Homotopy Manin Triples and Higher Current Algebras

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based on work with **Luigi Alfonsi**

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and [XXXX.XXXXX] in preparation

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Overview: Main idea of this talk

- **Manin triples** of Lie algebras appear ubiquitously in integrable systems, quantum groups, ...
- In particular, they underlie the mathematical definition of **rational conformal blocks**...
- ...and thence of **vertex algebras**
  (which capture physicists' notion of operator product expansions in chiral CFTs)
- The relevant Lie algebras here are essentially **current algebras**

$$\mathfrak{g} \otimes \mathbb{C}((t))$$

  and their near relations.
- All of this appears to be closely tied to complex dimension one...
  (accords with general wisdom about Kac-Moody algebras being special/
  CFT in complex dimension one being special)

  ...but this intuition begins to break down in potentially fruitful ways, if one is prepared to work with "up to homotopy" with "higher" algebras.

Plan of talk

Manin triples, rational conformal blocks, vertex algebras (review)

Beyond complex dimension one?

Higher current algebras

Local and global homotopy Manin triples

Manin triples

A **Manin triple** $(\mathfrak{a}, \mathfrak{a}_+, \mathfrak{a}_-)$ of Lie algebras over $\mathbb{C}$ consists of

- a Lie algebra $\mathfrak{a}$
  equipped with a symmetric nondegenerate invariant bilinear form $\langle - \mid - \rangle$
- two isotropic Lie subalgebras $\mathfrak{a}_+$, $\mathfrak{a}_-$ such that

$$\mathfrak{a} =_{\mathbb{C}} \mathfrak{a}_+ \oplus \mathfrak{a}_-$$

  as vector spaces.

Example: Current Algebras

with $\mathfrak{g}$ a simple finite-dimensional Lie algebra over $\mathbb{C}$, let

$$\mathfrak{a} = \mathfrak{g} \otimes \mathbb{C}((x)), \qquad \mathfrak{a}_+ = \mathfrak{g} \otimes \mathbb{C}[[x]], \qquad \mathfrak{a}_- = \mathfrak{g} \otimes x^{-1}\mathbb{C}[x^{-1}]$$

and

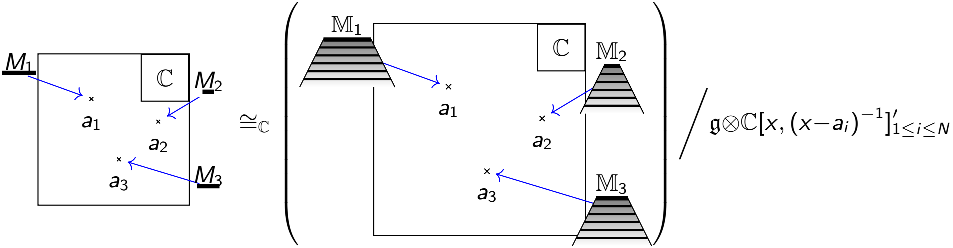
$$\langle f(t) \mid g(t) \rangle := \oint_{x=0} \kappa\big(f(t)|g(t)\big)\mathrm{d}t$$

Example: Rational Conformal Blocks

Take $a_1, \ldots, a_N \in \mathbb{C}$ distinct and let

$$\mathfrak{a} = \mathfrak{g} \otimes \bigoplus_{i=1}^{N} \mathbb{C}((x - a_i)), \qquad \mathfrak{a}_+ = \mathfrak{g} \otimes \bigoplus_{i=1}^{N} \mathbb{C}[[x - a_i]], \qquad \mathfrak{a}_- = \mathfrak{g} \otimes \mathbb{C}[x, (x - a_i)^{-1}]'_{1 \le i \le N}$$

Space of rational **coinvariants**:

