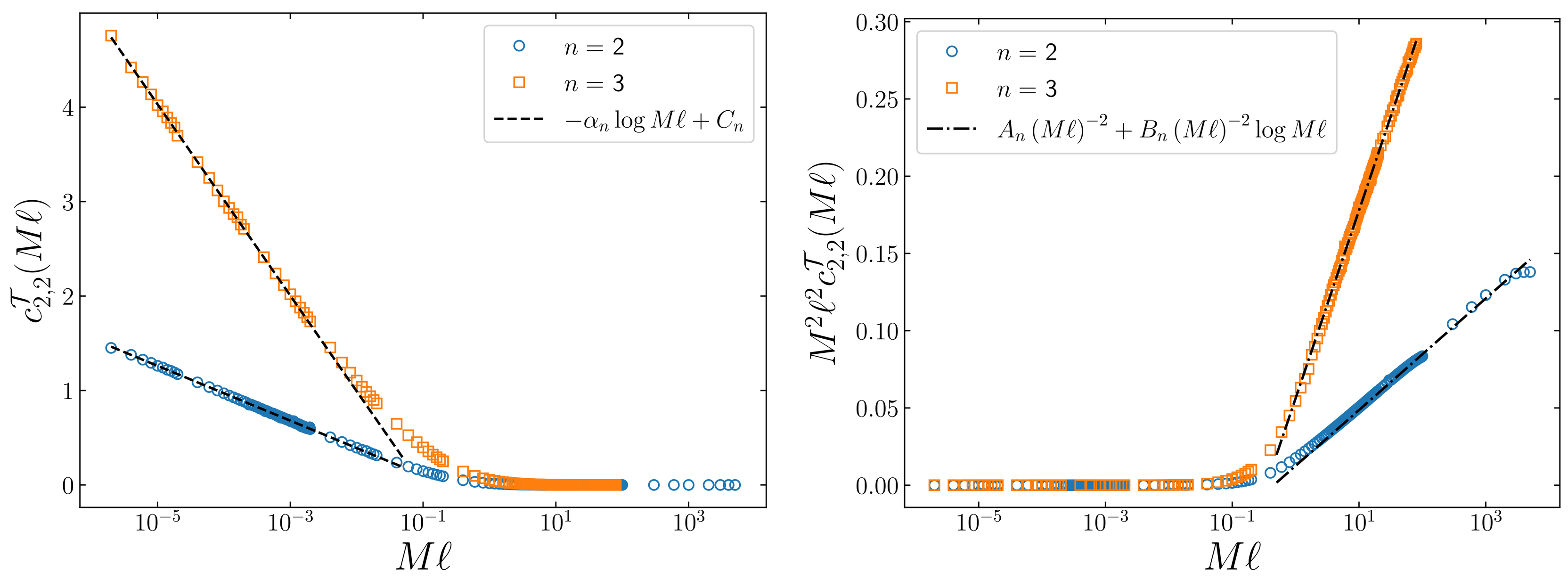
$$c_{r,0}^{\mathcal{T}}(M\ell,\Lambda;n) \approx -\frac{z_r(n)}{2}\log\frac{M\ell}{\Lambda} + \text{const}, \tag{13}$$

$$z_r(n) = \frac{2n}{r!\,(2\pi)^r}\sum_j\int_{-\infty}^{+\infty}\prod_{j=1}^{r-1}\mathrm{d}\theta_{j,j+1}\, f_{r,0}^{\mathcal{T}}(\theta_{12},\dots;n)\,. \tag{14}$$

- We conjecture that the non-interacting cumulants reproduce the logarithmic entanglement entropy of the IR Ising CFT, while the interacting ones provide the low distance UV corrections.

First correction to IR entanglement entropy

- The first correction to the Ising IR entanglement entropy is given by the cumulant $c_{2,2}^{\mathcal{T}}$ that we integrate numerically in units $M = 1$.



(a) Semilogarithmic plot of $c_{2,2}^{\mathcal{T}}$ for $n = 2, 3$ replicas. The dashed lines are the result of the fit to the logarithmic function $-\alpha_n \log\ell + C_n$ reported in Eqs. (15), (16).

(b) Semilogarithmic plot of the product of $c_{2,2}^{\mathcal{T}}$ for $n = 2, 3$ replicas times ℓ^2. The dash-dotted lines are the the fit to the function $A_n\,\ell^{-2} + B_n\,\ell^{-2}\log\ell$ reported in Eqs. (18), (19).

Figure 2. Plot of the cumulant $c_{2,2}^{\mathcal{T}}$, providing the first correction to the IR entanglement entropy.

- For small ℓ, as shown in Fig. 2b, the cumulant $c_{2,2}^{\mathcal{T}}$ decays logarithmically, as expected since it needs to contribute to the UV entanglement entropy of the tricritical Ising CFT.
- We fit for small ℓ the cumulant to the function $-\alpha_n\log\ell + C_n$, finding

$$\begin{aligned} \alpha_2 &= 0.126 \pm 0.002, & C_2 &= -0.19 \pm 0.02, & &\text{for } \ell \le 2\times 10^{-4}, & (15)\\ \alpha_3 &= 0.440 \pm 0.005, & C_3 &= -1.03 \pm 0.05, & &\text{for } \ell \le 6\times 10^{-4}. & (16) \end{aligned}$$

- For large ℓ the model is approximated by $T\bar{T}$-deformed Ising model [3, 4].
- In perturbation theory the first correction to the entanglement entropy was found in [8] to have the peculiar functional form $A_n\,\ell^{-2} + B_n\,\ell^{-2}\log\ell$

$$(1-n)\delta S_n^{(1)}(\ell, M) = \frac{\pi(n-1)^2(n+1)^2}{9\,n^3}\left[\frac{1}{16M^2\epsilon^2} - \frac{5}{4}\frac{1}{M^2\ell^2} + \frac{\log\frac{\ell}{2\epsilon}}{M^2\ell^2}\right] + \mathcal{O}\left(M^{-4}\ell^{-4}\right). \tag{17}$$

- In Fig. 2a we fit the cumulant $c_{2,2}^{\mathcal{T}}$ for large ℓ to the expected functional form, finding good qualitative agreement and obtaining parameters

$$\begin{aligned} A_2 &= 0.013 \pm 0.002, & B_2 &= 0.0157 \pm 0.0004, & &\text{for } \ell \ge 50, & (18)\\ A_3 &= 0.056 \pm 0.005, & B_3 &= 0.053 \pm 0.001, & &\text{for } \ell \ge 20. & (19) \end{aligned}$$

- The result of the fit is not in quantitative agreement with the prediction for perturbation theory in Eq. (17). This can be due to the fact that the perturbative and the form factor expansion do not agree order-by-order.

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