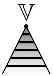


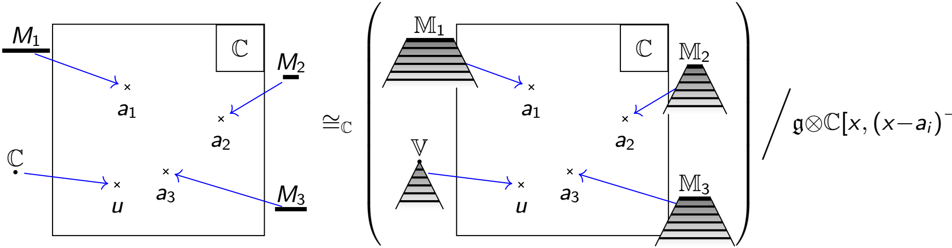
- Dual is space of **rational conformal blocks**.
- Which is the fibre of a trivial vector bundle over configuration space $\mathbf{Conf}_N(\mathbb{A}^1_{\mathbb{C}}) = \mathbb{A}^N_{\mathbb{C}} \setminus \{a_i = a_j\}$, which comes with the flat **KZ connection**

  . . .

Special case: **Vacuum module** induced from trivial module $\mathbb{C}|0\rangle$:

$$\mathbb{V} := \mathrm{Ind}_{\mathfrak{g}\otimes\mathbb{C}[[x]]}^{\mathfrak{g}\otimes\mathbb{C}((x))}\, \mathbb{C}|0\rangle$$





leads to at least two important constructions:

1. Structure of $\mathbb{V}$ as a **vertex algebra**, and $\mathbb{M}_i$ as modules over it
2. Gaudin models, opers, Bethe ansatz and geometric Langlands correspondence

Gaudin models and geometric Langlands correspondence

1. States $X \in \mathbb{V}$ go to linear operators $X(u) \in \mathrm{End}(M_1 \otimes \cdots \otimes M_N)$.
2. Can introduce central extension

$$0 \to \mathbb{C}k \to \widehat{\mathfrak{g}} \to \mathfrak{g} \otimes \mathbb{C}((u)) \to 0$$

3. Then. . .

[E. Frenkel, . . .]
[Feigin Frenkel Reshetikhin] [Mukhin Tarasov Varchenko] [Masoero Raimondo Valeri]

	Local		Global
$\mathfrak{g}$ (simple Lie algebra)	**singular vectors** in $\mathbb{V}$ at critical level $k = -h^\vee$	$\underset{\text{Coinvariants}}{\rightsquigarrow}$	commuting **Hamiltonians** of quantum Gaudin model
	$\updownarrow$		$\updownarrow$ Bethe Ansatz
${}^L\mathfrak{g}$ (its Langlands dual)	**opers** on the formal disc $\mathrm{Disc}_1^\times = \mathrm{Spec}\,\mathbb{C}((u))$	$\rightsquigarrow$	**opers** on $\mathbb{P}^1$

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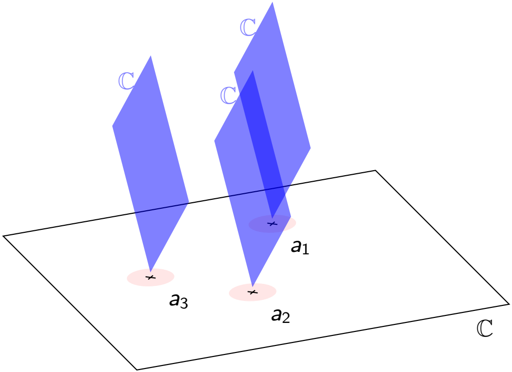
- the constructions above were all associated to the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ or the complex affine line $\mathbb{A}^1_{\mathbb{C}}$
- can we generalize to complex dimensions 2 or more?

Motivations include:

1. *(broad motivation)*
   vertex algebras/chiral CFTs/holomorphic field theory in higher dimensions
   [work of B. Williams, M. Szczesny, . . . , building on Costello-Gwilliam Factorization Algebras]

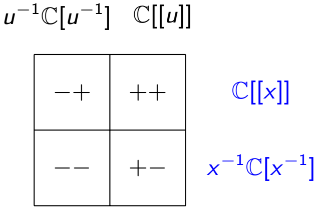
2. *(specialized motivation, this talk)*
   Gaudin models for affine Lie algebras
   Should describe integrals of motion of integrable quantum field theories.
   [Feigin, Frenkel] [Vicedo] [Masoero Raimondo Valeri] [. . .]



Worldsheet

$\times$

Spectral Plane



Toroidal algebras?
$\mathfrak{g} \otimes \mathbb{C}((x)) \otimes \mathbb{C}((u))$

Highest weight representations?

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[Faonte, Hennion, Kapranov]

- Observe
$$\mathbb{C}[[z]] \cong \Gamma(\mathrm{Disc}_1, \mathcal{O}) \qquad \mathbb{C}((z)) \cong \Gamma(\mathrm{Disc}_1^\times, \mathcal{O}),$$
where $\mathrm{Disc}_1 := \mathrm{Spec}\, \mathbb{C}[[z]]$ is the **formal 1-disc**, and
where $\mathrm{Disc}_1^\times := \mathrm{Disc}_1 \setminus \{\mathrm{pt.}\}$ is the **punctured formal 1-disc**.
- Natural to try same in higher dimensions, but...
$$\mathbb{C}[[w,z]] \cong \Gamma(\mathrm{Disc}_2, \mathcal{O}) \underset{!}{\cong} \Gamma(\mathrm{Disc}_2^\times, \mathcal{O}),$$
where $\mathrm{Disc}_2 := \mathrm{Spec}\, \mathbb{C}[[w,z]]$ is the formal 2-disc, and
where $\mathrm{Disc}_2^\times := \mathrm{Disc}_2 \setminus \{\mathrm{pt.}\}$ is the punctured formal 2-disc
(cf. Hartog's theorem)
- But there is higher sheaf cohomology!
$$H^\bullet(\mathrm{Disc}_2^\times, \mathcal{O}) \cong \begin{cases} \mathbb{C}[[w,z]] & \bullet = 0 \\ w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] & \bullet = 1 \\ 0 & \text{otherwise.} \end{cases}$$

- Consider **derived sections** $R\Gamma^\bullet(\mathrm{Disc}_2^\times, \mathcal{O})$.
  By definition $R\Gamma^\bullet(X, \mathcal{O})$ is a cochain complex such that

$$H^\bullet(R\Gamma(X, \mathcal{O})) \cong H^\bullet(X, \mathcal{O})$$

- Moreover, since $\mathcal{O}$ is a sheaf in commutative algebras,

$$R\Gamma^\bullet(X, \mathcal{O}) = \mathrm{holim}_{U \subset X}\, \Gamma(X, \mathcal{O})$$

  is canonically a **dg commutative algebra**, defined up to zigzags of quasi-isomorphisms.
- Get a **dg Lie algebra**

$$\mathfrak{g} \otimes R\Gamma^\bullet(\mathrm{Disc}_2^\times, \mathcal{O})$$

  – a **higher current algebra**

  [Faonte, Hennion, Kapranov]
- Need a good model of sheaf $U \mapsto R\Gamma^\bullet(U, \mathcal{O})$:
  Options: Dolbeault complex, in complex analytic setting
  Adelic complexes
  Cech complexes

Reminder: sheaf cohomology on a manifold/variety/scheme M

- Find a "good" cover $\{U_i\}$ of M. Then "replace" M by Cech nerve

$$\check{C}(\{U_i\}) := \left(\ \dots \ \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bigsqcup_{i<j} U_i \cap U_j \rightrightarrows \bigsqcup_i U_i \ \right),$$

  a semisimplicial object in manifolds/varieties/schemes, whose colimit is M.
- Apply global sections functor $\Gamma(-,\mathcal{O})$. Get

$$\Gamma(\check{C}(\{U_i\}),\mathcal{O}) = \left(\ \dots \quad \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \prod_{i<j} \Gamma(U_i \cap U_j,\mathcal{O}) \leftleftarrows \prod_i \Gamma(U_i,\mathcal{O}) \ \right),$$

  a semi<u>co</u>simplicial object in commutative algebras.

Example: $\mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\}$

Let $U_x = \{(x \neq 0)\}$, $U_y = \{(y \neq 0)\}$.

$$\left(\ \dots \mathbb{C}[x,y,x^{-1},y^{-1}] \leftleftarrows \mathbb{C}[x,y,x^{-1}] \times \mathbb{C}[x,y,y^{-1}] \ \right)$$

- Usual corresponding cochain complex computes the sheaf cohomology. But we lost the associative commutative algebra structure!
- Instead, work in **dg commutative** algebras, and take homotopy limit.

Thom-Sullivan functor $\mathrm{Th}^\bullet$

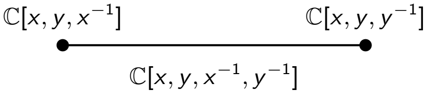
(rough idea) "Paint" polynomial differential forms onto the simplicial set, valued in the correct algebras.

Defines a functor

$$\mathrm{Th}^\bullet : [\Delta, \mathbf{CAlg}_{\mathbb{C}}] \to \mathbf{dgCAlg}_{\mathbb{C}}$$

from semicosimplicial algebras to dg algebras.

Example: $\mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\}$ continued...



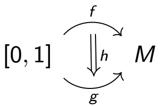
$$\mathrm{Th}^\bullet(\Gamma(\check{C}(\{U_x, U_y\}, \mathcal{O}))) = \big\{\omega \in \mathbb{C}[x, y, x^{-1}, y^{-1}] \otimes \mathbb{C}[v, \mathrm{d}v] \\ : \omega|_{v=0} \in \mathbb{C}[x, y, x^{-1}], \omega|_{v=1} \in \mathbb{C}[x, y, y^{-1}]\big\}$$

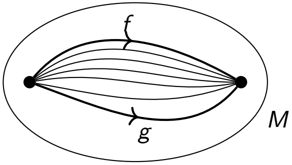
Gives model of $R\Gamma^\bullet(\mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\}, \mathcal{O})$ with dg commutative algebra structure.

Reminder: notions of homotopy

Prototypical example:

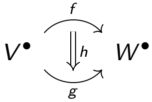
homotopies between maps of topological spaces,
e.g. between parameterized paths in a manifold M

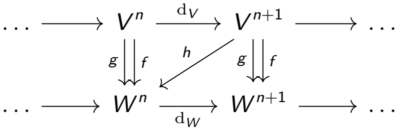




Relevant example in this talk:

homotopies between maps in $\mathbf{dgVect}_{\mathbb{C}}$, i.e. maps of cochain complexes in $\mathbb{C}$-vector spaces





$$f - g = \mathrm{d}_V \circ h + h \circ \mathrm{d}_W$$

Reminder: "sameness": notions of (weak) homotopy equivalence

When are two objects "the same"? Isomorphism might be too strong.

isomorphism	$A \underset{G}{\overset{F}{\rightleftarrows}} B$	$G \circ F = \mathrm{id}_A$ $F \circ G = \mathrm{id}_B$
$\Downarrow$		
deformation retract	$A \underset{G}{\overset{F}{\rightleftarrows}} B \circlearrowleft h$	$G \circ F = \mathrm{id}_A$ $F \circ G - \mathrm{id}_B = [\mathrm{d}_B, h]$
$\Downarrow$		
homotopy equivalence	$k \circlearrowright A \underset{G}{\overset{F}{\rightleftarrows}} B \circlearrowleft h$	$G \circ F - \mathrm{id}_A = [\mathrm{d}_A, k]$ $F \circ G - \mathrm{id}_B = [\mathrm{d}_B, h]$
$\Downarrow$		
weak equivalence aka **quasi-isomorphism**	$A \xrightarrow{F} B$	$H^\bullet(A) \xrightarrow{H(F)} H^\bullet(G)$ is an isomorphism

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Homotopy Manin triples

A strong homotopy Manin triple $(\mathfrak{a}, \mathfrak{a}_\pm, \iota_\pm)$ in $\mathbf{LieAlg}(\mathbf{dgVect}_\mathbb{C})$ is the data of:

- three objects $\mathfrak{a}$, $\mathfrak{a}_+$, $\mathfrak{a}_-$ in $\mathbf{LieAlg}(\mathbf{dgVect}_\mathbb{C})$,
- maps

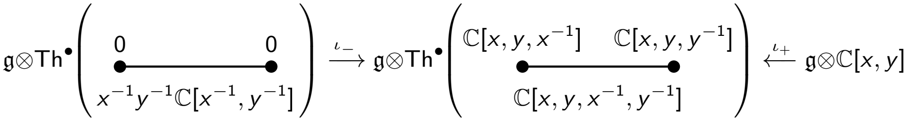
$$\mathfrak{a}_- \xrightarrow{\iota_+} \mathfrak{a} \xleftarrow{\iota_+} \mathfrak{a}_+ \qquad \text{in} \quad \mathbf{LieAlg}(\mathbf{dgVect}_\mathbb{C})$$

such that the resulting map $\mathfrak{a}_- \oplus \mathfrak{a}_+ \to \mathfrak{a}$ in $\mathbf{dgVect}_\mathbb{C}$, participates in a strong deformation retract

$$\mathfrak{a}_- \oplus \mathfrak{a}_+ \underset{\pi_- \oplus \pi_+}{\overset{(\iota_-, \iota_+)}{\rightleftarrows}} \mathfrak{a} \circlearrowleft h$$

(together with a pairing such that...)

Example: $\mathbb{A}^2_\mathbb{C} \setminus \{(0,0)\}$ continued...

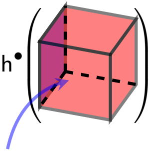
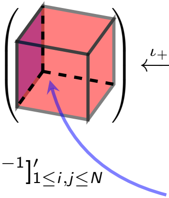


Homotopy Manin triples

"Global" example: $\mathbb{A}^2_{\mathbb{C}} \setminus \{(x_i, y_i)\}_{1\le i\le N}$

$$R\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{(w_i, z_i)\}_{1\le i\le N}, \mathcal{O})' \xrightarrow{\iota_-} \bigoplus_{i=1}^N R\Gamma(\mathrm{Disc}_2^\times(w_i, x_i), \mathcal{O}) \xleftarrow{\iota_+} \bigoplus_{i=1}^N \Gamma(\mathrm{Disc}_2(w_i, x_i), \mathcal{O})$$

Theorem [L. Alfonsi, CY]
There are models of these which form a strong homotopy Manin triple.
Flavour of the construction: polynomial differential forms on hypercubes:

$\mathrm{Th}^\bullet$  $\xrightarrow{\iota_-} \bigoplus_{i=1}^N \mathrm{Th}^\bullet$  $\xleftarrow{\iota_+} \bigoplus_{i=1}^N \mathbb{C}[[w - x_i]] \otimes \mathbb{C}[[z - y_i]]$

$\mathbb{C}[w, z, (w - x_j)^{-1}, (z - y_j)^{-1}]'_{1\le i,j\le N}$ $\qquad$ $\mathbb{C}((w - x_i)) \otimes \mathbb{C}((z - y_i))$

Moreover, can do a (dg analog of) extension of scalars, $\mathbb{C} \to R\Gamma(\mathbb{A}^{2N} \setminus \overline{(x_i = x_j, y_i = y_j)}, \mathcal{O})$ and work over dg algebra of functions on a higher analog of configuration space.

Conclusions and outlook

Main message:

Important Manin triples of Lie algebras associated to the punctured disc/complex plane have "higher" analogs, if one goes to dg Lie algebras. In particular, get a notion of **higher rational conformal blocks**.

Main open question:

Of the many constructions that start with such triples, which generalize to the higher setting?

- "Quantization" a la Drinfeld Yangians??
- (Shifted) central extensions?
- Higher vertex algebras?
  (and KZ equations? hypergeometric functions? quantum groups??)
- Higher Gaudin / Affine Gaudin models + integrable QFT?
- Higher analog of Feigin-Frenkel centre??